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College Precalculus, Evans, et al

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Printed: December 24, 2017

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CHAPTER 1 Prerequisites

Chapter Outline

1.1 Introduction: Prerequisites

What is "precalculus"? The course title gives you a clue. The prefix "pre" means "before." "Calculus" is the study of how things change. This course provides the knowledge required to study the ideas of integrals and derivatives in calculus courses.

This prerequisite chapter for precalculus addresses key concepts from algebra. How does algebraic thinking provide the tools necessary to describe and work with higher levels of mathematics? This chapter gives you a refresher of mathematics you've already studied that is essential to success in precalculus. Some of the ideas are based on ensuring that your knowledge of the terminology is proficient so you can apply the vocabulary in context. Others relate to formulas you have seen before but will need again. There are also procedures and steps you must apply to the new material found in this precalculus text.

1.2 Real Numbers

Learning Objectives

Learn how to classify, order, and graph real numbers.

Introduction

Suppose you and three friends were playing a game in which you each drew a number from a hat, and the person with the highest number won. Let's say you drew the number $\frac{3}{2}$, while your friends drew the numbers $\sqrt{3}$, 1.7, and $\frac{\pi}{3}$, respectively. Could you figure out who won the game? You'll learn how to classify, order, and graph real numbers so you can compare values such as these.

Real Numbers

Real numbers have always played an important role in mathematics, and their classification provides greater insight into their use. Real numbers are all the numbers on the number line, and there are infinitely many of them. Their types and categories are important because they can give you more information about the problem you are looking at. There are also imaginary numbers, a topic to be discussed later in this chapter. In fact, certain types of numbers can direct you to further formulas or definitions in mathematics.

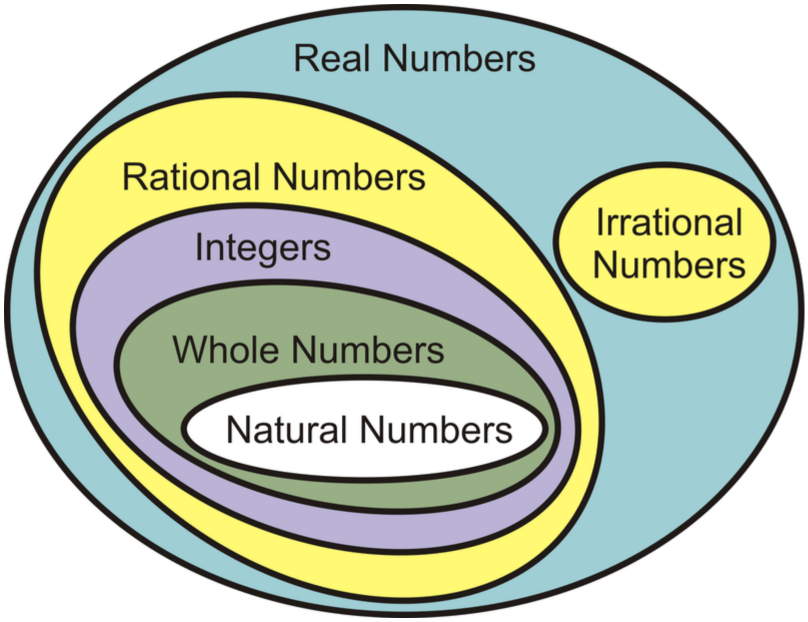
When you began to learn numbers, or when you teach numbers to a child, you count beginning with the number 1. For instance, 1, 2, 3, 4, ... These are classified as counting or natural numbers.

Not long after understanding how to count, you learn the idea of "none," when you are counting an item for which there are none to count. The number zero is introduced to identify quantities of no objects. This number is added to the counting numbers to create the set of whole numbers: 0, 1, 2, 3, ...

Then we discover numbers that are not positive when we attempt to take a larger number away from a smaller number. A debt is created that can be represented by negative numbers. The whole numbers along with their opposites (negatives) make up the set of integers: ... -3, -2, -1, 0, 1, 2, 3, ...

Next we explore numbers that are part of a whole. These numbers, which may come in the form of a fraction or decimal, represent portions of the whole, or a combination of wholes and portions of the whole. These are called rational numbers because each one can be represented as a ratio. Rational numbers include all of the integers as well. Here are some examples of integers represented as fractions: $\frac{12}{1}, -\frac{150}{3}, \frac{1}{-1}$, and $\frac{-74}{2}$. Other examples of rational numbers include $\frac{1}{4}, \frac{5}{4}$, and $\frac{3}{7}$.

Finally, there are some very unique numbers in that they cannot be represented as a ratio, and are thus called irrational numbers. They are very important to mathematics, so they are not disregarded. For example, π is essential to calculating the dimensions and the area of circles, spheres, and cylinders. There are also square (and higher-degree) roots that cannot be written as decimals, but are essential to solving problems, such as calculating the sides of a triangle. Some examples of irrational roots are $\sqrt{2}$ and $\sqrt[3]{7}$.



Types of Real Numbers

TABLE 1.1:

Type of Number	Definition	Examples
Real Numbers	Numbers that correspond to numbers on the number line	6, -.9785, π, $-\frac{17}{432}$
Irrational Numbers	Numbers that cannot be written as a fraction or ratio of integers	$\frac{\pi}{6}, \sqrt{2}, \sqrt[9]{67}, -13\sqrt{14}$
Rational Numbers	Numbers that can be written as a fraction or ratio, where *a* and *b* are integers and $\left\{\frac{a}{b} \text{ such that } b \neq 0\right\}$	$\frac{17}{19}$, 6, -19, $5.\overline{5}$
Integers	Positive and negative whole numbers and zero	-2, 14, -399, 0
Whole Numbers	Numbers that are not negative and have no fractional or decimal part	0, 1, 2, 3, ...
Counting or Natural Numbers	Numbers that are used to count: whole numbers and do not include zero or any negatives	1, 2, 3, ...

Classifying Real Numbers

See the video below for examples of how the set of real numbers is broken down into the subsets of rational and irrational numbers. It further explains how rational numbers are partitioned into integers, whole numbers, and counting or natural numbers. It also demonstrates how to compare real numbers using $>$, $<$, $\geq$, $\leq$, and $=$ symbols.

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Graphing and Ordering Real Numbers

Every non-integer real number can be positioned between two integers. Many times you will need to organize real numbers to determine the least value, greatest value, or both. This is usually done on a number line.

Examples

Example 1

Classify the following numbers*:*

a) 0

Solution:

Zero is a whole number, an integer, a rational number, and a real number.

b) -1

Solution:

-1 is an integer, a rational number, and a real number.

c) $\frac{\pi}{3}$

Solution:

Even though $\frac{\pi}{3}$ is a fraction, π is not an integer, so $\frac{\pi}{3}$ is an irrational number and a real number.

d) $\frac{\sqrt{36}}{9}$

Solution:

$$\frac{\sqrt{36}}{9} = \frac{6}{9} = \frac{2}{3}$$

This is a rational number and a real number.

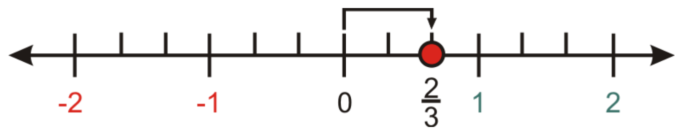
Example 2

Plot the following rational numbers on a number line:

a) $\frac{2}{3}$

Solution:

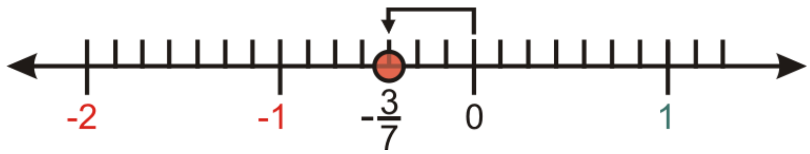
Upon dividing 2 by 3, the result is a repeating decimal, $0.\overline{6}$. So $\frac{2}{3} = 0.\overline{6}$, which is between 0 and 1.



b) $-\frac{3}{7}$

Solution:

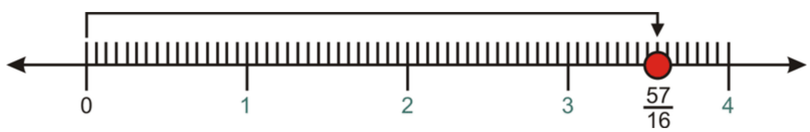
Upon dividing -3 by 7, the result is a repeating decimal, $-0.\overline{428571}$. So $-\frac{3}{7} = -0.\overline{428571}$, which is between -1 and 0.



c) $\frac{57}{16}$

Solution:

Upon dividing 57 by 16, the result is a finite decimal, 3.5625. So $\frac{57}{16} = 3.5625$, which is between 3 and 4.



Example 3

Compare $\frac{\pi}{15}$ and $\frac{\sqrt{3}}{\sqrt{75}}$.

Solution:

Step 1: Simplify the 2nd fraction in order to better compare the two numbers.

First, the denominator can be simplified by rewriting 75 into its factors, 3 and 25:

$$\sqrt{75} = \sqrt{3 \cdot 25}.$$

Next, take the square root of 25, because the square root of 25 is a whole number:

$$\sqrt{3 \cdot 25} = \sqrt{3} \cdot \sqrt{25} = 5\sqrt{3}.$$

Thus, the 2nd fraction simplifies to:

$$\frac{\sqrt{3}}{\sqrt{75}} = \frac{\sqrt{3}}{5\sqrt{3}} = \frac{1}{5}.$$

Step 2: Rewrite $\frac{\pi}{15}$ to compare it to $\frac{1}{5}$:

$$\frac{\pi}{15} = \frac{\pi}{3 \times 5} = \frac{\pi}{3} \times \frac{1}{5}.$$

Since $\pi > 3$, $\frac{\pi}{3} > 1$. Then, $\frac{\pi}{3} \times \frac{1}{5} > \frac{1}{5}$. Therefore, $\frac{\pi}{15} > \frac{\sqrt{3}}{\sqrt{75}}$.

Example 4

For the numbers $\frac{\sqrt{12}}{2}, 1.5 \cdot \sqrt{3}, \frac{3}{2}, \frac{2\sqrt{5}}{\sqrt{20}}$:

a) Classify each number:

Solution:

Simplify the numbers in order to classify them:

$\frac{\sqrt{12}}{2} = \frac{\sqrt{4 \times 3}}{2} = \frac{2\sqrt{3}}{2} = \sqrt{3}$. This is an irrational number. An irrational number is a type of real number.

$1.5 \cdot \sqrt{3}$. This number cannot be simplified, but since it is a multiple of an irrational number, it is also irrational. In other words, we cannot get rid of the irrational part, so we cannot write it as a rational number. It is also a real number.

$\frac{3}{2}$. Since this number is in the form of a proper fraction, it is also a rational number and a real number.

$\frac{2\sqrt{5}}{\sqrt{20}} = \frac{2\sqrt{5}}{\sqrt{4 \times 5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$. This number can be simplified to an integer. All positive integers can be expressed as natural numbers, whole numbers, and rational numbers. Integers are a special kind of real number.

b) Order the four numbers:

Solution:

The four numbers are ordered as follows: $1 < \frac{3}{2} < \sqrt{3} < 1.5 \cdot \sqrt{3}$.

$1 < \frac{3}{2}$ since the numerator is larger than the denominator and $\frac{3}{2} = 1.5$.

$\frac{3}{2} < \sqrt{3}$ since we can see on our calculators that $\sqrt{3} \approx 1.7$.

$\sqrt{3} < 1.5 \cdot \sqrt{3}$ since multiplying by 1.5 makes any positive number larger.

Review

Classify the numbers below as natural, whole, integer, irrational, rational, and/or real. Include all the categories that apply to the number.

1. -19
2. $\sqrt{0.25}$
3. $\frac{0}{4}$
4. $\sqrt{1.35}$
5. $-\frac{1}{9}$
6. $\frac{1}{5}$
7. $4\sqrt{2}$
8. 5
9. -2π
10. $\sqrt{100}$

Order the following numbers from least to greatest:

11.

$$\frac{\sqrt{6}}{2} \qquad \frac{61}{50} \qquad \sqrt{1.5} \qquad \frac{16}{13}$$

12.

$$-7\frac{1}{2} \qquad -3 \qquad 5.5 \qquad 7\frac{1}{2} \qquad -1.5$$

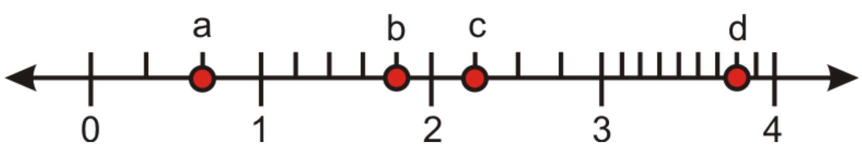
13.

$$\tfrac{9}{10} \qquad -2.2 \qquad \sqrt{4} \qquad -0.3$$

14.

$$\sqrt{3} \qquad 1.85 \qquad \tfrac{26}{10} \qquad 1 \qquad \tfrac{3}{4}$$

15. Find the value of each marked point:



Review (Answers)

Please see the Appendix.

Resource: Classifying Numbers

Watch the following video for additional review on the classification of numbers.

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1.3 Exponents

Learning Objectives

Learn how to write numbers as powers of other numbers and simplify expressions with exponents.

Introduction

Exponents are used to express quantities in a variety of fields. Economists use exponents to calculate compound interest. Biologists use exponents to model population growth. Chemists use exponents to express the kinetics involved in chemical reactions. Physicists use exponents to model wave behavior. Computer scientists use exponents in public-key cryptography. The properties and laws of exponents enable us to efficiently simplify exponential expressions. In this section, you will develop an understanding of the properties and laws of exponents through notes, videos, and practice examples.

Exponents

If $a \in \mathbb{R}$ (read as "a is an element in the set of real numbers") and $n \in \mathbb{N}$ ("n is an element in the set of natural numbers"), then the nth power of a is written as

$$a^n = \underbrace{a \times a \times a \times \ldots \times a}_{\substack{\downarrow \\ n \text{ factors}}},$$

where n is called the exponent and a is called the base. The term a^n is known as a power. In other words, $4^3 = 4 \times 4 \times 4 = 64$ and $2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$.

Multiplying and Dividing Exponential Terms

To multiply two powers with the same base, add the exponents:

$$a^m \times a^n = \underbrace{(a \times a \times \ldots \times a)}_{\substack{\downarrow \\ m \text{ factors}}} \underbrace{(a \times a \times \ldots \times a)}_{\substack{\downarrow \\ n \text{ factors}}}$$

$$a^m \times a^n = \underbrace{(a \times a \times a \ldots \times a)}_{\substack{\downarrow \\ m+n \text{ factors}}}$$

$$a^m \times a^n = a^{m+n}$$

To divide two powers with the same base, $a \neq 0$, subtract the exponents:

Raising Terms to an Exponent

To raise a power to a new exponent, multiply the exponents:

$$(a^m)^n = \underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}}{}^n$$

$$(a^m)^n = \underbrace{\underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}} \times \underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}} \, \underbrace{(a \times a \times \ldots \times a)}_{m \text{ factors}}}_{n \text{ times}}$$

$$(a^m)^n = \underbrace{a \times a \times a \ldots \times a}_{mn \text{ factors}}$$

$$(a^m)^n = a^{mn}$$

To raise a product to an exponent, raise each of the factors to the exponent:

$$(ab)^n = \underbrace{(ab) \times (ab) \times \ldots \times (ab)}_{n \text{ factors}}$$

$$(ab)^n = \underbrace{(a \times a \times \ldots \times a)}_{n \text{ factors}} \times \underbrace{(b \times b \times \ldots \times b)}_{n \text{ factors}}$$

$$(ab)^n = a^n b^n$$

To raise a quotient to an exponent, raise both the numerator and the denominator to the exponent:

$$\left(\frac{a}{b}\right)^n = \underbrace{\frac{a}{b} \times \frac{a}{b} \times \ldots \times \frac{a}{b}}_{n \text{ factors}}$$

$$\left(\frac{a}{b}\right)^n = \frac{\overbrace{(a \times a \times \ldots \times a)}^{n \text{ factors}}}{\underbrace{(b \times b \times \ldots \times b)}_{n \text{ factors}}}$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \; (b \neq 0)$$

Laws of Exponents

The following are basic rules or laws that govern powers and exponents:

Basic Laws of Exponents

If $a, b, m,$ and n are real numbers, then:

1. Multiplying Powers with the Same Base

   $a^m \times a^n = a^{m+n}$
2. Dividing Powers with the Same Base

   $\frac{a^m}{a^n} = a^{m-n}\ (a \neq 0)$
3. Raising a Power to a New Exponent

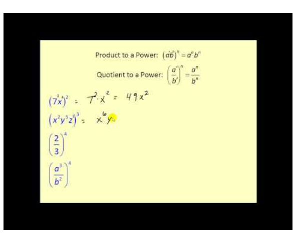
   $(a^m)^n = a^{mn}$
4. Raising a Product to an Exponent

   $(ab)^n = a^n b^n$
5. Raising a Quotient to an Exponent

   $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad (b \neq 0)$

Properties of Exponents

Watch the following video for examples of using the basic rules of exponents and an introduction to the zero exponent:



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Zero Exponent

The procedure for reducing exponents with like bases in a fraction is as follows:

$$\frac{a^m}{a^n} = a^{m-n}$$

If $m = n$, then the following would be true:

$$\frac{a^m}{a^n} = a^{m-n} = a^0$$
$$\frac{3^3}{3^3} = 3^{3-3} = 3^0$$

Any quantity divided by itself is equal to 1. For example, $\frac{3^3}{3^3} = 1$ so $3^0 = 1$.

Zero as an Exponent

In general, $a^0 = 1$ if $a \neq 0$.
Note that if $a = 0$, 0^0 is not defined.

Negative Exponents

If we apply the general rule for multiplying exponents with a common base, $a^m \times a^n = a^{m+n}$, we can determine the property of negative exponents. Review the following example:

$$4^2 \times 4^{-2} = 4^{2+(-2)} = 4^0 = 1$$

Therefore,

$$4^2 \times 4^{-2} = 1$$
$$\frac{4^2 \times 4^{-2}}{4^2} = \frac{1}{4^2} \qquad \text{Divide both sides by } 4^2.$$
$$\frac{\cancel{4^2} \times 4^{-2}}{\cancel{4^2}} = \frac{1}{4^2} \qquad \text{Simplify the equation.}$$
$$4^{-2} = \frac{1}{4^2}$$

This is true in general, and creates the following laws for negative exponents:

Laws for Negative Exponents

If $a \neq 0$, then:

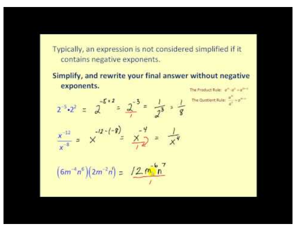
1. $a^{-m} = \frac{1}{a^m}$
2. $\frac{1}{a^{-m}} = a^m$

These laws for negative exponents can be expressed in many ways:

- If a term has a negative exponent, write it as 1 over the term with a positive exponent. For example, $a^{-m} = \frac{1}{a^m}$ and $\frac{1}{a^{-m}} = a^m$.
- If a term has a negative exponent, write the reciprocal with a positive exponent. For example, $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2$ and $a^{-m} = \frac{a^{-m}}{1} = \frac{1}{a^m}$.
- If the term is a factor in the numerator with a negative exponent, write it in the denominator with a positive exponent. For example, $3x^{-3}y = \frac{3y}{x^3}$ and $a^{-m}b^n = \frac{1}{a^m}(b^n) = \frac{b^n}{a^m}$.
- If the term is a factor in the denominator with a negative exponent, write it in the numerator with a positive exponent. For example, $\frac{2x^3}{x^{-2}} = 2x^3(x^2)$ and $\frac{b^n}{a^{-m}} = b^n\left(\frac{a^m}{1}\right) = b^n a^m$.

These ways of understanding negative exponents provide algorithms for arriving at solutions without doing multiple steps of calculations. The results will be the same.

Watch the following video for examples of simplifying expressions that involve negative exponents:



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Writing a Number as a Power of Another Number

In many of the previous examples, we evaluated powers with numerical bases by expanding the power into its factors and determining the product of the factors.

2^4 was expanded to $2 \times 2 \times 2 \times 2$.

The product was determined:

$$2 \times 2 = 4$$
$$4 \times 2 = 8$$
$$8 \times 2 = 16$$

Therefore, $\boxed{2^4 = 16}$

This concept can also be reversed. Write 32 as a power of 2:

$$32 = 2 \times 2 \times 2 \times 2 \times 2 = 2^5$$

There are 5 twos. Therefore, $\boxed{32 = 2^5}$

Examples

Example 1

Evaluate the following using the laws of exponents:

a) $\left(\frac{3}{4}\right)^{-2}$

Solution:

Method 1: Apply the negative exponent rule.

$$\left(\frac{3}{4}\right)^{-2} = \frac{1}{\left(\frac{3}{4}\right)^2}$$

Write the expression with a positive exponent by applying $\boxed{a^{-m} = \frac{1}{a^m}}$.

$$\frac{1}{\left(\frac{3}{4}\right)^2} = \frac{1}{\frac{3^2}{4^2}}$$

Apply the law of exponents for raising a quotient to a power $\boxed{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$.

$$\frac{1}{\frac{3^2}{4^2}} = \frac{1}{\frac{9}{16}}$$

Evaluate the powers.

$$\frac{1}{\frac{9}{16}} = 1 \div \frac{9}{16}$$

Divide.

$$1 \div \frac{9}{16} = 1 \times \frac{16}{9} = \frac{16}{9}$$

$$\boxed{\left(\frac{3}{4}\right)^{-2} = \frac{16}{9}}$$

Method 2: Apply a shortcut and write the reciprocal with a positive exponent.

$$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$$

Write the reciprocal with a positive exponent.

$$\left(\frac{4}{3}\right)^2 = \frac{4^2}{3^2}$$

Apply the law of exponents for raising a quotient to a power $\boxed{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$.

$$\frac{4^2}{3^2} = \frac{16}{9}$$

Simplify.

$$\boxed{\left(\frac{3}{4}\right)^{-2} = \frac{16}{9}}$$

Applying this shortcut facilitates the process for obtaining the solution.

b) $3^2 \times 3^3$

Solution:

$3^2 \times 3^3$	The base is 3.
3^{2+3}	Keep the base of 3 and add the exponents.
3^5	This answer is in exponential form.

The answer can be taken one step further. The base is numerical so the term can be evaluated:

$$3^5 = 3 \times 3 \times 3 \times 3 \times 3$$
$$3^5 = 243$$
$$\boxed{3^2 \times 3^3 = 3^5 = 243}$$

c) $2^7 \div 2^3$

Solution:

$2^7 \div 2^3$	The base is 2.
2^{7-3}	Keep the base of 2 and subtract the exponents.
2^4	The answer is in exponential form.

The answer can be taken one step further. The base is numerical so the term can be evaluated:

$$2^4 = 2 \times 2 \times 2 \times 2$$
$$2^4 = 16$$
$$\boxed{2^7 \div 2^3 = 2^4 = 16}$$

d) $(2^3 \times 3^2)^2$

Solution:

$(2^3 \times 3^2)^2$	The base is $2^3 \times 3^2$.
$(2)^{3\times2} \cdot (3)^{2\times2}$	Keep the base of $2^3 \times 3^2$ and raise each factor of the base to the power of 2.
$2^6 \times 3^4$	Simplify. Apply the exponent to each factor of the base.
$2^6 \times 3^4$	The answer is in exponential form.

The answer can be taken one step further. The base of each factor is numerical so each term can be evaluated. The final answer will be the product of the two answers:

$$2^6 = 2\times2\times2\times2\times2\times2$$
$$2^6 = 64$$
$$3^4 = 3\times3\times3\times3$$
$$3^4 = 81$$
$$64\times81 = 5184$$
$$\boxed{(2^3\times3^2)^2 = 2^6\times3^6 = 5184}$$

e) $\left(\frac{2}{3}\right)^2$

Solution:

$\left(\frac{2}{3}\right)^2$ The base is $\frac{2}{3}$.

$\frac{2^{1\times2}}{3^{1\times2}}$ Keep the base of $\frac{2}{3}$ and multiply the exponents of both the numerator and the denominator by 2.

$\frac{2^2}{3^2}$ The answer is in exponential form.

The answer can be taken one step further. The base is numerical so each term can be evaluated:

$$2^2 = 2\times2 \qquad 3^2 = 3\times3$$
$$2^2 = 4 \qquad 3^2 = 9$$
$$\frac{4}{9}$$
$$\boxed{\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}}$$

Example 2

Simplify the following (using only positive exponents and, if possible, shortcuts):

a) y^{-6}

Solution:

y^{-6} Write the expression with a positive exponent by applying $\boxed{a^{-m} = \frac{1}{a^m}}$.

$$\boxed{y^{-6} = \frac{1}{y^6}}$$

b) $\left(\frac{a}{b}\right)^{-3}$

Solution:

$\left(\frac{a}{b}\right)^{-3}$ Write the reciprocal with a positive exponent.

$\left(\frac{a}{b}\right)^{-3} = \left(\frac{b}{a}\right)^{3}$ Apply the law of exponents for raising a quotient to a power $\boxed{\left(\frac{a}{b}\right)^{n} = \frac{a^n}{b^n}}$.

$\left(\frac{b}{a}\right)^{3} = \frac{b^3}{a^3}$

$$\boxed{\left(\frac{a}{b}\right)^{-3} = \frac{b^3}{a^3}}$$

c) $(-3x)^2$

Solution:

$(-3x)^2$ The base is $-3x$.

$(-3)^{1\times 2} \cdot (x)^{1\times 2}$ Keep the base of $-3x$ and raise each factor of the base to the power of 2.

$(-3)^2 \cdot (x)^2$ Simplify. Apply the exponent to each factor of the base.

$9x^2$ The answer is in exponential form.

$$\boxed{(-3x)^2 = 9x^2}$$

d) $\frac{y^3}{y^{-5}}$

Solution:

$\frac{y^3}{y^{-5}}$ The base is y.

$y^{3-(-5)}$ Keep the base of y and subtract the exponents.

y^8 The answer is in exponential form.

$$\boxed{\frac{y^3}{y^{-5}} = y^8}$$

e) $\left(\frac{4a^5b^3}{6ab}\right)^3$

Solution:

$\left(\frac{4a^5b^3}{6ab}\right)^3$	The base is $\frac{4a^5b^3}{6ab}$. Begin by simplifying the base.
$\frac{2}{3}a^{5-1}b^{3-1} = \frac{2a^4b^2}{3}$	Write the problem with the simplified base.
$\left(\frac{2a^4b^2}{3}\right)^3$	
$\frac{2^{1\times3}a^{4\times3}b^{2\times3}}{3^{1\times3}}$	Keep the base of $\frac{2a^4b^2}{3}$ and multiply the exponents of both the numerator and the denominator by 3.
$\frac{2^3a^{12}b^6}{3^3}$	The answer is in exponential form.

The answer can be taken one step further. The denominator and the numerator both have numerical coefficients to be evaluated:

$$2^3 = 2\times2\times2 \quad 3^3 = 3\times3\times3$$
$$2^3 = 8 \quad 3^3 = 27$$
$$\frac{8a^{12}b^6}{27}$$
$$\boxed{\left(\frac{4a^5b^3}{6ab}\right)^3 = \left(\frac{2a^4b^2}{3}\right)^3 = \frac{2^3a^{12}b^6}{3^3} = \frac{8a^{12}b^6}{27}}$$

Example 3

Use the above concept to answer the following:

a) Write 81 as a power of 3.

Solution:

$81 = 3\times3\times3\times3 = 3^4$

There are 4 threes. Therefore, $\boxed{81 = 3^4}$

b) Write $(9)^3$ as a power of 3.

Solution:

$9 = 3\times3 = 9$

There are 2 threes. Therefore, $\boxed{9 = 3^2}$

$(3^2)^3$ Apply the law of exponents for power to a power—multiply the exponents.

$3^{2\times3} = 3^6$

Therefore, $\boxed{(9)^3 = 3^6}$

c) Write $(4^3)^2$ as a power of 2.

Solution:

$4 = 2\times2 = 4$

There are 2 twos. Therefore, $\boxed{4 = 2^2}$

$\left((2^2)^3\right)^2$ Apply the law of exponents for power to a power—multiply the exponents:

$\boxed{2^{2\times 3} = 2^{6}}$

$(2^6)^2$ Apply the law of exponents for power to a power—multiply the exponents:

$\boxed{2^{6\times 2} = 2^{12}}$

Therefore, $\boxed{(4^3)^2 = 2^{12}}$

Example 4

a) Use the laws of exponents to simplify the following: $(-3x^2)^3(9x^4y)^{-2}$

Solution:

Apply the law of exponents to simplify:

$$\begin{aligned}(-3x^2)^3(9x^4y)^{-2} &= (-3)^3x^6 \cdot \frac{1}{(9x^4y)^2}\\ &= -27x^6 \cdot \frac{1}{9^2x^8y^2}\\ &= \frac{-27x^6}{81x^8y^2}\\ &= -\frac{1x^{-2}}{3y^2}\\ &= -\frac{1}{3x^2y^2}\end{aligned}$$

b) Rewrite the following using only positive exponents: $(x^2y^{-1}-1)^2$

Solution:

$(x^2y^{-1}-1)^2$ Begin by expanding the binomial.

$(x^2y^{-1}-1)(x^2y^{-1}-1)$ Then multiply the terms by distribution. Remember to apply the product rule for exponents $\boxed{a^m \times a^n = a^{m+n}}$.

$(x^{2+2}y^{-1+(-1)} - 1x^2y^{-1} - 1x^2y^{-1} + 1)$ Simplify.

$(x^4y^{-2} - 2x^2y^{-1} + 1)$ Apply the negative exponent rule $\boxed{a^{-m} = \frac{1}{a^m}}$.

$$\left(\frac{x^4}{y^2} - \frac{2x^2}{y} + 1\right)$$

$$\boxed{(x^2y^{-1}-1)^2 = \left(\frac{x^4}{y^2} - \frac{2x^2}{y} + 1\right)}$$

c) Use the laws of exponents to evaluate the following: $[5^{-4} \times (25)^3]^2$

Solution:

$[5^{-4} \times (25)^3]^2$

Try to do this one by applying the laws of exponents.

$[5^{-4} \times (25)^3]^2 = [5^{-4} \times (5^2)^3]^2$

$[5^{-4} \times (5^2)^3]^2 = [5^{-4} \times 5^6]^2$

$[5^{-4} \times 5^6]^2 = (5^2)^2$

$(5^2)^2 = 5^4$

$5^4 = 625$

$$\boxed{[5^{-4} \times (25)^3]^2 = 5^4 = 625}$$

Summary

- In an algebraic expression, the **base** is the variable, number, product, or quotient to which the exponent refers.
- In an algebraic expression, the **exponent** is the number to the upper right of the base that tells how many times to multiply the base by itself.
- In the expression 2^5, "2" is the base and "5" is the exponent. This means multiply 2 by itself 5 times, as shown here: $2 \times 2 \times 2 \times 2 \times 2$.
- A **power** is simply the name given to an algebraic expression that is raised to an exponent. 2^5 and $(-3y)^4$ are both examples of a power.
- The **laws of exponents** are the algebraic rules and formulas that tell us the operation to perform on the exponents when dealing with exponential expressions. These laws are:

Laws of Exponents

Multiplication, Division, and Power Rules:

- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad (b \neq 0)$
- $(ab)^n = a^n b^n$
- $(a^m)^n = a^{mn}$
- $\frac{a^m}{a^n} = a^{m-n} \qquad (a \neq 0)$
- $a^m \cdot a^n = a^{m+n}$

Zero and Negative Exponent Rules:

- $a^{-n} = \frac{1}{a^n} \qquad (a \neq 0)$
- $a^0 = 1 \qquad (a \neq 0)$

Review

Evaluate each of the following expressions:

1. $(-3)^{-3}$
2. $6\times\left(\frac{1}{2}\right)^{-2}$
3. $7^{-4}\times 7^{4}$
4. $(4^{0}+4^{-1})^{-1}$
5. $-(3^{2})^{3}$
6. $\left(\frac{1}{3}\right)^{6}\div\left(\frac{1}{3}\right)^{8}$

Simplify the expressions below. (Your answers should have only positive exponents.)

7. $(x^{-1}+y^{-1})^{2}$
8. $(4wx^{-2}y^{3}z^{-4})^{3}$
9. $\frac{a^{2}b^{3}c^{-2}}{d^{-2}bc^{-6}}$
10. $m^{4}(m^{2}+m-5m^{-2})$
11. $(x^{3}y^{2})(xy^{3})(x^{5}y)$
12. $\frac{x^{6}y^{8}}{x^{4}y^{-2}}$
13. $\left(\frac{2x^{10}}{3y^{20}}\right)^{3}$
14. $(10x^{8})\div(2x^{4})$
15. $(-2x)^{5}(2x^{2})$
16. $(16x^{10})\left(\frac{3}{4}x^{5}\right)$
17. $\frac{(x^{15})(x^{24})(x^{25})}{(x^{7})^{8}}$

Express each of the following as a power of 3. Do not evaluate.

18. $(3^{3})^{5}$
19. 9^{4}
20. $(9)(27^{2})$

Review (Answers)

Please see the Appendix.

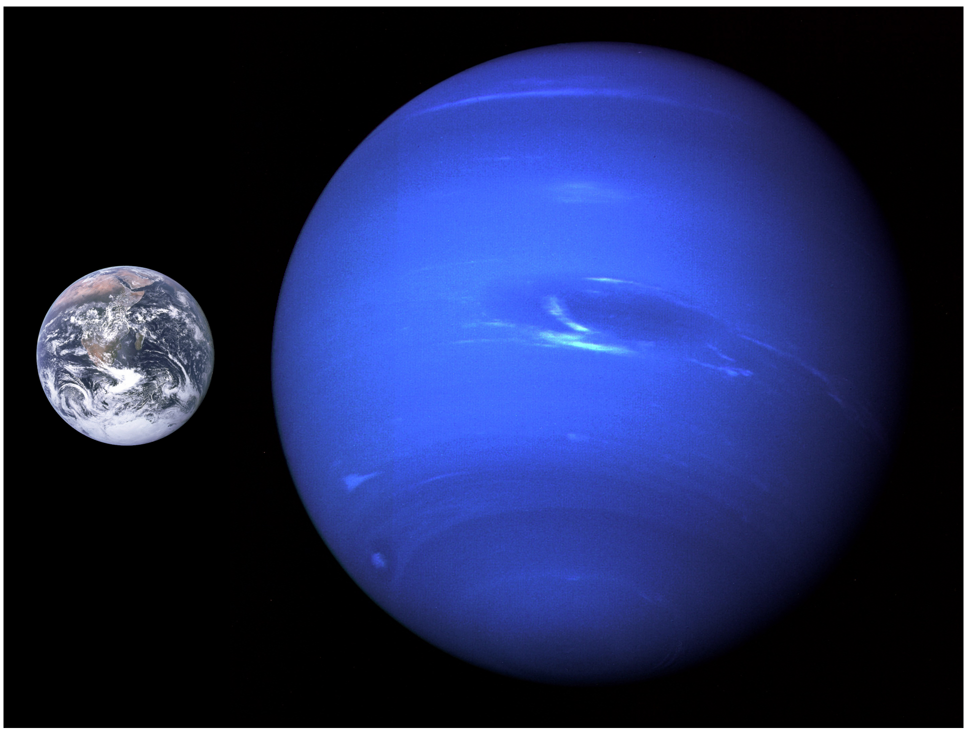
1.4 Scientific Notation

Learning Objectives

Learn how to write numbers in scientific notation and calculate operations on numbers in scientific notation.

Introduction

Very large and very small quantities and measures are often used to provide information for magazines, textbooks, television, newspapers, and the internet.



Some examples are:

- The distance between the sun and Neptune is 4,500,000,000 km.
- The diameter of an electron is approximately 0.000 000 000 000 22 inches.

Scientific notation is a convenient way to represent such numbers. How can you write the numbers above using scientific notation?

Operations with Scientific Notation

To represent a number in scientific notation means to express the number as a product of two factors: a number between 1 and 10 (including 1) and a power of 10. A positive real number x is said to be written in **scientific**

notation if it is expressed as

$$x = a \times 10^n$$

where

$$1 \leq a < 10 \text{ and } n \in I \text{ } (n \text{ is an integer}).$$

In other words, a number in scientific notation is a single, nonzero digit followed by a decimal point and other digits, all multiplied by a power of 10.

When working with numbers written in scientific notation, you can use the rules below. These rules are verified by example in Examples 2 and 3, below.

Scientific Notation Rules

For $m, n \in I$,

$$(A \times 10^n) + (B \times 10^n) = (A + B) \times 10^n$$

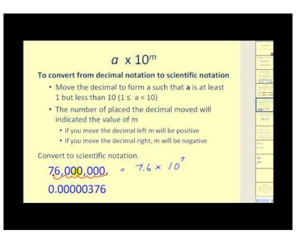
$$(A \times 10^n) - (B \times 10^n) = (A - B) \times 10^n$$

$$(A \times 10^m) \times (B \times 10^n) = (A \times B) \times (10^{m+n})$$

$$(A \times 10^m) \div (B \times 10^n) = (A \div B) \times (10^{m-n})$$

Writing and Operating with Scientific Notation

Watch the following video for an overview on converting between decimal notation and scientific notation, and on multiplying and dividing numbers in scientific notation:



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Examples

Example 1

Write the following numbers using scientific notation:

a) 2679000

Solution:

$$2679000 = 2.679 \times 1000000$$
$$2.679 \times 1000000 = 2.679 \times 10^{6}$$

The exponent, $n = 6$, represents the position of the decimal point that is 6 places to the right of the standard position of the decimal point.

b) 0.00005728

Solution:

$$0.00005728 = 5.728 \times 0.00001$$
$$5.728 \times 0.00001 = 5.728 \times \frac{1}{100000}$$
$$5.728 \times \frac{1}{100000} = 5.728 \times \frac{1}{10^{5}}$$
$$5.728 \times \frac{1}{100000} = 5.728 \times 10^{-5}$$

The exponent, $n = -5$, represents the position of the decimal point that is 5 places to the left of the standard position of the decimal point.

One advantage of scientific notation is that calculations with large or small numbers can be done by applying the laws of exponents.

Example 2

Complete the following table:

TABLE 1.2:

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$1.3 \times 10^5 + 2.5 \times 10^5$	$130,000 + 250,000$	$380,000$	3.8×10^5
$3.7 \times 10^{-2} + 5.1 \times 10^{-2}$	$0.037 + 0.051$	0.088	8.8×10^{-2}
$4.6 \times 10^4 - 2.2 \times 10^4$	$46,000 - 22,000$	$24,000$	2.4×10^4
$7.9 \times 10^{-2} - 5.4 \times 10^{-2}$	$0.079 - 0.054$	0.025	2.5×10^{-2}

Note that the numbers in the last column have the same power of 10 as those in the 1st column.

Example 3

Complete the following table:

TABLE 1.3:

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$(3.6 \times 10^2) \times (1.4 \times 10^3)$	$360 \times 1,400$	$504,000$	5.04×10^5
$(2.5 \times 10^3) \times (1.1 \times 10^{-6})$	$2,500 \times 0.0000011$	0.00275	2.75×10^{-3}
$(4.4 \times 10^4) \div (2.2 \times 10^2)$	$44,000 \div 220$	200	2.0×10^2

TABLE 1.3: (continued)

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$(6.8 \times 10^{-4}) \div (3.2 \times 10^{-2})$	$0.00068 \div 0.032$	0.02125	2.125×10^{-2}

Note that for multiplication, the power of 10 is the result of adding the exponents of the powers in the 1st column. For division, the power of 10 is the result of subtracting the exponents of the powers in the 1st column.

Example 4

Calculate each of the following:

a) $4.6 \times 10^4 + 5.3 \times 10^5$

Solution:

Before the rule

$$(A \times 10^n) + (B \times 10^n) = (A + B) \times 10^n$$

can be used, one of the numbers must be rewritten so that the powers of 10 are the same.

Rewrite 4.6×10^4

$4.6 \times 10^4 = (0.46 \times 10^1) \times 10^4$ The power 10^1 indicates the number of places to the right that the decimal point must be moved to return 0.46 to the original number of 4.6.

$(0.46 \times 10^1) \times 10^4 = 0.46 \times 10^5$ Add the exponents of the power.

Rewrite the question and substitute 4.6×10^4 with 0.46×10^5.

$0.46 \times 10^5 + 5.3 \times 10^5$

Apply the rule $(A \times 10^n) + (B \times 10^n) = (A + B) \times 10^n$.

$$(0.46 \times 10^5) + (5.3 \times 10^5) = (0.46 + 5.3) \times 10^5$$
$$(0.46 + 5.3) \times 10^5 = 5.76 \times 10^5$$
$$4.6 \times 10^4 + 5.3 \times 10^5 = 5.76 \times 10^5$$

b) $4.7 \times 10^{-3} - 2.4 \times 10^{-4}$

Solution:

Before the rule

$$(A \times 10^n) - (B \times 10^n) = (A - B) \times 10^n$$

can be used, one of the numbers must be rewritten so that the powers of 10 are the same.

Rewrite 4.7×10^{-3}

$4.7 \times 10^{-3} = (47 \times 10^{-1}) \times 10^{-3}$ The power 10^{-1} indicates the number of places to the left that the decimal point must be moved to return 47 to the original number of 4.7.

$(47 \times 10^{-1}) \times 10^{-3} = 47 \times 10^{-4}$ Add the exponents of the power.

Rewrite the question and substitute 4.7×10^{-3} with 47×10^{-4}.

$47 \times 10^{-4} - 2.4 \times 10^{-4}$

Apply the rule $(A \times 10^n) - (B \times 10^n) = (A - B) \times 10^n$.

$$(47 \times 10^{-4}) - (2.4 \times 10^{-4}) = (47 - 2.4) \times 10^{-4}$$
$$(47 \times 10^{-4}) - (2.4 \times 10^{-4}) = 44.6 \times 10^{-4}$$

The answer must be written in scientific notation.

$$44.6 \times 10^{-4} = (4.46 \times 10^{1}) \times 10^{-4}$$

Apply the law of exponents
– add the exponents of the power.

$$4.46 \times 10 \times 10^{-4} = 4.46 \times 10^{-3}$$
$$4.7 \times 10^{-3} - 2.4 \times 10^{-4} = 4.46 \times 10^{-3}$$

c) $(7.3 \times 10^5) \times (6.8 \times 10^4)$

Solution:

$$7.3 \times 10^5 \times 6.8 \times 10^4$$

Apply the rule
$(A \times 10^m) \times (B \times 10^n) = (A \times B) \times (10^{m+n})$.

$$= (7.3 \times 6.8) \times (10^{5+4})$$
$$= (49.64) \times (10^9)$$
$$= 49.64 \times 10^9$$

Write the answer in scientific notation.

$$= (4.964 \times 10^1) \times 10^9$$

Apply the law of exponents
– add the exponents of the power.

$$49.64 \times 10^9 = 4.964 \times 10^{10}$$
$$(7.3 \times 10^5) \times (6.8 \times 10^4) = 4.964 \times 10^{10}$$

d) $(4.8 \times 10^9) \div (5.79 \times 10^7)$

Solution:

$$(4.8 \times 10^9) \div (5.79 \times 10^7)$$

Apply the rule
$(A \times 10^m) \div (B \times 10^n) = (A \div B) \times (10^{m-n})$.

$$= (4.8 \div 5.79) \times 10^{9-7}$$

Apply the law of exponents
– subtract the exponents of the power.

$$= (0.829) \times 10^2$$

Write the answer in scientific notation.

$$= (8.29 \times 10^{-1}) \times 10^2$$

Apply the law of exponents
– add the exponents of the power.

$$= 8.29 \times 10^1$$

Example 5

a) Express the following product in scientific notation: $(4 \times 10^{12})(9.2 \times 10^{7})$.

Solution:

Apply the rule:

$$(A \times 10^{m}) \times (B \times 10^{n}) = (A \times B) \times (10^{m+n}).$$

$$\begin{aligned}
&(4 \times 10^{12}) \times (9.2 \times 10^{7}) = (4 \times 9.2) \times (10^{12+7}) \\
&(4 \times 9.2) \times (10^{12+7}) = 36.8 \times 10^{19}
\end{aligned}$$

Express the answer in scientific notation:

$$\begin{aligned}
&36.8 \times 10^{19} = (3.68 \times 10^{1}) \times 10^{19} \\
&(3.68 \times 10^{1}) \times 10^{19} = 3.68 \times 10^{20} \\
&(4 \times 10^{12})(9.2 \times 10^{7}) = 3.68 \times 10^{20}
\end{aligned}$$

b) Express the following quotient in scientific notation: $\frac{6400000}{0.008}$.

Solution:

Begin by expressing the numerator and the denominator in scientific notation:

$$\frac{6.4 \times 10^{6}}{8.0 \times 10^{-3}}$$

Apply the rule $(A \times 10^{m}) \div (B \times 10^{n}) = (A \div B) \times (10^{m+n})$.

$$\begin{aligned}
(6.4 \times 10^{6}) \div (8.0 \times 10^{-3}) &= (6.4 \div 8.0) \times (10^{6-(-3)}) \\
&= (0.8) \times (10^{9}) \\
&= 0.8 \times 10^{9} \\
&= (8.0 \times 10^{-1}) \times 10^{9} \\
&= 8.0 \times 10^{-1} \times 10^{9} \\
&= 8.0 \times 10^{8} \\
\frac{6400000}{0.008} &= 8.0 \times 10^{8}
\end{aligned}$$

c) If

$$\begin{aligned}
a &= 0.000415 \\
b &= 521 \\
c &= 71640
\end{aligned}$$

calculate the value for $\frac{ab}{c}$. Express the answer in scientific notation.

Solution:

$$0.000415 = 4.15 \times 10^{-4}$$
$$521 = 5.21 \times 10^{2}$$
$$71640 = 7.1640 \times 10^{4}$$

Use the values in scientific notation to determine the value for $\frac{ab}{c}$:

$$\frac{ab}{c} = \frac{(4.15 \times 10^{-4})(5.21 \times 10^{2})}{7.1640 \times 10^{4}}$$

In the numerator, apply the rule $(A \times 10^m) \times (B \times 10^n) = (A \times B) \times (10^{m+n})$.

$$\frac{(4.15 \times 10^{-4})(5.21 \times 10^{2})}{7.1640 \times 10^{4}} = \frac{(4.15 \times 5.21) \times (10^{-4} \times 10^{2})}{7.1640 \times 10^{4}}$$
$$\frac{(4.15 \times 5.21) \times (10^{-4} \times 10^{2})}{7.1640 \times 10^{4}} = \frac{21.6215 \times 10^{-2}}{7.1640 \times 10^{4}}$$

Apply the rule $(A \times 10^m) \div (B \times 10^n) = (A \div B) \times (10^{m-n})$.

$$\frac{21.6215 \times 10^{-2}}{7.1640 \times 10^{4}} = (21.6215 \div 7.1640) \times (10^{-2-4})$$
$$(21.6215 \div 7.1640) \times (10^{-2-4}) = 3.018 \times 10^{-6}$$

Review

Express each of the following in scientific notation:

1. 42000
2. 0.00087
3. 150.64
4. 56789
5. 0.00947

Express each of the following in standard form:

6. 4.26×10^{5}
7. 8×10^{4}
8. 5.967×10^{10}
9. 1.482×10^{-6}
10. 7.64×10^{-3}

Perform the indicated operations and express the answer in scientific notation:

11. $8.9 \times 10^{4} + 4.3 \times 10^{5}$

12. $8.7 \times 10^{-4} - 6.5 \times 10^{-5}$
13. $(5.3 \times 10^{6}) \times (7.9 \times 10^{5})$
14. $(3.9 \times 10^{8}) \div (2.8 \times 10^{6})$

For the given values, perform the indicated operations for $\frac{ab}{c}$, and express the answer in scientific notation and standard form:

15.

$$\begin{aligned} a &= 76.1 \\ b &= 818000000 \\ c &= 0.000016 \end{aligned}$$

16.

$$\begin{aligned} a &= 9.13 \times 10^{9} \\ b &= 5.45 \times 10^{-23} \\ c &= 1.62 \end{aligned}$$

Review (Answers)

Please see the Appendix.

1.5 Radicals

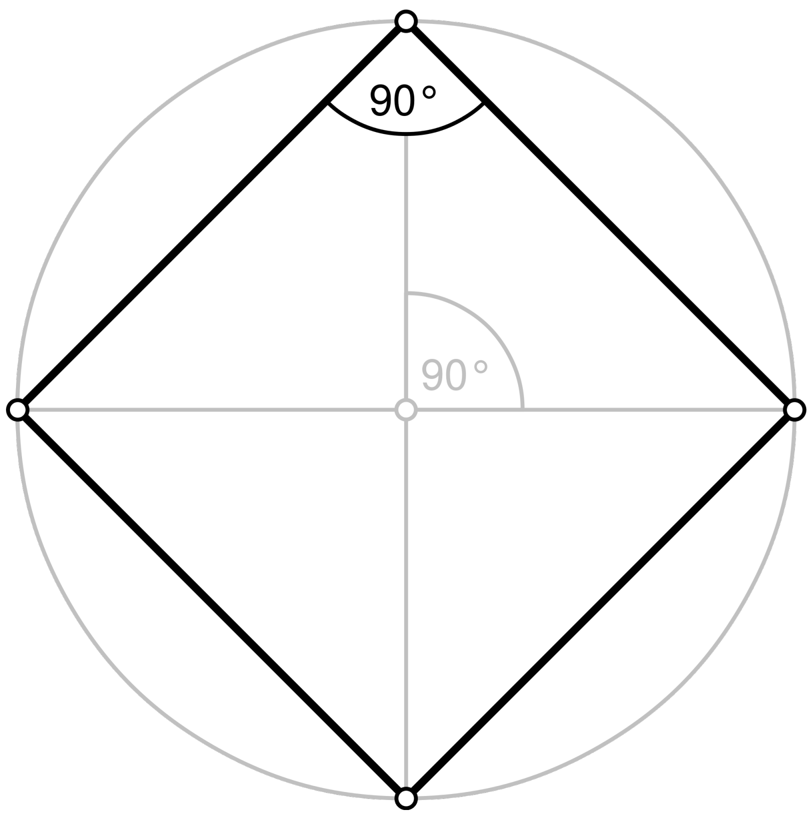
Learning Objectives

Learn how to add, subtract, multiply, and divide radicals in order to simplify expressions.

Learn how to write roots as fractional exponents.

Introduction

What if you knew the area of a square was 1,000 square meters, and you wanted to find the length of its side? After completing this concept, you'll be able to find square roots like this one by hand and with a calculator.



Roots and Radicals

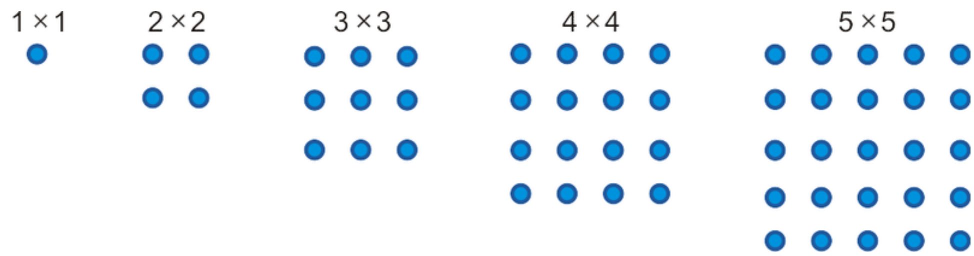
Square Roots

A **square root** of a number is a number that, when multiplied by itself, gives the original number. In other words, if $a = b^2$, we say that b is the square root of a.

Note: Negative numbers and positive numbers both yield positive numbers when squared, so each positive number has both a positive and a negative square root. (For example, 3 and -3 can both be squared to yield 9.) The positive square root of a number is called the **principal square root**.

The square root of a number x is written as $\sqrt{x}$ or sometimes as $\sqrt[2]{x}$. The symbol $\sqrt{\ }$ is called a **radical sign.**

Numbers with whole-number square roots are called **perfect squares**. The 1st five perfect squares (1, 4, 9, 16, and 25) are shown below.

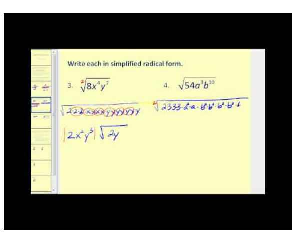


You can determine whether a number is a perfect square by looking at its prime factors. If every number in the factor tree appears an even number of times, the number is a perfect square. To find the square root of that number, simply take one of each pair of matching factors and multiply them together.

Simplifying Radical Expressions

In general, the nth root of a number x is written as $\sqrt[n]{x}$. The n is called the **index**, $\sqrt{\ }$ is called the **radical sign**, and x is called the **radicand**. In the example $\sqrt[3]{5}$, "3" is the index and "5" is the radicand.

The video below provides an overview of how to simplify radicals that are not perfect roots. The narrator provides learners with a four-step process that can be used to simplify radicals and models the process with examples.



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Square Root Rules

Here are four rules that govern how we treat square roots:

- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
- $A\sqrt{a} \times B\sqrt{b} = AB\sqrt{ab}$
- $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}, b \neq 0$
- $\frac{A\sqrt{a}}{B\sqrt{b}} = \frac{A}{B}\sqrt{\frac{a}{b}}, b \neq 0$

Approximating Square Roots

Terms like $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{7}$ (square roots of prime numbers) cannot be written as **rational numbers**. That is to say, they cannot be expressed as the ratio of two integers. We call them **irrational numbers**. In decimal form, an irrational number is a nonterminating and nonrepeating string of numbers after the decimal point.

To find approximate values for square roots, we use the $\sqrt{\ }$ or $\sqrt{x}$ button on a calculator. When the number we plug in is a perfect square, or the square of a rational number, we will get an exact answer. When the number is a non-perfect square, the answer will be irrational and will look like a random string of digits. Since the calculator can only show some of the infinitely many digits that are actually in the answer, it is really showing us an **approximate answer**—not exactly the right answer, but as close as it can get.

Multiplying Radicals

The following video provides an overview with examples of how to multiply and divide radicals:

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The rules for multiplication of square roots apply to nth roots provided they have the same index.

Multiplying Radicals

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab} \quad \text{where } n \text{ is the index}$$

Dividing Radicals

Dividing radicals is more complicated. A radical in the denominator of a fraction is not considered simplified by mathematicians, because traditional division requires dividing by an integer and not a irrational number. Since most radicals are irrational numbers, you must **rationalize the denominator** in order to simplify the fraction.

To **rationalize the denominator** means to remove any radical signs from the denominator of the fraction using multiplication.

Remember:

$$\sqrt{a} \times \sqrt{a} = \sqrt{a^2} = a$$

Writing the Square Root as an Exponent

1. Evaluate $\left(\sqrt{x}\right)^2$. What happens?

The $\sqrt{\ }$ and the ^{2} cancel each other out, $\left(\sqrt{x^2}\right) = x$.

2. Recall that when a power is raised to another power, we multiply the exponents:

$$\left(\sqrt{x}\right)^2 = (x^n)^2 = x^{n \cdot 2} = x^1 = x$$

3. Thus, we can rewrite the exponents and root as an equation, $n \cdot 2 = 1$. Solve for n:

$$\begin{aligned} \frac{n \cdot \cancel{2}}{\cancel{2}} &= \frac{1}{2} \\ n &= \frac{1}{2} \end{aligned}$$

4. Therefore, $\sqrt{x} = x^{\frac{1}{2}}$.

$$\left(\sqrt{x}\right)^2 = \left(x^{\frac{1}{2}}\right)^2 = x^{\left(\frac{1}{2}\right) \cdot 2} = x^1 = x$$

Similarly, evaluate $\left(\sqrt[3]{x}\right)^3 = x$.

$$\left(\sqrt[3]{x}\right)^3 = \left(x^{\frac{1}{3}}\right)^3 = x^{\left(\frac{1}{3}\right) \cdot 3} = x^1 = x$$

Thus, $\sqrt[3]{x} = x^{\frac{1}{3}}$.

From this investigation, we can extend this idea to the other roots as well:

$$\sqrt[4]{x} = x^{\frac{1}{4}}, \ \sqrt[5]{x} = x^{\frac{1}{5}}, \ \ldots \ \sqrt[n]{x} = x^{\frac{1}{n}}.$$

The Rational Exponent Theorem:

For any real number a, index n, and exponent m, the following is always true:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Using a Calculator to Solve for Roots

To find $5^{\frac{2}{3}}$ using a calculator (if rounding your answer to the nearest hundredth).

To type this into a calculator, the keystrokes would probably look like: $5\hat{}\left(\frac{2}{3}\right)$.

The “^” symbol is used to indicate a power. Anything in parenthesis after the “^” would be in the exponent. Type: 5^(2/3). Evaluating this, the calculator would read 2.924017738... Round to the nearest hundredth for 2.92.

Some calculators might have a x^y button. This button has the same purpose as the ^ and would be used in place of ^.

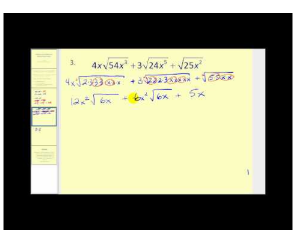
Adding and Subtracting Radicals

Suppose you’re taking a trip, and you’ll be making two stops. This distance from your starting point to your 1st stop is $14\sqrt{2}$ miles, and the distance from your 1st stop to your 2nd stop is $9\sqrt{2}$ miles. How far will you travel in total? What operation would you have to perform to find the answer to this question? To add or subtract radicals, they must have the same root and radicand.

Adding and Subtracting Radicals

$$a\sqrt[n]{x} + b\sqrt[n]{x} = (a+b)\sqrt[n]{x}$$

The following video provides an overview of adding and subtracting radicals, including relevant vocabulary, steps, and examples:



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Examples

Example 1

Determine the principal square root of each of these perfect squares:

a) 121

Solution:

$121 = 11 \times 11$, so $\sqrt{121} = 11$.

b) 225

Solution:

$225 = (5 \times 5) \times (3 \times 3)$, so $\sqrt{225} = 5 \times 3 = 15$.

c) 324

Solution:

$324 = (2 \times 2) \times (3 \times 3) \times (3 \times 3)$, so $\sqrt{324} = 2 \times 3 \times 3 = 18$.

When the prime factors don't pair up neatly, we "factor out" the ones that do pair up and leave the rest under a radical sign. We write the answer as $a\sqrt{b}$, where a is the product of half the paired factors we pulled out and b is the product of the leftover factors.

Example 2

Determine principal or positive square root of the following numbers:

a) 8

Solution: $\sqrt{8} = \sqrt{(2 \times 2) \times 2} = 2\sqrt{2}$. This gives us one pair of 2s and one leftover 2 inside the square root.

b) 48

Solution:

$\sqrt{48} = \sqrt{(2 \times 2 \times 2 \times 2) \times 3} = 4\sqrt{3}$

c) 75

Solution:

$\sqrt{75} = \sqrt{(5 \times 5) \times 3} = 5\sqrt{3}$

Example 3

Simplify the following square-root problems:

a) $\sqrt{8} \times \sqrt{2}$

Solution:

$\sqrt{8} \times \sqrt{2} = \sqrt{16} = 4$

b) $3\sqrt{4} \times 4\sqrt{3}$

Solution:

$3\sqrt{4} \times 4\sqrt{3} = 12\sqrt{12} = 12\sqrt{(2 \times 2) \times 3} = 12 \times 2\sqrt{3} = 24\sqrt{3}$

c) $\sqrt{12} \div \sqrt{3}$

Solution:

$\sqrt{12} \div \sqrt{3} = \sqrt{\frac{12}{3}} = \sqrt{4} = 2$

d) $12\sqrt{10} \div 6\sqrt{5}$

Solution:

$12\sqrt{10} \div 6\sqrt{5} = \frac{12}{6}\sqrt{\frac{10}{5}} = 2\sqrt{2}$

Example 4

Use a calculator to find the following square roots. Round your answer to three decimal places.

a) $\sqrt{99}$

Solution:

≈ 9.950

b) $\sqrt{0.5}$

Solution:

≈ 0.707

c) $\sqrt{1.75}$

Solution:

≈ 1.323

d) $\sqrt[7]{12}$

Solution:

Using rational exponents, the 7^{th} root becomes the $\frac{1}{7}$ power;$12^{\frac{1}{7}} \approx 1.426$.

Example 5

Simplify $\sqrt{3} \cdot \sqrt{12}$.

Solution:

$\sqrt{3} \cdot \sqrt{12} = \sqrt{36} = 6$

Example 6

Simplify $\frac{2}{\sqrt{3}}$.

Solution:

We must clear the denominator of its radical using the property above. Remember, what you do to one piece of a fraction, you must do to all pieces of the fraction:

$$\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3^2}} = \frac{2\sqrt{3}}{3}$$

Let's now consider how to convert roots into exponents. Look at the square root and see if we can use the properties of exponents to determine what exponential number it is equivalent to.

Example 7

Simplify the following radical expressions:

a) $256^{\frac{1}{4}}$

Solution:

Rewrite this expression as a radical expression. A number to the one-fourth power is the same as the 4th root:

$256^{\frac{1}{4}} = \sqrt[4]{256} = \sqrt[4]{4^4} = 4$

Therefore, $256^{\frac{1}{4}} = 4$.

b) $49^{\frac{3}{2}}$

Solution:

In this problem, the root is written as an exponent. Rewrite the problem as a radical expression:

$49^{\frac{3}{2}} = \left(49^3\right)^{\frac{1}{2}} = \sqrt{49^3}$ or $\left(\sqrt{49}\right)^3$

It may be easier to evaluate the 2nd option above: $\left(\sqrt{49}\right)^3 = 7^3 = 343$ or $\left(\sqrt{49}\right)^3 = \sqrt{49} \cdot \sqrt{49} \cdot \sqrt{49} = 7 \cdot 7 \cdot 7 = 343$.

c) $125^{\frac{4}{3}}$

Solution:

$125^{\frac{4}{3}} = \left(\sqrt[3]{125}\right)^4 = 5^4 = 625$

d) $256^{\frac{5}{8}}$

Solution:

$256^{\frac{5}{8}} = \left(\sqrt[8]{256}\right)^5 = 2^5 = 32$

Example 8

Simplify $\frac{7}{\sqrt[3]{5}}$.

Solution:

In this case, we need to make the number inside the cube root a perfect cube. We need to multiply the numerator and the denominator by $\sqrt[3]{5^2}$.

$$\frac{7}{\sqrt[3]{5}} \cdot \frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}} = \frac{7\sqrt[3]{25}}{\sqrt[3]{5^3}} = \frac{7\sqrt[3]{25}}{5}$$

Example 9

A planet's maximum distance from the sun (in astronomical units) is given by the formula $d = p^{\frac{2}{3}}$ where p is the period (in years) of the planet's orbit around the sun. If a planet's orbit around the sun is 27 years, what is its distance from the sun?

Solution:

Substitute 27 for p and solve.

$d = 27^{\frac{2}{3}}$

Rewrite the problem:

$27^{\frac{2}{3}} = \left(27^2\right)^{\frac{1}{3}} = \sqrt[3]{27^2}$ or $\sqrt[3]{27}^2$

$\left(\sqrt[3]{27}\right)^2 = 3^2 = 9.$

Therefore, the planet's distance from the sun is 9 astronomical units.

Example 10

Simplify the following radical expressions:

a) $3\sqrt{5}+6\sqrt{5}$.

Solution:

Since both terms in the sum have a factor of "$\sqrt{5}$", this factor is treated similar to like terms. Using the rule above:

$$3\sqrt{5}+6\sqrt{5} = (3+6)\sqrt{5} = 9\sqrt{5}$$

b) $2\sqrt[3]{13}+6\sqrt[3]{12}$.

Solution:

Since the two terms do have have the same radical factor, there can be no further simplification.

In some cases, the radical may need to be reduced before addition/subtraction is possible.

c) $4\sqrt{3}+2\sqrt{12}$.

Solution:

The first term is already simplified and $\sqrt{12}$ simplifies to $2\sqrt{3}$.

$$4\sqrt{3}+2\sqrt{12} = 4\sqrt{3}+2\left(2\sqrt{3}\right)$$
$$4\sqrt{3}+4\sqrt{3} = 8\sqrt{3}$$

d) $3\sqrt[3]{2}+5\sqrt[3]{16}$.

Solution:

Step 1: Factor the 2nd radical to simplify:

$$\begin{aligned}3\sqrt[3]{2}+5\sqrt[3]{16} &= 3\sqrt[3]{2}+5\sqrt[3]{2\cdot 8}\\ &= 3\sqrt[3]{2}+5\sqrt[3]{2\cdot 2^3}\\ &= 3\sqrt[3]{2}+5\sqrt[3]{2^3}\cdot\sqrt[3]{2}\\ &= 3\sqrt[3]{2}+5\cdot 2\sqrt[3]{2}\\ &= 3\sqrt[3]{2}+10\sqrt[3]{2}\end{aligned}$$

Step 2: Combine the two radicals:

$$\begin{aligned}3\sqrt[3]{2}+10\sqrt[3]{2} &= (3+10)\sqrt[3]{2}\\ &= 13\sqrt[3]{2}\end{aligned}$$

Review

For 1-10, find the following square roots exactly without using a calculator, giving your answer in the simplest form:

1. $\sqrt{25}$
2. $\sqrt{24}$
3. $\sqrt{2000}$
4. $\sqrt{\frac{1}{4}}$ (Hint: The division rules you learned can be applied backwards!)
5. $\sqrt{\frac{9}{4}}$
6. $\sqrt{0.16}$
7. $\sqrt{0.01}$

For 8-15, use a calculator to solve. Round to two decimal places.

8. $\sqrt{13}$
9. $\sqrt{99}$
10. $\sqrt{2}$
11. $\sqrt{2000}$
12. $\sqrt{.25}$
13. $\sqrt{1.35}$
14. $\sqrt{0.37}$
15. $\sqrt{0.01}$

Rewrite using rational exponents or roots, and use a calculator for the problems below. Round to two decimal places.

16. $\sqrt[5]{45}$
17. $\sqrt[9]{140}$
18. $\sqrt[8]{50}^3$
19. $72^{\frac{5}{3}}$
20. $125^{\frac{3}{4}}$

Multiply the following expressions:

21. $\sqrt{6}\left(\sqrt{10}+\sqrt{8}\right)$
22. $\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)$
23. $(2\sqrt{x}+5)(2\sqrt{x}+5)$

Rationalize the denominator:

24. $\frac{7}{\sqrt{15}}$
25. $\frac{9}{\sqrt{10}}$
26. $\frac{2x}{\sqrt{5x}}$

Evaluate the following without a calculator:

27. $64^{\frac{2}{3}}$
28. $27^{\frac{4}{3}}$

29. $16^{\frac{5}{4}}$
30. $\sqrt{25^3}$
31. $\sqrt[2]{9}^5$
32. $\sqrt[5]{32^2}$

For the following problems, rewrite the expressions with rational exponents and then simplify the exponent and evaluate without a calculator:

33. $\sqrt[4]{\left(\frac{2}{3}\right)^8}$
34. $\sqrt[3]{\frac{7}{2}}^6$
35. $\sqrt{(16)^{\frac{1}{2}}}^6$

Write the following expressions in simplest radical form:

36. $\sqrt[3]{48a^3b^7}$
37. $\sqrt[3]{\frac{16x^5}{135y^4}}$
38. True or false? $\sqrt[7]{5} \cdot \sqrt[6]{6} = \sqrt[42]{30}$
39. $3\sqrt{8} - 6\sqrt{32}$
40. $\sqrt{6} - \sqrt{27} + 2\sqrt{54} + 3\sqrt{48}$
41. $\sqrt{8x^3} - 4x\sqrt{98x}$
42. $\sqrt{48a} + \sqrt{27a}$
43. $\sqrt[3]{4x^3} + x\sqrt[3]{256}$

Solve the following:

44. The volume of a spherical balloon is $950 cm^3$. Find the radius of the balloon. (Volume of a sphere $= \frac{4}{3}\pi R^3$)
45. The volume of a soda can is 355 cm^3. The height of the can is 4 times the radius of the base. Find the radius of the base of the cylinder.

Review (Answers)

Please see the Appendix.

1.6 Evaluating Expressions

Learning Objectives

Learn how to plug the values into the variable(s) in an expression and simplify the expression.

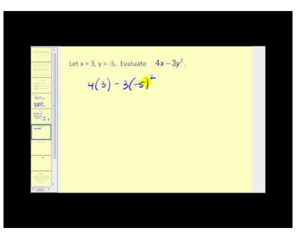
Introduction

Suppose you know the area of a circle is approximately $3.14r^2$, where r is the radius of the circle. What if a circle has a radius of 25 inches? How would you find its area? In this section, you'll learn how to substitute 25 inches into the expression, in place of the r (radius), and then evaluate the expression.

Evaluating Expressions

In algebra, evaluating an expression commonly means replacing any variables (letters) in the expression with given values, and then simplifying the expression by performing any operations involved. When simplifying, make sure to follow the order of operations.

The following video provides an overview of evaluating expressions, including vocabulary definitions, a general procedure, and examples:



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Examples

Example 1

Evaluate $7y - 11$, when $y = 4$.

Solution:

Evaluate means to follow the directions, which is to take 7 times y and subtract 11. Because y is the number 4, we can evaluate our expression as follows:

$7 \times (y) - 11$	First, substitute the number 4 in place of y.
$7 \times 4 - 11$	Then multiply 7 by 4.
$28 - 11$	Subtract 11 from 28.
17	The solution is 17.

Because algebra uses variables to represent unknown quantities, the multiplication symbol $\times$ is often confused with the variable x. To help avoid confusion, mathematicians replace the multiplication symbol with parentheses ( ) or the multiplication dot •, or by eliminating the symbol entirely if a number is being multiplied by a variable. For example, all four of these expressions have the same meaning:

$$4 \times a + 3 \times b$$
$$4(a) + 3(b)$$
$$4 \cdot a + 3 \cdot b$$
$$4a + 3b$$

Example 2

Rewrite $P = 2 \times l + 2 \times w$ with alternative multiplication symbols.

Solution:

$P = 2 \times l + 2 \times w$ can be written as $P = 2 \cdot l + 2 \cdot w$.

It can also be written as $P = 2l + 2w$.

The following is a real-life example that shows the importance of evaluating a mathematical expression:

Example 3

To prevent major accidents or injuries, horses are to be fenced in a rectangular pasture. If the dimensions of the pasture are 300 feet by 225 feet, how much fencing should the ranch hand purchase to enclose the pasture?

Solution:

Begin by drawing a diagram of the pasture and labeling what you know:

To find the amount of fencing needed, you must add all the sides together—find the perimeter of

the pasture:

$$P = L + L + W + W = 2L + 2W$$

Substituting the values of the variables L and W yields

$$P = 2 \cdot 300 + 2 \cdot 225 = 1,050$$

feet of fencing.

Example 4

a) Write the expression $2 \times a$ in a more condensed form using alternate multiplication symbols and then evaluate it for $3 = a$.

Solution:

$2 \times a$ can be written as $2(a),\ 2a,$ or $2 \cdot a$. We can substitute 3 for a: 2(3) = 6.

b) If it costs \$9.25 for each movie ticket, how much does it cost for 4 people to see a movie?

Solution:

Since each movie ticket is \$9.25, we multiply this price by 4 people to get the total cost:

$\$9.25 \times 4 = \$37$

It costs \$37 for 4 people to see a movie.

Example 5

Evaluate the following if $a = \text{-}2,\ c = 4,$ and $d = \text{-}7$.

$$\frac{2a}{c-d}$$

Solution:

Start by substituting in for the variables:

$$\frac{2(\text{-}2)}{4-(\text{-}7)}$$

Simplify:

$$\frac{2(\text{-}2)}{4-(\text{-}7)} = \frac{\text{-}4}{4+7} = \frac{\text{-}4}{11}$$

Review

In 1-4, write the expression in a more condensed form by leaving out the multiplication symbol:

1. $2 \times 11x$
2. $1.35 \cdot y$
3. $3 \times \frac{1}{4}$
4. $\frac{1}{4} \cdot z$

In 5-8, evaluate the expression:

5. $5m+7$, when $m=3$
6. $\frac{1}{3}(c)$, when $c=63$
7. $(k-11) \div 8$, when $k=43$
8. $(-2)^2+3(j)$, when $j=-3$

In 9-13, evaluate the expression. Let $a=\text{-}3,\ b=2,\ c=5,$ and $d=\text{-}4$.

9. $4c+d$
10. $5ac-2b$
11. $\frac{3b}{d}$
12. $\frac{a-4b}{3c+2d}$
13. $\frac{ab}{cd}$

In 14-18, evaluate the expression. Let $x=\text{-}1,\ z=\text{-}3,$ and $w=4$.

14. $8x^3$
15. $3z^2-5w^2$
16. $\frac{z^3+w^3}{z^3-w^3}$
17. $2x^2-3x^2+5x-4$
18. $3+\frac{1}{z^2}$

In 19-23, evaluate the expression in each real-life problem:

19. The measurement around the widest part of these holiday bulbs is called their *circumference.* The formula for circumference is $2(r)\pi$, where $\pi \approx 3.14$ and r is the radius of the circle. Suppose the radius is 1.25 inches. Find the circumference .

20. The dimensions of a piece of notebook paper are 8.5 inches by 11 inches. Evaluate the area of the paper. The formula for the area of a rectangle is length $\times$ width.
21. Sonya purchased 16 cans of soda at $0.99 each. What is the amount Sonya spent on soda?
22. Mia works at a job earning $4.75 per hour. How many hours should she work to earn $124?
23. The area of a square is the side length squared. Evaluate the area of a square with a side length of 10.5 miles.

Review (Answers)

Please see the Appendix.

1.7 Factoring Polynomials

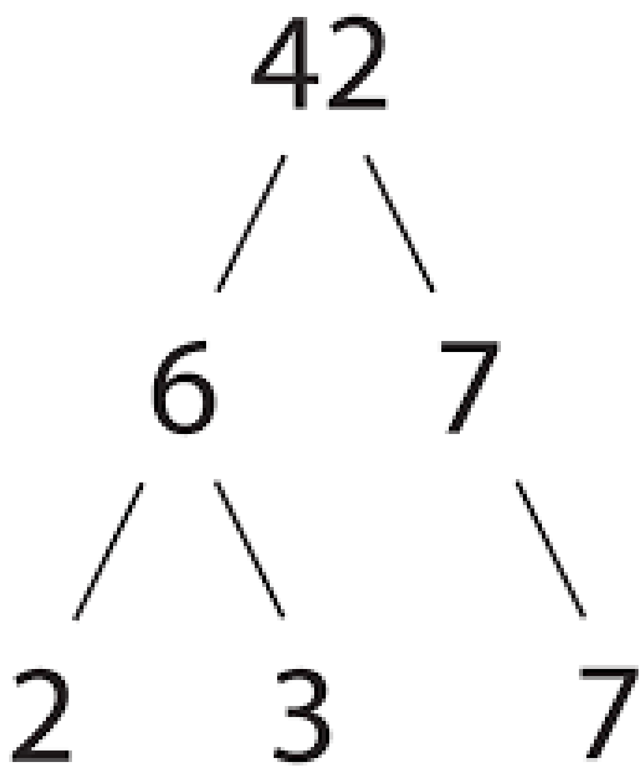
Learning Objectives

Learn how to combine factoring techniques to factor polynomials completely.

Introduction

What if you came across a polynomial like $3x^2 - 27$ with multiple factors? How could you factor it *completely*?

The process is related to the process of factoring whole numbers. If you were asked to find the prime factorization of 42, you might start with the factors 6 x 7 or 2 x 21. However, each of these sets of factors includes a number that can also be factored (6 and 21, respectively). The prime factorization of 42 is 2 x 3 x 7.



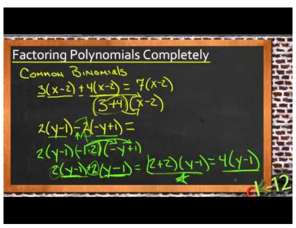
Factoring Polynomials

Remember that a **polynomial** is an expression with at least one algebraic term. **Monomials** have a single term and **binomials** have two terms. These algebraic terms contain variables which can have exponents and can be adjoined together by addition or subtraction.

We say that a polynomial is **factored completely** when all prime factors have been found. The following steps will help in the process of factoring completely:

- First, factor all common monomials.
- Identify special products, such as difference of squares or the square of a binomial. Factor according to their formulas.
- If there are no special products, determine the factors of the polynomial using trinomial factoring techniques.
- Double check each factor to see if any of these can be factored further.

The following video works through three example problems involving factoring polynomials completely:



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Greatest Common Factor

When factoring, start by pulling out the greatest common factor (or GCF).

Special Forms

Below are special forms that will help you multiply and factor binomials and trinomials that specifically fit these patterns. Note that these factors work only if they fit these rules exactly.

Special Forms for Factoring

- Perfect Square Trinomials:
  $a^2+2ab+b^2=(a+b)^2$

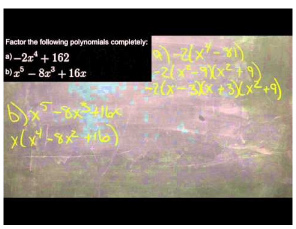
  $a^2-2ab+b^2=(a-b)^2$
- Difference of two squares:

  $a^2-b^2=(a+b)(a-b)$
- Sum of two cubes:

  $a^3+b^3=(a+b)(a^2-ab+b^2)$
- Difference of two cubes:

  $a^3-b^3=(a-b)(a^2+ab+b^2)$

For help with factoring polynomials such as in the previous examples, see this video for two multistep examples, including factoring out the GCF and factoring perfect square trinomials:



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Factor Out a Common Binomial

The 1st step in the factoring process is often factoring out the common monomials from a polynomial. But sometimes polynomials have common terms that are binomials. For example, consider the following expression:

$$x(3x+2)-5(3x+2)$$

Since the term $(3x+2)$ appears in both terms of the polynomial, we can factor it out. We write that term in front of a set of parentheses containing the terms that are left over:

$$(3x+2)(x-5)$$

This expression is now completely factored.

Factoring by Grouping

In some cases, there is not a a GCF for *all* the terms in a polynomial. If you have four terms with no GCF, then try factoring by grouping.

Step 1: Group the 1st two terms together, and then the last two terms together.

Step 2: Factor out a GCF from each separate binomial.

Step 3: Factor out the common binomial.

Factoring Using the a-c Method

When factoring a quadratic that doesn't fit a special form, you can use the a-c Method. The standard form of a quadratic is ax^2+bx+c. The a-c Method starts by multiplying a and c. Then find the factors of that product that add up to b. You can then split the trinomial into 4 terms and use grouping to solve.

Examples

Example 1

Factor the following using greatest common factor:

a) $3x+6$

Solution:

$3x+6=3(x+2)$, since $6=3\times2$ and thus 3 is the greatest common factor.

b) $4x^2+2x$

Solution:

$4x^2+2x=2x(2x+1)$, since $4=2\times2$ and $x^2=x\times x$, and thus $2x$ is the GCF.

Example 2

Factor the following polynomials completely:

a) $2x^2-8$

Solution:

a) Factor out common monomials: $2x^2-8=2(x^2-4)$

We recognize x^2-4 as a difference of squares. We factor it as $(x+2)(x-2)$.

Each factor is prime. We have factored completely.

The answer is $2(x+2)(x-2)$.

b) x^3+6x^2+9x

Solution:

Factor out common monomials: $x^3+6x^2+9x=x(x^2+6x+9)$

We recognize x^2+6x+9 as a perfect square and factor it as $(x+3)^2$.

Each factor is prime. We have factored completely.

The answer is $x(x+3)^2$.

c) $-2x^4+162$

Solution:

Factor out the common monomial. In this case, factor out -2 rather than 2. (It's always easier to factor out the negative number, so that the highest degree term is positive.)

$$-2x^4+162=-2(x^4-81).$$

We recognize the expression in parentheses as a difference of squares. We factor and get

$$-2(x^2-9)(x^2+9).$$

If we look at each factor we see that the 1st parentheses is a difference of squares. We factor and get

$$-2(x+3)(x-3)(x^2+9).$$

Each factor is prime, which means each factor cannot be factored any further. We have factored completely.

The answer is $-2(x+3)(x-3)(x^2+9)$.

d) x^5-8x^3+16x

Solution:

Factor out the common monomial: $x^5-8x^3+14x=x(x^4-8x^2+16)$

We recognize x^4-8x^2+16 as a perfect square, and we factor it as $x(x^2-4)^2$.

We look at each term and recognize that the term in parentheses is a difference of squares.

We factor it and get $((x+2)(x-2))^2$, which we can rewrite as $(x+2)^2(x-2)^2$.

Each factor is prime. We have factored completely.

The final answer is $x(x+2)^2(x-2)^2$.

Example 3

Factor out the common binomials:

a) $3x(x-1)+4(x-1)$

Solution:

$3x(x-1)+4(x-1)$ has a common binomial of $(x-1)$.

When we factor out the common binomial, we get $(x-1)(3x+4)$.

b) $x(4x+5)+(4x+5)$

Solution:

$x(4x+5)+(4x+5)$ has a common binomial of $(4x+5)$.

When we factor out the common binomial, we get $(4x+5)(x+1)$.

Example 4

Factor completely: $2x^3+3x^2+10x+15$

Solution:

Group the 1st two terms and the last two terms:

$(2x^3+3x^2)+(10x+15)$

Factor out the GCF from each binomial:

$x^2(2x+3)+5(2x+3)$

Factor out the common binomial:

$(2x+3)(x^2+5)$

Example 5

Factor completely using the a-c method:

a) $24x^3-28x^2+8x$.

Solution:

First, notice that each term has $4x$ as a factor. Start by factoring out $4x$:

$24x^3-28x^2+8x=4x(6x^2-7x+2)$

Next, factor the trinomial in the parentheses. Since $a \neq 1$, find $a \cdot c$: $6 \cdot 2=12$. Find the factors of 12 that add up to -7. Since 12 is positive and -7 is negative, the two factors should be negative:

$$\begin{array}{lll} 12=-1\cdot-12 & and & -1+-12=-13 \\ 12=-2\cdot-6 & and & -2+-6=-8 \\ 12=-3\cdot-4 & and & -3+-4=-7 \end{array}$$

Rewrite the trinomial using $-7x=-3x-4x$, and then factor by grouping:

$$\begin{aligned} 6x^2-7x+2 &= 6x^2-3x-4x+2 \\ &= 3x(2x-1)-2(2x-1) \\ &= (3x-2)(2x-1) \end{aligned}$$

The final factored answer is

$4x(3x-2)(2x-1)$.

b) $6x^2-30x+36$

Solution:

Factor out the common monomial. In this case, 6 can be divided from each term:

$$6(x^2-5x+6)$$

There are no special products. We factor x^2-5x+6 as a product of two binomials, $(x\ )(x\ )$.

The two numbers that multiply to 6 and add to -5 are -2 and -3, so

$$6(x^2-5x+6)=6(x-2)(x-3).$$

Each factor is prime. We have factored completely.

The answer is $6(x-2)(x-3)$.

Review

Factor completely:

1. $7x-14$
2. $3x+9xy$
3. $xyz+y^2z^3$
4. $2x^2+16x+30$
5. $5x^2-70x+245$
6. $-x^3+17x^2-70x$
7. $2x^4-512$
8. $25x^4-20x^3+4x^2$
9. $12x^3+12x^2+3x$
10. $12c^2-75$
11. $6x^2-600$
12. $-5t^2-20t-20$
13. $6x^2+18x-24$
14. $-n^2+10n-21$
15. $2a^2-14a-16$

Review (Answers)

Please see the Appendix.

Resource

The WTAMU Virtual Math Lab has a detailed page on factoring polynomials here: http://www.wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/col_alg_tut7_factor.htm .

This page contains many videos showing example problems being solved.

1.8 Solving Multi-Step Equations

Learning Objectives

Learn to solve equations using inverse operations.

Introduction

In this year's election for student government president, there were two candidates. The winner received $\frac{1}{3}$ more votes than the loser. If there were 584 votes cast for president, how many votes did each of the two candidates receive?

Solving Equations

To solve the above problem, you will need to create an equation and solve it using multiple steps.

One-Step Equations

The goal of solving an equation is to get the variable by itself on one side of the equation.

In order to do this, you need to remember two main approaches:

- Use properties of equality to keep equations balanced.
- Use inverse operations.

Properties of Equality

For real numbers a, b, and c:

- Addition Property of Equality

  If $a = b$, then $a + c = b + c$.
- Subtraction follows from addition:

  If $a = b$, then $a + (\text{-}c) = b + (\text{-}c)$ or $a - c = b - c$.
- Multiplication Property of Equality:

  If $a = b$, then $a \times c = b \times c$.
- Division follows from multiplication:

  If $a = b$, then $a \times \frac{1}{c} = b \times \frac{1}{c}$ or $\frac{a}{c} = \frac{b}{c}$ for $c \neq 0$.

Multi-Step Equations

To solve equations with multiple steps, use the reverse of the **order of operations** to help you solve. For more complex equations, **combine like terms**. Like terms are terms with the same variables with matching exponents.

For example, $3x^2$ and x^2 are like terms because they both have the same variable and exponent. Similarly, xy^3z^2 and $-4xy^3z^2$ are also like terms. However, xy^2 and x^2y are **not** like terms because the exponents are switched based on the variables.

Equations with Fractions

When introducing fractions into an equation, the same rules for solving any equation apply. You need to keep the equations in balance by adding, subtracting, multiplying, or dividing on both sides of the equal sign in order to isolate the variable. The goal still remains to get your variable alone on one side of the equal sign, with your constant terms on the other, to solve for this variable. If there are multiple fractions that don't have the same denominator, you must 1st find the least common denominator (LCD) before combining like terms. Or, sometimes you need to multiply or divide the equation by the numerator and denominator to solve for the variable.

Examples

Example 1

Solve $k-5=12$.

Solution:

Since the inverse of subtraction is addition, add 5 to both sides of the equation:

$$\begin{aligned} k-5 &= 12 \\ \underline{+5 \quad} & \underline{+5} \end{aligned}$$

The **additive inverse property** notes that the sum of a number and its opposite equal zero, so $-5+5=0$, which will cancel out the number next to the variable. The addition property of equality allows you to add a number to a side, as long as you add the **same number** to both sides:

$$\begin{aligned} k\cancel{-5} &= 12 \\ \underline{\cancel{+5} \quad} & \underline{+5} \\ k &= 17 \end{aligned}$$

Example 2

Solve $\frac{m}{4}=3$.

Solution:

The inverse of division is multiplication, so multiply by 4 on both sides of the equation:

$$4 \times \frac{m}{4} = 3 \times 4$$

The **multiplicative inverse property** notes that $4 \times \frac{1}{4}=1$, which will cancel out the number under the variable. The multiplication property of equality allows you to multiply a side by a number, as long as you multiply both sides by the **same number:**

$$\begin{aligned} \cancel{4} \times \frac{m}{\cancel{4}} &= 3 \times 4 \\ m &= 12 \end{aligned}$$

Example 3

Solve $3x+5=11$.

Solution:

If we were evaluating this, we would multiply and then add. However, when solving the equation for x, start by working backwards by first dealing with the addition and then the multiplication. Subtract 5 from both sides of the equation to cancel out the addition by the additive inverse property:

$$\begin{aligned} 3x+\cancel{5} &= 11 \\ \underline{\cancel{-5} \quad} & \underline{\text{-5}} \\ 3x &= 6 \end{aligned}$$

Divide 3 from both sides of the equation to cancel out the multiplication by the multiplicative inverse property:

$$\frac{\cancel{3}x}{\cancel{3}} = \frac{6}{3}$$
$$x = 2$$

Example 4

Solve $5x - 10 = 3x + 14$.

Solution:

Bring like terms to the same side and combine, using inverse operations:

$$\begin{aligned} 5x - 10 &= \cancel{3x} + 14 \\ \underline{-3x \qquad} &\underline{\quad \cancel{-3x} \qquad} \\ 2x - 10 &= 14 \end{aligned}$$

Now solve as before:

$$\begin{aligned} 2x - \cancel{10} &= 14 \\ \underline{+\cancel{10}} &\underline{\ +10} \\ \frac{\cancel{2}x}{\cancel{2}} &= \frac{24}{2} \\ x &= 12 \end{aligned}$$

Example 5

Solve $\frac{1}{3}t + 5 = -1$.

Solution:

$$\begin{aligned} \frac{1}{3}t + 5 &= -1 \\ \frac{1}{3}t + 5 - 5 &= -1 - 5 && \text{(Subtract 5 from both sides to isolate the variable.)} \\ \frac{1}{3}t &= -6 && \text{(Simplify.)} \\ (\cancel{3})\frac{1}{\cancel{3}}t &= -6(3) && \text{(Multiply both sides by the denominator (3) in the fraction.)} \\ t &= -18 && \text{(Simplify.)} \end{aligned}$$

Therefore, $t = -18$.

$$\begin{aligned}
&\text{Check:}\\
\frac{1}{3}t+5&=-1\\
\frac{1}{3}({\color{red}-18})+5&=-1\\
-6+5&=-1\\
-1&=-1
\end{aligned}$$

Example 6

Solve $\frac{2}{5}w-4=-\frac{1}{5}w+8$.

Solution:

$$\begin{aligned}
\frac{2}{5}w-4&=-\frac{1}{5}w+8\\
\frac{2}{5}w-4{\color{red}+4}&=-\frac{1}{5}w+8{\color{red}+4}\\
\frac{2}{5}w&=-\frac{1}{5}w+12\\
\frac{2}{5}w+{\color{red}\frac{1}{5}w}&=-\frac{1}{5}w+{\color{red}\frac{1}{5}w}+12\\
\frac{3}{5}w&=12\\
\frac{3}{5}w\cdot{\color{red}\frac{5}{3}}&=12\cdot{\color{red}\frac{5}{3}}\\
w&=20
\end{aligned}$$

Therefore, $w=20$.

You can always check your answer by plugging it in and evaluating both sides to make sure they are equal.

$$\begin{aligned}
\frac{2}{5}w-4&=-\frac{1}{5}w+8\\
\frac{2}{5}({\color{red}20})-4&=-\frac{1}{5}({\color{red}20})+8\\
\frac{40}{5}-4&=-\frac{20}{5}+8\\
8-4&=-4+8\\
4&=4
\end{aligned}$$

Example 7

Earlier, you were given a problem about a student election for president:

In this year's student election for president, there were two candidates. The winner received $\frac{1}{3}$ more votes. If there were 584 votes cast for president, how many votes did each of the two candidates receive?

Solution:

Let $x=$ votes for candidate 1 (the winner)

Let $y =$ votes for candidate 2

$x+y = 584$

You must have only one variable in the equation in order to solve it. Let's look at another relationship from the problem:

$$
\begin{aligned}
x &= y+\frac{1}{3}y && \text{(Candidate 1 received } \tfrac{1}{3} \text{ more votes than candidate 2.)}\\
x &= \frac{3}{3}y+\frac{1}{3}y && \text{(Make denominator common for both } y \text{ variables.)}\\
x &= \frac{4}{3}y && \text{(Simplify.)}
\end{aligned}
$$

Now substitute into the original problem and simplify:

$$
\begin{aligned}
\frac{4}{3}y+y &= 584\\
\frac{4}{3}y+\frac{3}{3}y &= 584\\
\frac{7}{3}y &= 584\\
(\cancel{3})\frac{7}{\cancel{3}}y &= 584(3)\\
7y &= 1752\\
\frac{\cancel{7}y}{\cancel{7}} &= \frac{1752}{7}\\
y &= \frac{1752}{7}
\end{aligned}
$$

So, candidate 2 received 250 votes. Candidate 1 would then receive $584-250 = 334$ votes.

Example 8

Solve for x: $\frac{2}{3}x = 12$.

Solution:

$$
\begin{aligned}
\frac{2}{3}x &= 12\\
(\cancel{3})\frac{2}{3}x &= 12(3)\\
2x &= 36\\
\frac{\cancel{2}x}{\cancel{2}} &= \frac{36}{2}\\
x &= 18
\end{aligned}
$$

Therefore, $x = 18$.

Check:

$$\begin{aligned}
\frac{2}{3}x &= 12 \\
\frac{2}{3}(18) &= 12 \\
\frac{36}{3} &= 12 \\
12 &= 12
\end{aligned}$$

Review

Solve for the variable in each of the following equations:

1. $\frac{1}{3}p = 5$
2. $\frac{3}{7}j = 8$
3. $2b + 4 = 6$
4. $\frac{2}{3}x - 2 = 1$
5. $x + 3 = -3$

6. $k + \frac{2}{3} = 5k$
7. $\frac{1}{6}c + \frac{1}{3} = -2$
8. $4x + 3 = 19$
9. $\frac{3}{4}x - \frac{2}{5} = \frac{1}{2}$
10. $\frac{t}{4} + 3 = 2$

11. $\frac{1}{3}x + \frac{1}{4}x = 1$
12. $d + 2d = \frac{5}{3}$
13. $7x - 1 = 3x - 5$
14. $\frac{1}{3}x - \frac{1}{2} = \frac{3}{4}x$
15. $\frac{2}{3}j - \frac{1}{2} = \frac{3}{4}j + \frac{1}{3}$

Review (Answers)

Please see the Appendix.

1.9 Inequalities

Learning Objectives

Learn how to solve inequalities and write your solution in inequality notation and set notation.

Introduction

An inequality is a relation between two expressions that are not equal. Inequality signs include $\neq$ (not equal to), >(greater than), $\geq$ (greater than or equal to), <(less than), and $\leq$ (less than or equal to). These can be used to describe and solve situations such as the following:

Ms. Jain wants to buy identical boxes of art supplies for her 25 students. If she can spend no more than $375 on art supplies, what inequality describes the price can she afford for each individual box of supplies?

Expressing Solutions of an Inequality

The solution of an inequality can be expressed in four different ways:

1. **Inequality notation.** The answer is simply expressed as $x < 15$.

2. **Set notation.** The answer is expressed as a set: $\{x|x < 15\}$. The brackets indicate a set, and the vertical line means "such that," so we read this expression as "the set of all values of x such that x is a real number less than 15."

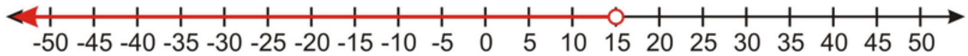
3. **Interval notation.** This uses brackets to indicate the range of values in the solution. For example, the answer to our problem would be expressed as $(-\infty, 15)$, meaning "the interval containing all the numbers from $-\infty$ to 15, but not actually including $-\infty$ or 15."

a) Square or **closed brackets** "[" and "]" indicate that the number next to the bracket is included in the solution set.

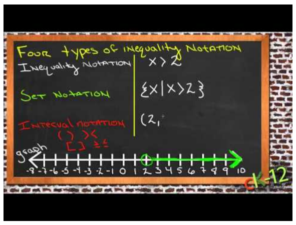
b) Round or **open brackets** "(" and ")" indicate that the number next to the bracket is not included in the solution set. When using **infinity** and **negative infinity** (∞ and $-\infty$), we always use open brackets because infinity isn't an actual number and so it can never be included in an interval.

4. **Solution graph.** This shows the solution on the real number line. A closed circle on a number indicates that the number is included in the solution set, while an open circle indicates that the number is not included in the set. For

our example, the solution graph is:



The following video reviews these four types of notation:



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Identify the Number of Solutions of an Inequality

Inequalities can have:

- A set that has an infinite number of solutions.
- A set that has a finite number of solutions.
- No solutions.

Inequalities often have an infinite number of solutions, at least in theory. For example, the inequality $\frac{5x-1}{4} > -2(x+5)$ has the solution $x > -3$. This solution says that all real numbers greater than -3 make this inequality true, and there are infinitely many such numbers.

However, in real life, sometimes we are trying to solve a problem that can only have positive integer answers, because the answers describe numbers of discrete objects.

For example, suppose you are trying to figure out how many $8 pens you can buy if you want to spend less than $50. An inequality to describe this situation would be $8x < 50$, and if you solved that inequality you would get $x < \frac{50}{8}$, or $x < 6.25$.

But could you really buy any number of pens as long as it's less than 6.25? No; you couldn't buy 6.1 pens, or -5 pens, or any other fractional or negative number of pens. So if we wanted to express our solution in set notation, we couldn't express it as the set of all numbers less than 6.25, or $\{x|x < 6.25\}$. Instead, the solution is just the set containing all the nonnegative whole numbers less than 6.25, or $\{0, 1, 2, 3, 4, 5, 6\}$. When we're solving a real-world problem dealing with discrete objects like pens, our solution set will often be a finite set of numbers instead of an infinite interval.

An inequality can also have **no solutions** at all. For example, consider the inequality $x-5 > x+6$. When we subtract x from both sides, we end up with $-5 > 6$, which is not true for any value of x. We say that this inequality has no solution.

The opposite can also be true. If we flip the inequality sign in the above inequality, we get $x-5 < x+6$, which simplifies to $-5 < 6$. That's always true no matter what x is, so the solution to that inequality would be **all real numbers**, or $(-\infty, \infty)$.

Examples

Example 1

Describe what the following solutions in interval notation mean:

a) [-4, 6]

Solution:

The solution is all numbers between -4 and 6, including -4 and 6.

b) (8, 24)

Solution:

The solution is all numbers between 8 and 24, not including the numbers 8 and 24.

c) [3, 12)

Solution:

The solution is all numbers between 3 and 12, including 3 but not including 12.

d) $(-10, \infty)$

Solution:

The solution is all numbers greater than -10, not including -10.

e) $(-\infty, \infty)$

Solution:

The solution is all real numbers.

Example 2

Solve the following inequality. Write the solution in inequality and set notation.

$$-2x < 12$$

Solution:

Divide both sides of the inequality by -2. Remember that the inequality sign needs to be switched from less than to greater than (or from greater than to less than) when you multiply or divide both sides of an inequality by a negative number:

$$\begin{aligned}\frac{-2x}{-2} &> \frac{12}{-2}\\ x &> -6\end{aligned}$$

In set notation, the solution is $\{x|x > -6\}$.

Example 3

Solve the inequality below. Write the solution in inequality and set notation.

$$8-4x \geq 2(3x-11)$$

Solution:

Step 1: Distribute the 2 on the right side of the inequality:

$$\begin{aligned}8-4x &\geq 2(3x-11)\\ 8-4x &\geq 6x-22\end{aligned}$$

Step 2: Collect the numbers on one side of the inequality and all the x terms to the other side of the inequality:

$$8-8-4x-6x \geq 6x-6x--22-8$$
$$-10x \geq -30$$

Step 3: Divide both sides of the inequality by -10 and switch the inequality sign:

$$\frac{-10x}{-10} \leq \frac{-30}{-10}$$
$$x \leq 3$$

In set notation, the solution is $\{x|x \leq 3\}$.

Example 4

In order to get a bonus this month, Lalit must sell at least 120 newspaper subscriptions. He sold 85 subscriptions in the 1st three weeks of the month. How many subscriptions must Lalit sell in the last week of the month?

Solution:

Let $x =$ the number of subscriptions Lalit sells in the last week of the month. The total number of subscriptions for the month must be greater than 120, so we write $85+x \geq 120$. We solve the inequality by subtracting 85 from both sides: $x \geq 35$.

Therefore, Lalit must sell 35 or more subscriptions in the last week to get his bonus.

To check the answer, we see that $85+35 = 120$. If he sells 35 or more subscriptions, the total number of subscriptions he sells that month will be 120 or more.

Example 5

A group of Class VI students is trying to raise at least $650 this summer. How many boxes of erasers (each box has only 2 erasers) must they sell at $4.50 per box in order to reach their goal?

Solution:

Let $x =$ number of boxes sold. Then the inequality describing this problem is $4.50x \geq 650$.

We solve the inequality by dividing both sides by 4.50: $x \geq 144.44$.

We round up the answer to 145, since only whole boxes can be sold.

Therefore, the group of Class VI students must sell at least 145 boxes.

If we multiply 145 by $4.50, we obtain $652.50. So if the group sells more than 144 boxes, they will raise more than $650. But if they sell 144 boxes, they will raise only $648, which is not enough. So they must indeed sell at least 145 boxes.

Example 6

The width of a rectangle is 20 cm. What must the length be if the perimeter is at least 180 cm?

Solution:

Let $x =$ length of the rectangle. The formula for perimeter is

$$\text{Perimeter} = 2 \times \text{length} + 2 \times \text{width}$$

Since the perimeter must be at least 180 cm, we have $2x + 2(20) \geq 180$.

Simplify: $2x + 40 \geq 180$

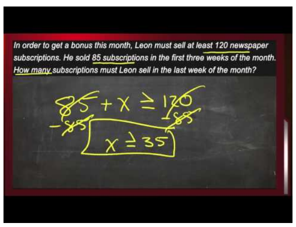
Subtract 40 from both sides: $2x \geq 140$

Divide both sides by 2: $x \geq 70$

Therefore, the length must be at least 70 cm.

If the length is at least 70 cm and the width is 20 cm, then the perimeter is at least $2(70) + 2(20) = 180\ cm$.

The following video provides real-world applications of solving inequalities:



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Review

Solve each inequality. Give the solution in inequality notation and interval notation.

1. $x + 15 < 12$
2. $x - 4 \geq 13$
3. $9x > -\frac{3}{4}$
4. $-\frac{x}{15} \leq 5$
5. $620x > 2400$
6. $\frac{x}{20} \geq -\frac{7}{40}$
7. $\frac{3x}{5} > \frac{3}{5}$
8. $x + 3 > x - 2$

Solve each inequality. Give the solution in inequality notation and set notation.

9. $x + 17 < 3$
10. $x - 12 \geq 80$
11. $-0.5x \leq 7.5$
12. $75x \geq 125$
13. $\frac{x}{-3} > -\frac{10}{9}$
14. $\frac{x}{-15} < 8$
15. $\frac{x}{4} > \frac{5}{4}$

16. $3x-7 \geq 3(x-7)$

Solve the following inequalities, give the solution in set notation, and show the solution graph:

17. $4x+3 < -1$

18. $2x < 7x-36$

19. $5x > 8x+27$

20. $5-x < 9+x$

21. $4-6x \leq 2(2x+3)$

22. $5(4x+3) \geq 9(x-2)-x$

23. $2(2x-1)+3 < 5(x+3)-2x$

24. $8x-5(4x+1) \geq -1+2(4x-3)$

25. $2(7x-2)-3(x+2) < 4x-(3x+4)$

26. $\frac{2}{3}x-\frac{1}{2}(4x-1) \geq x+2(x-3)$

27. At the Delhi Zoo, you can pay either $22 for the entrance fee or $71 for the yearly pass, which entitles you to unlimited admission.

a) At most, how many times can you enter the zoo for the $22 entrance fee before spending more than the cost of a yearly membership?

b) Are there infinitely many or finitely many solutions to this inequality?

28. Prateek's scores for four tests were 82, 95, 86, and 88. What will he have to score on his 5th and last test to average at least 90 for the term?

Review (Answers)

Please see the Appendix.

1.10 Absolute Value

Learning Objectives

Learn how to find the distance between two values on a number line and solve equations involving absolute values.

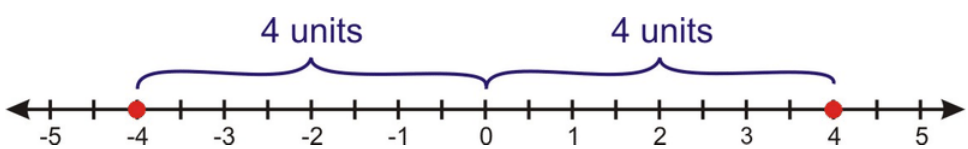
Introduction

What if you were given two points like -8 and 12? How could you find the distance between them on a number line?

Absolute value is used to find distances, whether the distance of a number from zero, or the distance between numbers, as in this case.

Absolute Value

The **absolute value** is the distance between a number and zero on a number line. There are always two numbers on the number line that are the same distance from zero. For instance, the numbers 4 and -4 are each a distance of 4 units away from zero.



$|4|$ represents the distance from 4 to zero, which equals 4.

$|-4|$ represents the distance from -4 to zero, which also equals 4.

Absolute Value

For any real number x,

$$|x| = x \; for \; all \; x \geq 0$$
$$|x| = -x \; (read \text{ the opposite of } x) \; for \; all \; x < 0$$

Absolute value has no effect on a positive number or zero, but changes a negative number into its positive additive inverse.

Absolute value situations can also involve unknown variables. For example, suppose the distance from zero is 16. What two points can this represent?

Begin by writing an absolute value sentence to represent this situation:

$$16 = |n|, \text{ where } n = \text{ the missing value}$$

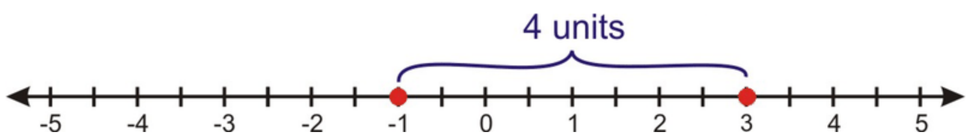
Which two numbers are 16 units from zero?

$$n = 16 \text{ or } n = -16$$

Absolute value situations can also involve distances from points other than zero. We treat such cases by separating the problem into two independent equations and solving them separately.

Absolute value is very useful in finding the distance between two points on the number line. The **distance** between any two points a and b on the number line is $|a-b|$ or $|b-a|$.

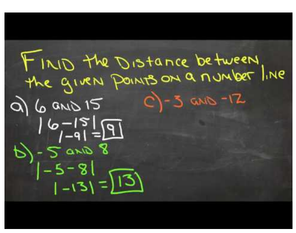
For example, the distance from 3 to -1 on the number line is $|3-(-1)| = |4| = 4$.



We could have also found the distance by subtracting in the opposite order: $|-1-3| = |-4| = 4$. This makes sense because the distance is the same whether you are going from 3 to -1 or from -1 to 3.

Note: When we compute the change in x and the change in y as part of a slope computation, these values are positive or negative, depending on the direction of movement. In this discussion, "distance" means a positive distance only.

The following video gives an overview of finding the distance between two points on a number line, and it also previews the next section on solving basic absolute value equations:



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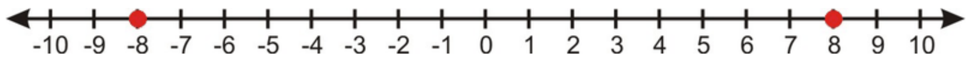
Solve an Absolute Value Equation

We now want to solve equations involving absolute values. Consider the following equation:

$$|x| = 8$$

This means that the distance from the number x to zero is 8. There are two numbers that satisfy this condition: 8 and -8.



When we solve absolute value equations we always consider two possibilities:

1. The expression inside the absolute value sign is not negative.
2. The expression inside the absolute value sign is negative.

Then we solve each equation separately.

Examples

Example 1

Evaluate the following absolute values:

a) $|25|$

Solution:

$|25| = 25$ Since 25 is a positive number, the absolute value does not change it.

b) $|-120|$

Solution:

$|-120| = 120$ Since -120 is a negative number, the absolute value makes it positive.

c) $|-3|$

Solution:

$|-3| = 3$ Since -3 is a negative number, the absolute value makes it positive.

d) $|55|$

Solution:

$|55| = 55$ Since 55 is a positive number, the absolute value does not change it.

e) $\left|-\frac{5}{4}\right|$

Solution:

$\left|-\frac{5}{4}\right| = \frac{5}{4}$ Since $-\frac{5}{4}$ is a negative number, the absolute value makes it positive.

Example 2

Find the distance between the following points on the number line:

a) 6 and 15

Solution:

Distance is the absolute value of the difference between the two points.

$\text{distance} = |6-15| = |-9| = 9$

b) -5 and 8

Solution:

$\text{distance} = |-5-8| = |-13| = 13$

c) -3 and -12

Solution:

distance $= |-3-(-12)| = |9| = 9$

Example 3

Solve the following absolute value equations:

a) $|x| = 3$

Solution:

There are two possibilities: $x = 3$ and $x = -3$.

b) $|x| = 10$

Solution:

There are two possibilities: $x = 10$ **and** $x = -10$.

Example 4

Solve $|2x-7| = 6$.

Solution:

Begin by separating this into its separate equations:

$$2x-7 = 6 \textit{ and } 2x-7 = -6$$

Solve each equation independently:

$$\begin{aligned} 2x-7 &= 6 \\ 2x-7+7 &= 6+7 \\ 2x &= 13 \\ x &= \frac{13}{2} \end{aligned} \qquad \begin{aligned} 2x-7 &= -6 \\ 2x-7+7 &= -6+7 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

Example 5

A company packs coffee beans in airtight bags. Each bag should weigh 16 ounces, but it is hard to fill each bag to the exact weight. After being filled, each bag is weighed, and if it is more than 0.25 ounces overweight or underweight, it is emptied and repacked. What are the lightest and heaviest acceptable bags?

Solution:

The varying quantity is the weight of the bag of coffee beans. Choosing a letter w to represent this quantity, and writing an absolute value equation, yields critical values of w:

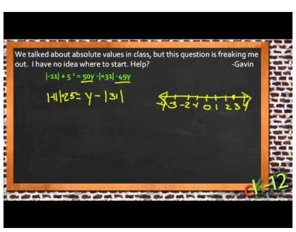
$$|w-16| = 0.25$$

Separate and solve:

$$w-16 = 0.25 \qquad\qquad w-16 = -0.25$$
$$w = 16.25 \qquad\qquad w = 15.75$$

The lightest bag acceptable is 15.75 ounces, and the heaviest bag acceptable is 16.25 ounces.

The following video is one more example of an equation with absolute values:



MEDIA

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Review

Evaluate the absolute values:

1. $|250|$
2. $|-12|$
3. $|-0.003|$
4. $\left|-\frac{2}{5}\right|$
5. $\left|\frac{1}{10}\right|$

Find the distance between the points:

6. 12 and -11
7. 5 and 22
8. -9 and -18
9. -2 and 3
10. -0.012 and 1.067
11. $-\frac{2}{3}$ and $\frac{7}{8}$

Solve the absolute value equations and interpret the results by graphing the solutions on a number line:

12. $|7u| = 77$
13. $|x-5| = 10$
14. $|5r-6| = 9$
15. $1 = \frac{|6+5z|}{5}$
16. $|8x| = 32$
17. $\left|\frac{m}{8}\right| = 1$
18. $|x+2| = 6$
19. $|5x-2| = 3$
20. $51 = |1-5b|$
21. $8 = 3+|10y+5|$
22. $|4x-1| = 19$
23. $8|x+6| = -48$

Solve the following:

24. A company manufactures rulers. Its 12-inch rulers pass quality control if they're within $\frac{1}{32}$ inch of the ideal length. What is the longest and shortest ruler that can leave the factory?
25. The height of a football goal crossbar is 10 feet. The height of the goal post is 30 feet. What is the absolute value of the distance from the crossbar to the top of the goal post?
26. The ideal selling price of a car is $17,000. The dealer allows this price to vary $850. What is the lowest price at which this dealer will sell this car?

Review (Answers)

Please see the Appendix.

1.11 Imaginary and Complex Numbers

Learning Objectives

Learn how to recognize and evaluate imaginary numbers.

Learn how to graph and simplify complex numbers.

Introduction

How do you evaluate the square root of a negative number? The square roots of negative numbers are referred to as imaginary numbers. Complex numbers are of the form $a+bi$, where a and b are real numbers and i is the imaginary unit. What is i? $i^2 = \text{-}1$. Therefore, $\sqrt{\text{-}1} = i$. This section will cover evaluating expressions and solving equations involving complex numbers.

Imaginary Numbers

What is the square root of -1?

You may recall finding roots of negatives in algebra when attempting to solve equations like $x^2+4=0$.

Since there are no real numbers that can be squared to equal -4, this equation has no real solution. What is the imaginary constant "i"?

The definition of "i" :

$$i = \sqrt{\text{-}1}$$

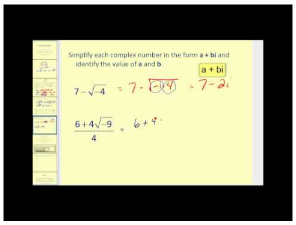
The use of the word *imaginary* does not mean these numbers are not very useful. For a long period in the history of mathematics, it was thought that the square root of a negative number was in fact only within the mathematical imagination. That has changed. Mathematicians now consider imaginary numbers as another set of numbers that have real significance, but do not fit on the number line. Engineers, scientists, and others solve real-world problems using combinations of real and imaginary numbers—called complex numbers—every day.

Imaginary values such as $\sqrt{-16}$ can be simplified by simplifying the radical into $\sqrt{16}\cdot\sqrt{-1}$, yielding $4\sqrt{-1}$ or $4i$.

The uses of i become more apparent when you begin working with increased powers of i, as you will see in the examples below.

Complex Numbers

The video below provides an overview of how to identify and simplify expressions involving imaginary and complex numbers. It includes relevant vocabulary and examples.



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When you combine imaginary numbers with real numbers, you get complex numbers.

The definition of complex numbers:

Complex numbers are of the form $a+bi$, where a is a real number, b is a real number, and i is the imaginary constant $\sqrt{-1}$.

Plotting points was something you may have done in another mathematics course. For instance, plotting the point (4, 5) meant starting at the origin and moving 4 units to the right, the x direction, and 5 units up, the y direction.

In this lesson, one of the things we will consider is the graphing of complex numbers such as $4+3i$.

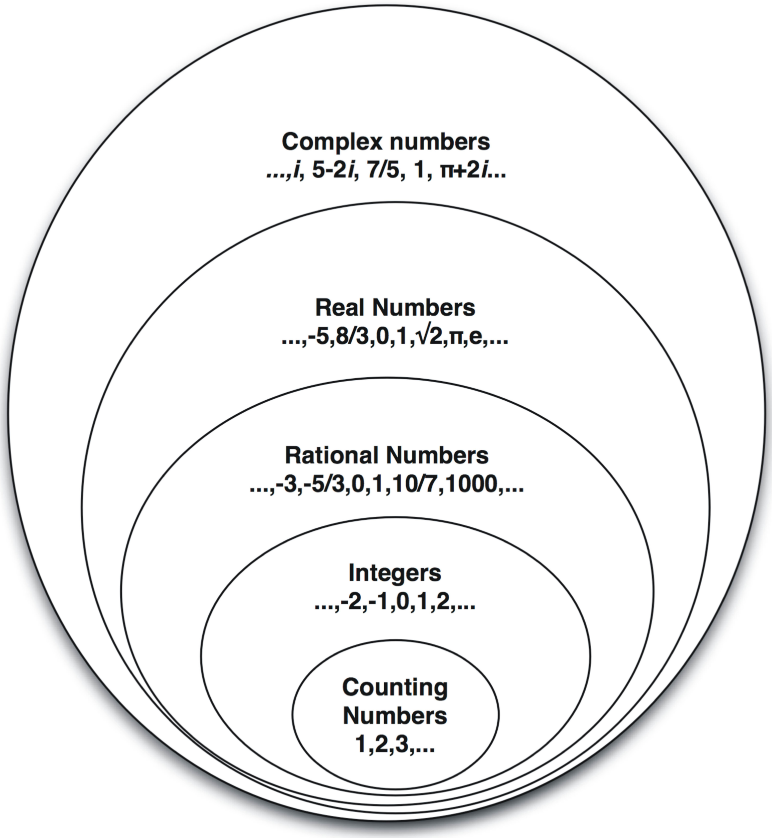
In essence it doesn't sound that hard, but which direction do you only imagine moving 3 units?

$a + bi$ is the **standard** or **rectangular** form of a complex number.

A complex number is a number that has a real part (in this case, a), and an imaginary part—that is, the imaginary number i with a coefficient b.

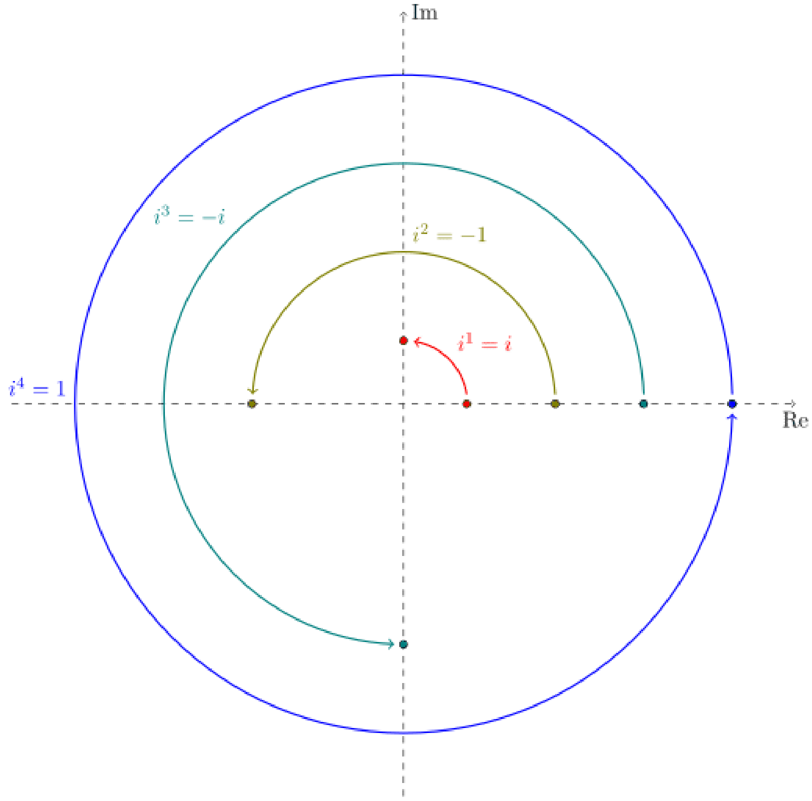
The complex numbers are a superset of the real numbers, meaning that all of the real numbers are part of the set of complex numbers.

Given $a + bi$, if $b = 0$ (meaning there is no imaginary part to the complex number), then all you have remaining is a real number. Viewed this way, every real number can be written as a complex number (just let it equal a), but there are many more complex numbers than real numbers. Hence, the complex numbers are a superset of the real numbers:



Graphing complex numbers

In standard form, $a + bi$, a complex number can be graphed using rectangular coordinates (a, b). The x-coordinate represents "real number" values, while the y-coordinate represents the "imaginary" values.



Examples

Example 1

Simplify $\sqrt{-5}$

Solution:

Step 1: Factor out the -1: $\sqrt{-5} = \sqrt{(-1)\cdot(5)}$

Step 2: Apply the multiplication rule for radicals: $= \sqrt{-1}\sqrt{5}$

Step 3: Convert to i: $= i\sqrt{5}$

Example 2

Simplify $\sqrt{-72}$

Solution:

Step 1: Factor out the -1: $\sqrt{-72} = \sqrt{(-1)\cdot(72)}$

Step 2: Apply the multiplication rule for radicals: $= \sqrt{-1}\sqrt{72}$

Step 3: Convert to i: $= i\sqrt{72}$

Step 4: We're not done yet! Since $72 = 36 \cdot 2$

$$\begin{aligned} i\sqrt{72} &= i\sqrt{36}\sqrt{2} \\ &= i(6)\sqrt{2} \\ &= 6i\sqrt{2} \end{aligned}$$

Example 3

Strange things happen when the imaginary constant i is multiplied by itself different numbers of times.

a) What is i^2?

Solution:

i^2 is the same as $(\sqrt{-1})^2$. When you square a square root, the two numbers cancel and you are left with the number originally inside the radical, in this case, -1.

$\therefore i^2 = -1$

b) What is i^3?

Solution:

i^3 is the same thing as $i^2 \cdot i$, which is $-1 \cdot i$ or $-i$.

$\therefore i^3 = -i$.

c) What is i^4?

Solution:

$i^4 = i^2 \cdot i^2$ which is $(-1) \cdot (-1)$

$\therefore i^4 = 1$

Example 4

Solve for x: $(x-1)^2 + 4 = 0$

Solution:

Subtract 4 from both sides of the equation:

$$(x-1)^2 = -4$$

Take the square root of both sides of the equation. Since the square root of a squared number is equal to the absolute value, you must include the positive and negative version of the solution:

$$\sqrt{(x-1)^2} = \pm\sqrt{-4}$$

$$x - 1 = \pm\sqrt{-4}$$

Convert $\sqrt{-1}$ to i:

$$x - 1 = \pm\sqrt{-1}\sqrt{4}$$

$$x-1=\pm 2i$$

Solve for x:

$$x=1\pm 2i$$

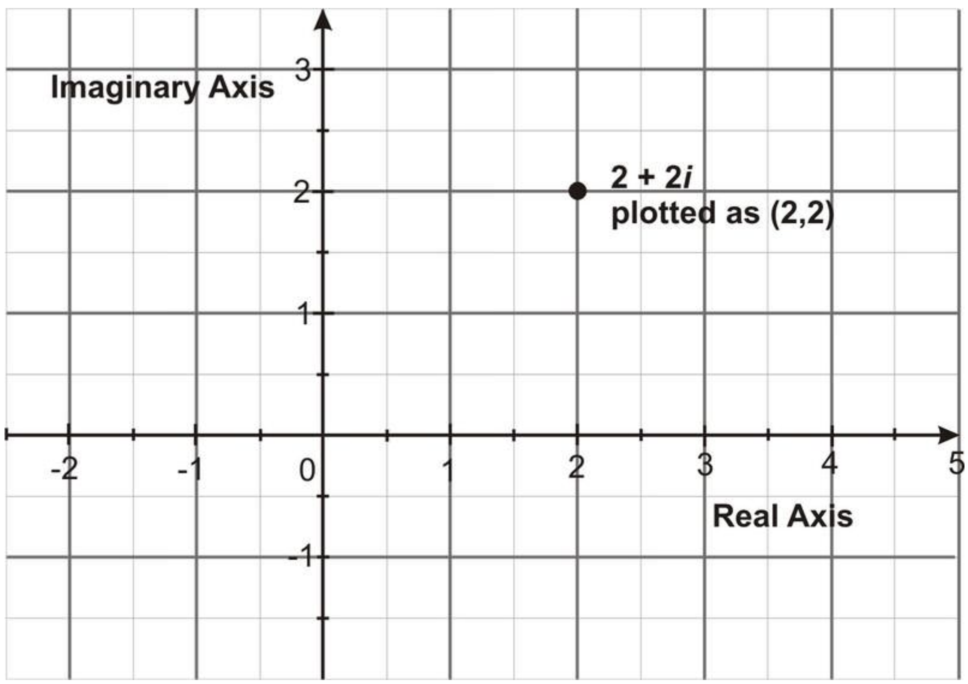
Therefore, $x=1+2i$ or $1-2i$.

Example 5

Graph the complex number $z=2+2i$ in rectangular form. Note that z is often used to denote complex numbers.

Solution:

The coordinate (2, 2) is graphed as shown below. First, move along the real (or horizontal) axis the number of units for the real part (2), and then go up or down along the imaginary (or vertical) axis the number of units for the imaginary part (the coefficient in front of i).



Example 6

Solve each equation and express it as a complex number. (Note: If the imaginary part is 0, express the solution as $a+0i$).

a) $x^2+24=0$

Solution:

$$\begin{aligned}x^2&=-24\\ \sqrt{x^2}&=\pm\sqrt{-24}\\ x&=\pm\sqrt{-24}\\ x&=\pm 2i\sqrt{6}\end{aligned}$$

b) $2x^2-4x+7=0$

Solution:

To solve the equation $2x^2-4x+7=0$, we need to use the quadratic formula.

Given $ax^2+bx+c=0$, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. In this example, $a=2$, $b=$ -4, and $c=7$. Start by substituting the values into the formula:

$$\begin{aligned} x &= \frac{4\pm\sqrt{(-4)^2-4\cdot 2\cdot 7}}{2\cdot 2} \\ &= \frac{4\pm\sqrt{16-56}}{4} \\ &= \frac{4\pm\sqrt{-40}}{4} \\ &= \frac{4\pm 2i\sqrt{10}}{4} \\ &= 1\pm\frac{\sqrt{10}}{2}i \end{aligned}$$

Example 7

Plot each of the following complex numbers in rectangular form:

a) (4 + 2*i*)

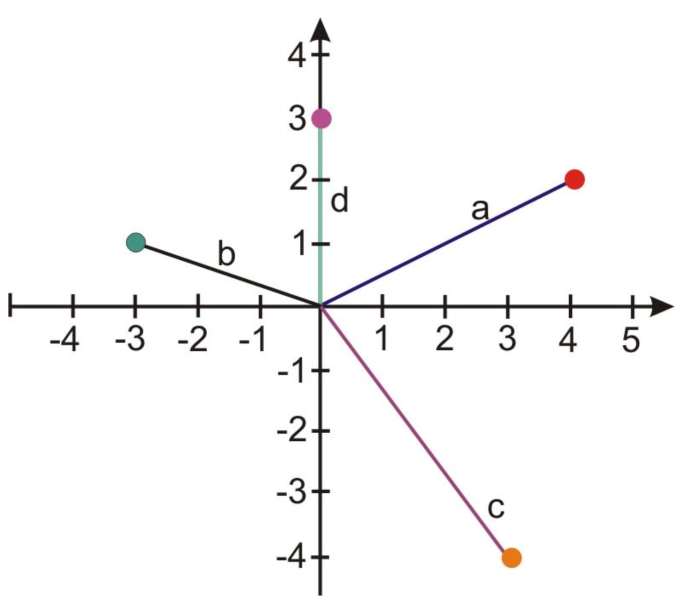
b) (-3 + *i*)

c) (3 - 4*i*)

d) 3*i*

Solutions:

Your graph should look like this:



Review

Simplify the following radicals:

1. $\sqrt{-9}$
2. $\sqrt{-17}$
3. $\sqrt{108-140}$

Multiply the imaginary numbers:

4. $4i \cdot 3i$
5. $\sqrt{16}i \cdot 3$
6. $\sqrt{4i^2} \cdot \sqrt{12}i$
7. Simplify and express as a complex number: $-\sqrt{60} + \sqrt{-121}$
8. Solve the equation and express the answer as a simplified complex number: $x(4x)+4=0$
9. Graph the complex numbers:

a) 3 + 2i

b) 2 - 3i

c) -2 + 2i

Simplify:

10. $\sqrt{-324}$
11. $\sqrt{-121}$
12. $-\sqrt{-16}$
13. $-\sqrt{-1}$
14. $\sqrt{-1.21}$

Simplify:

15. i^3
16. $24i^{20}$
17. i^{225}
18. i^{1024}

Multiply:

19. $i^4 \cdot i^{11}$
20. $5i^6 \cdot 5i^8$
21. $3\sqrt{-75} \cdot 5\sqrt{-3}$
22. $-4\sqrt{-10} \cdot 5\sqrt{-3} \cdot 6\sqrt{-18}$

23. What are complex numbers technically the sum of?

Express in simplest form in terms of i:

24. $13-\sqrt{-49}$

25. $10-\sqrt{\frac{-4}{36}}$

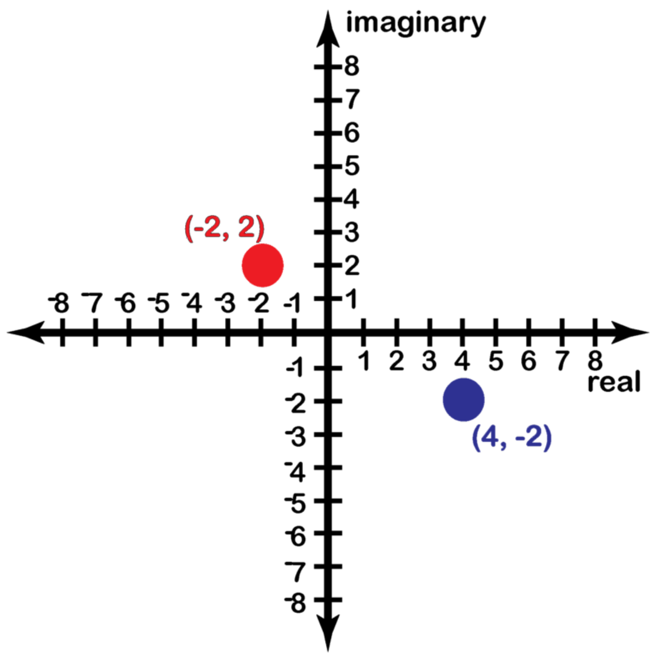
26. $4+\sqrt{-250}$

27. $3^2\sqrt{-0.0009}$

28. $\sqrt{-0.16}-(-\sqrt{27})$

29. $9\sqrt{-8i^9}+3\sqrt{25}$

30. Two complex numbers are graphed below. What are the numbers expressed in standard complex number form?



Review (Answers)

Please see the Appendix.

Resource

Note: For a very detailed explanation of i and the complex numbers, visit: http://betterexplained.com/articles/a-visual-intuitive-guide-to-imaginary-numbers/

1.12 Coordinate Geometry

Learning Objectives

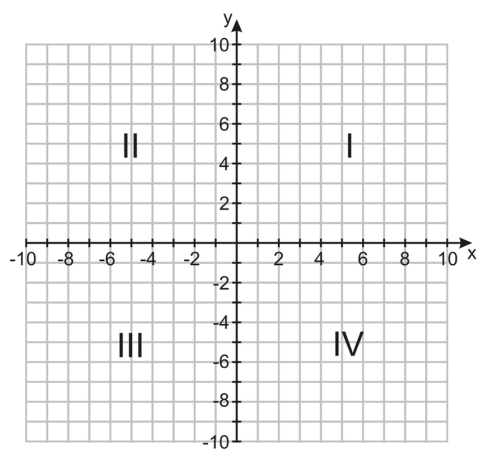
Learn how to identify the coordinates of points and plot points in the coordinate plane.

Introduction

Much of algebraic representation occurs within the coordinate (or Cartesian) plane. This lesson will review how to represent points, lines, and parallel and perpendicular lines graphically.

The Coordinate Plane

The coordinate plane can be thought of as two number lines that meet at right angles. The horizontal line is called the x-axis and the vertical line is called the y-axis. Together the lines are called the **axes**, and the point at which they cross is called the **origin**. The axes split the coordinate plane into four **quadrants**, which are numbered sequentially (I, II, III, IV) moving counterclockwise from the upper right.



Identify Coordinates of Points

When given a point on a coordinate plane, you can easily determine its **coordinates**. The coordinates of a point are two numbers, and written together they are called an **ordered pair**. The numbers describe how far along the x-axis and y-axis the point is. The ordered pair is written in parentheses, with the x-coordinate (also called the **abscissa**) 1st and the y-coordinate (also known as the **ordinate**) 2nd.

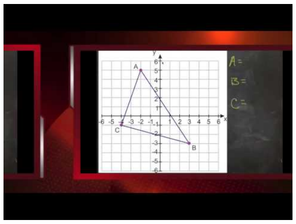
$(1,7)$An ordered pair with an x-value of 1 and a y-value of 7$(0,5)$An ordered pair with an x-value of 0 and a y-value of 5$(-2.5,4$

Identifying coordinates is just like reading points on a number line, except that the points do not actually lie **on** the number line!

Plot Points in a Coordinate Plane

Plotting points is simple, once you understand how to read coordinates and the scale on a graph. As a note on scale, in the next two examples pay close attention to the labels on the axes.

Watch this video for help with the examples below:



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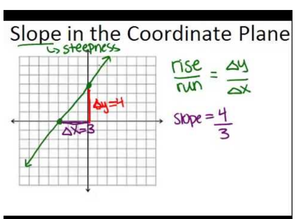
Slope

The following video provides an overview on the concept of the slope of a line. It defines slope, gives the formula, and works through examples:



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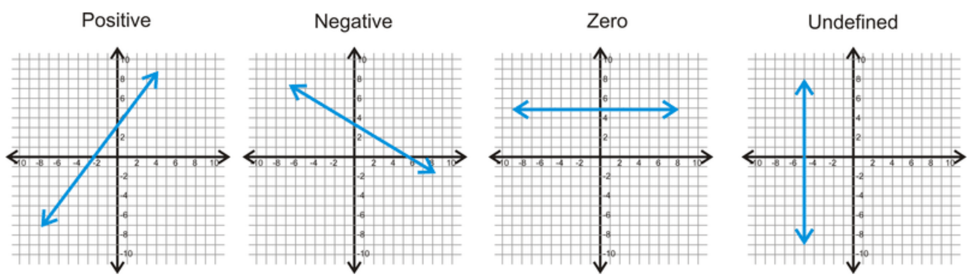
Slope is the change in vertical direction over the change in horizontal direction. It is defined as the $\frac{\text{rise}}{\text{run}}$.

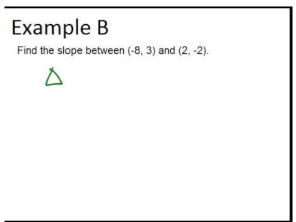
Slope of Line between Two Points

The slope of the line passing through two points (x_1, y_1) and (x_2, y_2) is $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$.

Different Types of Slope:



Watch this video for help with the examples below:



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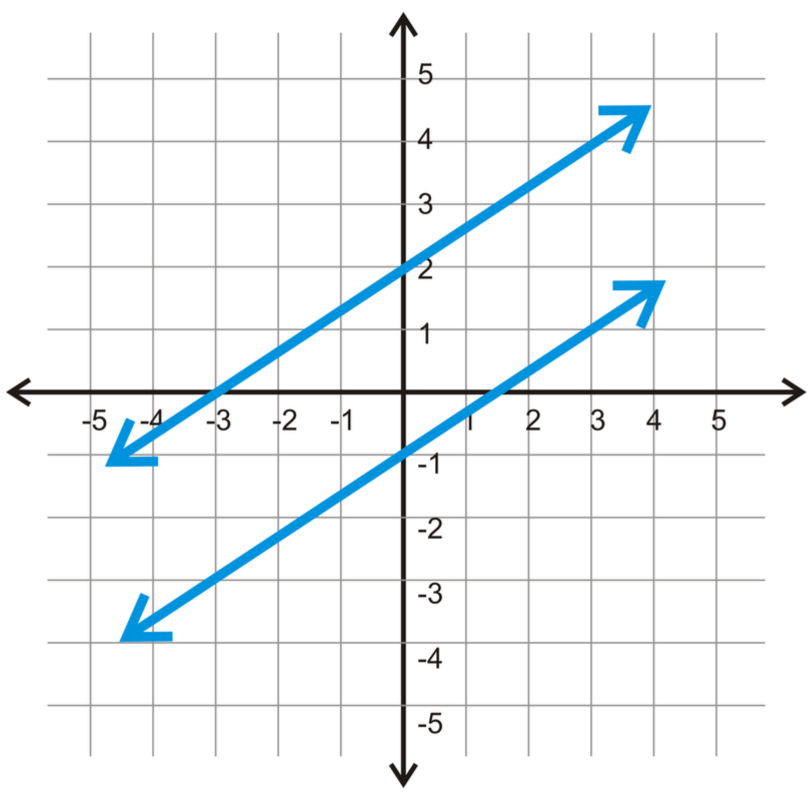
Parallel Lines in the Coordinate Plane

Parallel lines are two lines that never intersect. In the coordinate plane, that would look like this:



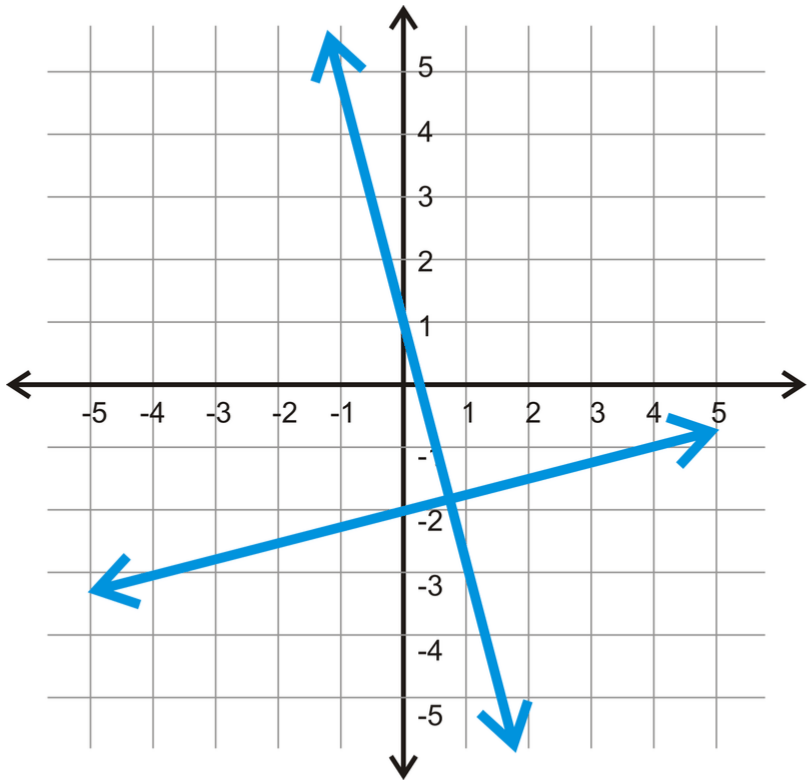
If we take a closer look at these two lines, we see that the slopes are both $\frac{2}{3}$.

This can be generalized to any pair of parallel lines. Parallel lines always have the same slope and different $y-$intercepts.

What if you were given two parallel lines in the coordinate plane? What could you say about their slopes?

Perpendicular Lines in the Coordinate Plane

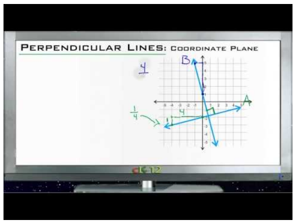
Perpendicular lines are two lines that intersect at a 90° or right angle. In the coordinate plane, that would look like this:



If we take a closer look at these two lines, we see the slope of one is -4 and the other is $\frac{1}{4}$.

This can be generalized to any pair of perpendicular lines in the coordinate plane. The slopes of perpendicular lines are **negative reciprocals** of each other.

This video explores the relationship between the slopes of perpendicular lines:



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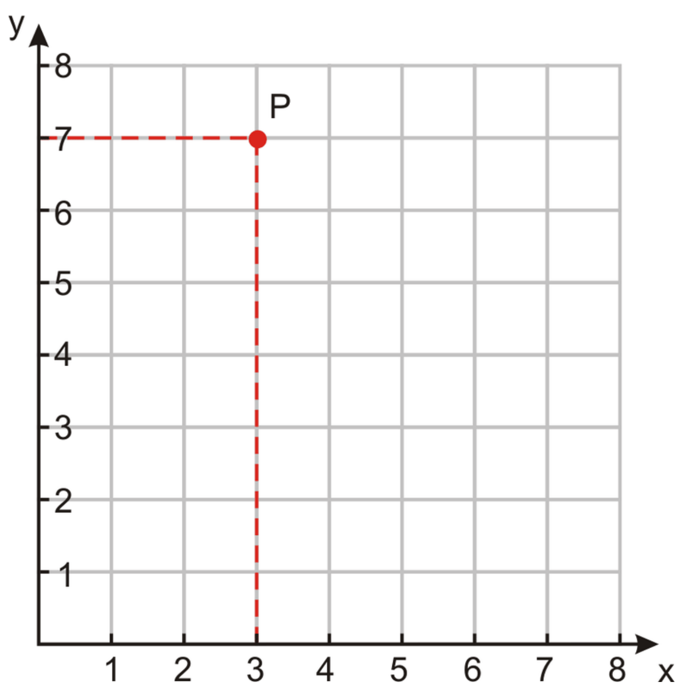
Examples

Example 1



Find the coordinates of the point labeled P in the diagram above.

Solution:

Imagine you are standing at the origin (the point where the x-axis meets the y-axis). In order to move to a position where P is directly above you, you would move 3 units to the right. (We say this is in the positive x direction.)

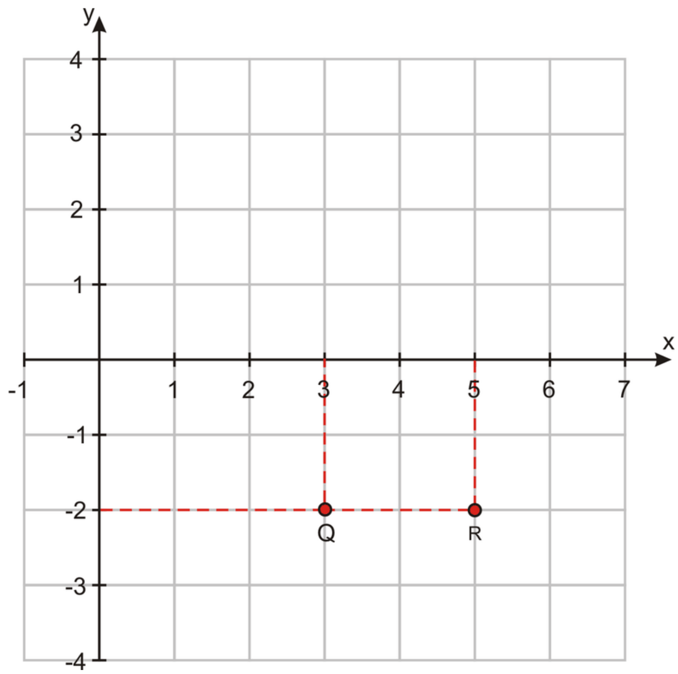
The x-coordinate of P is +3.

Now, if you were standing at the 3 marker on the x-axis, point P would be 7 units above you. (Above the axis means it is in the positive y direction.)

The y-coordinate of P is +7.

The coordinates of point P are (3, 7).

Example 2



Find the coordinates of the points labeled Q and R in the diagram above.

Solution:

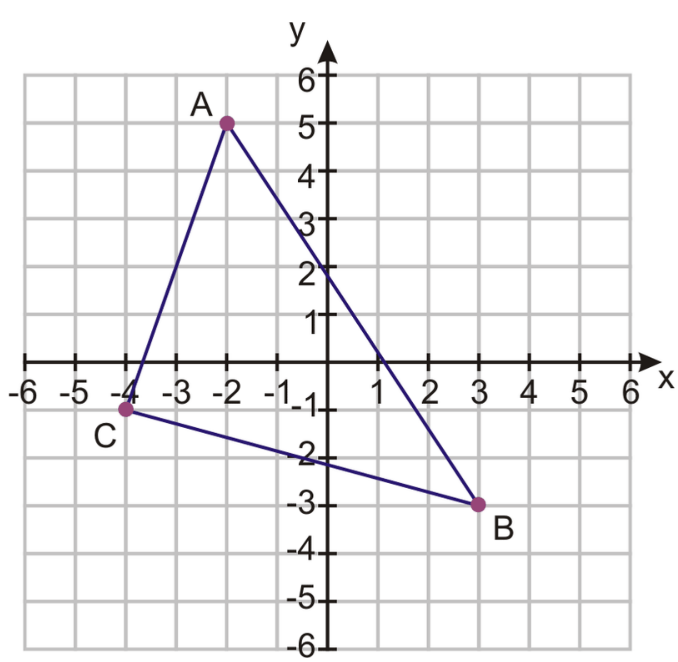
In order to get to Q, we move 3 units to the right, in the positive x direction, then 2 units down, in the negative y direction. The x-coordinate of Q is +3, and the y-coordinate of Q is -2.

The coordinates of R are found in a similar way. The x-coordinate is +5 (meaning 5 units in the positive x direction), and the y-coordinate is again -2.

The coordinates of Q are $(3, -2)$. The coordinates of R are $(5, -2)$.

Example 3

Triangle ABC is shown in the diagram below. Find the coordinates of the vertices A, B, and C.



Solution:

Point A: A (-2, 5)

x-coordinate = -2

y-coordinate = +5

Point B: B (3, -3)

x-coordinate = +3

y-coordinate = -3

Point C: C (-4, -1)

x-coordinate = -4

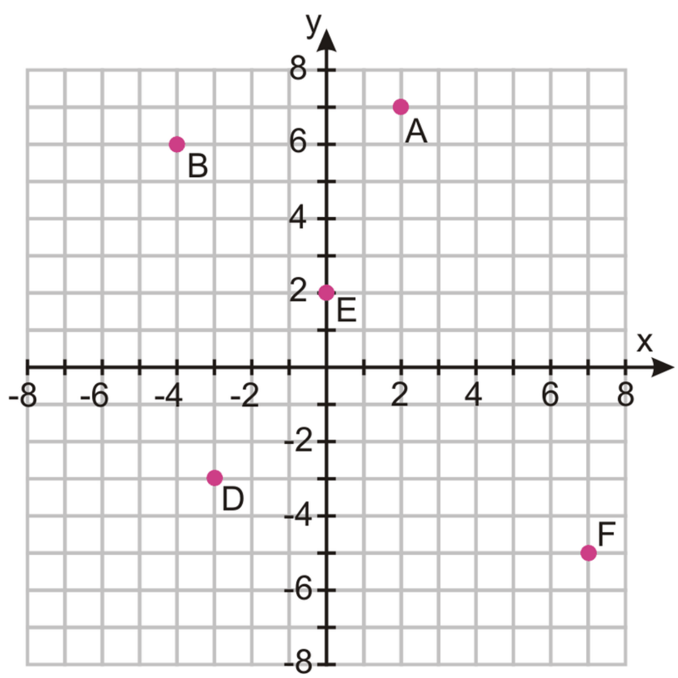
y-coordinate = -1

Example 4

Plot the following points on the coordinate plane:

$A(2,7)$ $B(-4,6)$ $D(-3,-3)$ $E(0,2)$ $F(7,-5)$

Solutions:



Point $A(2,7)$ is 2 units right, 7 units up. It is in Quadrant I.

Point $B(-4,6)$ is 4 units left, 6 units up. It is in Quadrant II.

Point $D(-3,-3)$ is 3 units left, 3 units down. It is in Quadrant III.

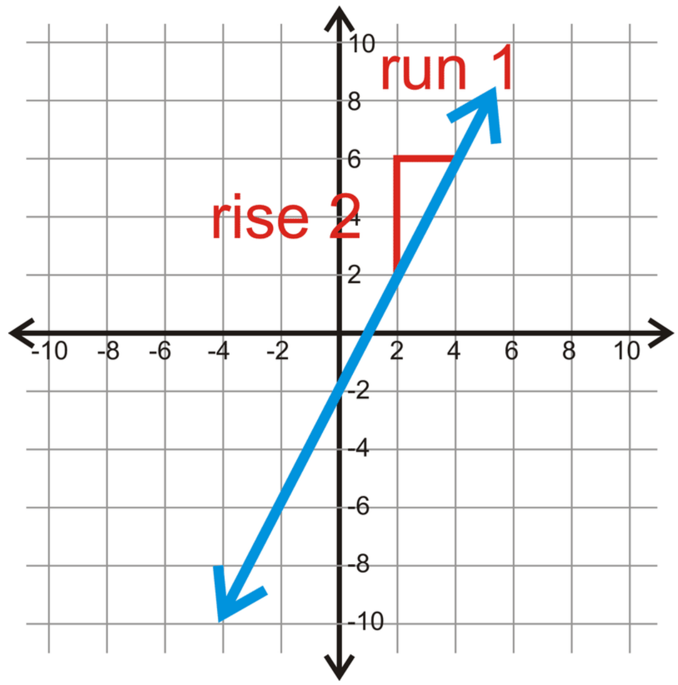
Point $E(0,2)$ is 2 units up from the origin. It is right on the y-axis, between Quadrants I and II.

Point $F(7,-5)$ is 7 units right, 5 units down. It is in Quadrant IV.

Example 5

a) What is the slope of the line through (2, 2) and (4, 6)?

Solution:



Use the slope formula to determine the slope. Use (2, 2) as (x_1, y_1) and (4, 6) as (x_2, y_2).

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{4-2} = \frac{4}{2} = 2$$

Therefore, the slope of this line is 2. This slope is positive.

b) Find the slope between (-8, 3) and (2, -2).

Solution:

$$m = \frac{-2-3}{2-(-8)} = \frac{-5}{10} = -\frac{1}{2}$$

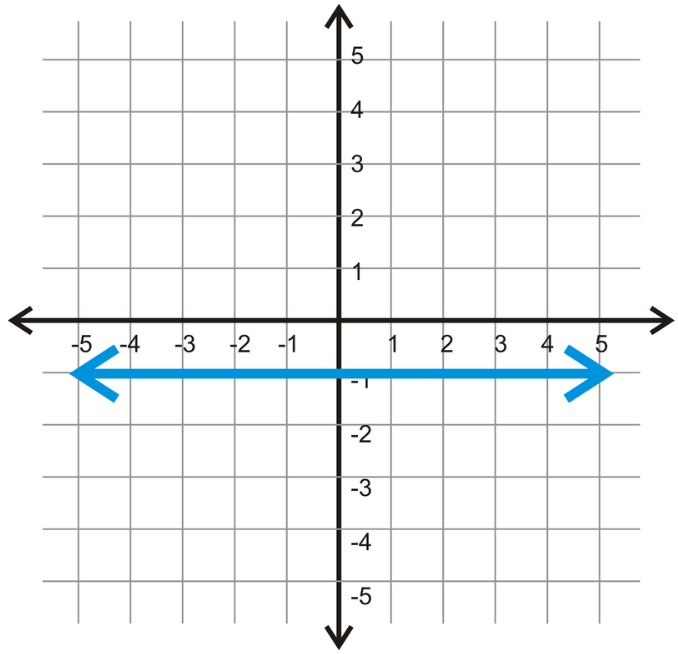
This is a negative slope.

c) Find the slope between (-5, -1) and (3, -1).

Solution:

$$m = \frac{-1-(-1)}{3-(-5)} = \frac{0}{8} = 0$$

Therefore, the slope of this line is 0, which means it is a horizontal line. Horizontal lines always pass through the $y-$axis. Notice that the $y-$coordinate for both points is -1. In fact, the $y-$coordinate for *any* point on this line is -1. This means the horizontal line must cross $y = -1$.

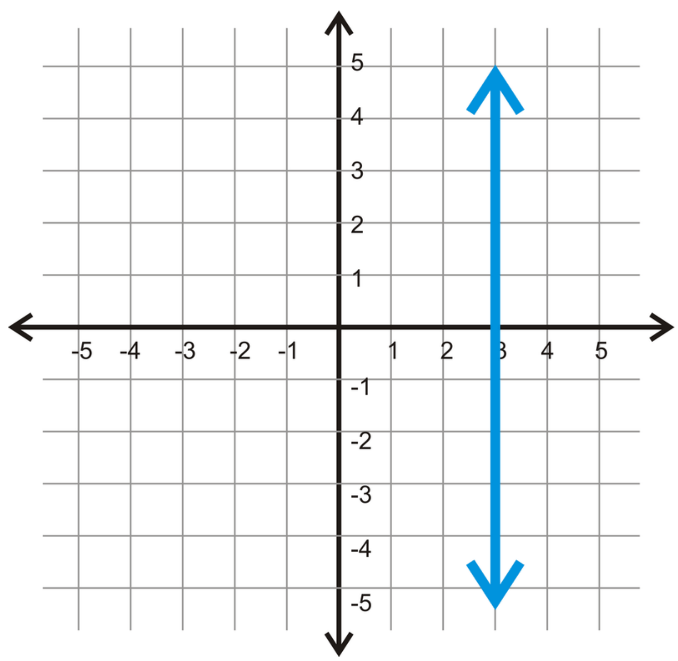


d) Find the slope of the line through (3, 2) and (3, 6).

Solution:

$$m = \frac{6-2}{3-3} = \frac{4}{0} = undefined$$

Therefore, the slope of this line is undefined, which means it is a vertical line. Vertical lines always pass through the $x-$axis. Notice that the $x-$coordinate for both points is 3. In fact, the $x-$coordinate for any point on this line is 3. This means the vertical line must cross $x = 3$.



Example 6

a) Find the slope of the line that is perpendicular to this line: $y = -\frac{2}{3}x - 5$.

Solution:

$m = -\frac{2}{3}$, take the reciprocal and change the sign, $m_{\perp} = \frac{3}{2}$ ($\perp$ is the notation for perpendicular).

b) Find the slope of the line that is perpendicular to this line: $y = x + 2$.

Solution:

Because there is no number in front of x, the slope is 1. The reciprocal of 1 is 1, so the only thing to do is make it negative, $m_{\perp} = -1$.

Example 7

Find the equation of the line that is perpendicular to $y = -\frac{1}{3}x + 4$ and passes through (9, -5).

Solution:

First, the slope is the opposite reciprocal of $-\frac{1}{3}$. So, $m = 3$.

The equation of a line can be written using **slope-intercept form,** $y = mx + b$**.**

Plug in 9 for x and -5 for y to solve for the *new* $y-$intercept (b):

$$
\begin{aligned}
-5 &= 3(9) + b \\
-5 &= 27 + b \\
-32 &= b
\end{aligned}
$$

Therefore, the equation of the perpendicular line is $y = 3x - 32$.

Note that you can use the same process, but keep the slope identical if solving for the equation of the line that is parallel.

Example 8

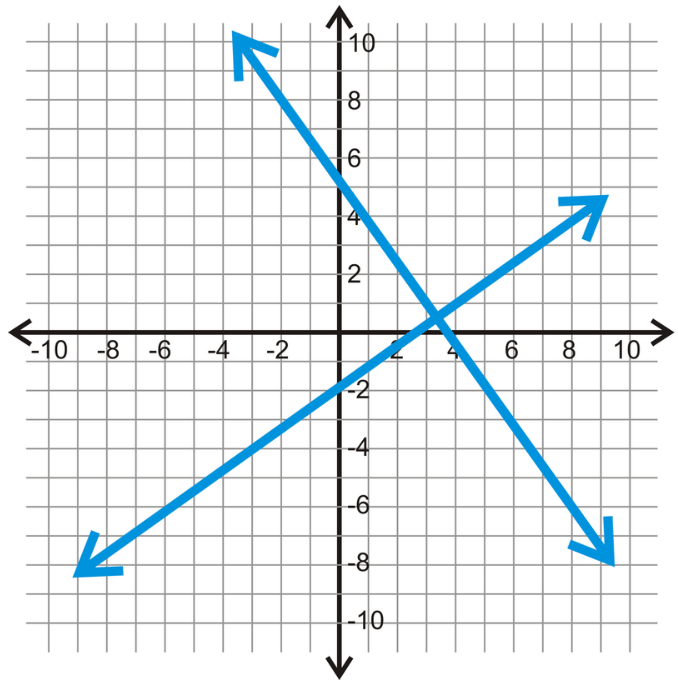
Graph $3x - 4y = 8$ and $4x + 3y = 15$. Determine if they are perpendicular.

Solution:

First we have to change each equation into slope-intercept form. In other words, we need to solve each equation for y:

$$
\begin{aligned}
3x - 4y &= 8 & 4x + 3y &= 15 \\
-4y &= -3x + 8 & 3y &= -4x + 15 \\
y &= \frac{3}{4}x - 2 & y &= -\frac{4}{3}x + 5
\end{aligned}
$$

Now that the lines are in slope-intercept form (also called $y-$intercept form), we can tell they are perpendicular because their slopes are opposite reciprocals.



Example 9

Find the slope of the interval between the two given points:

a) (3, -4) and (3, 7)

Solution:

These two points create a vertical line, so the slope is undefined.

b) (6, 1) and (4, 2)

Solution:

The slope is $\frac{(2-1)}{(4-6)} = -\frac{1}{2}$.

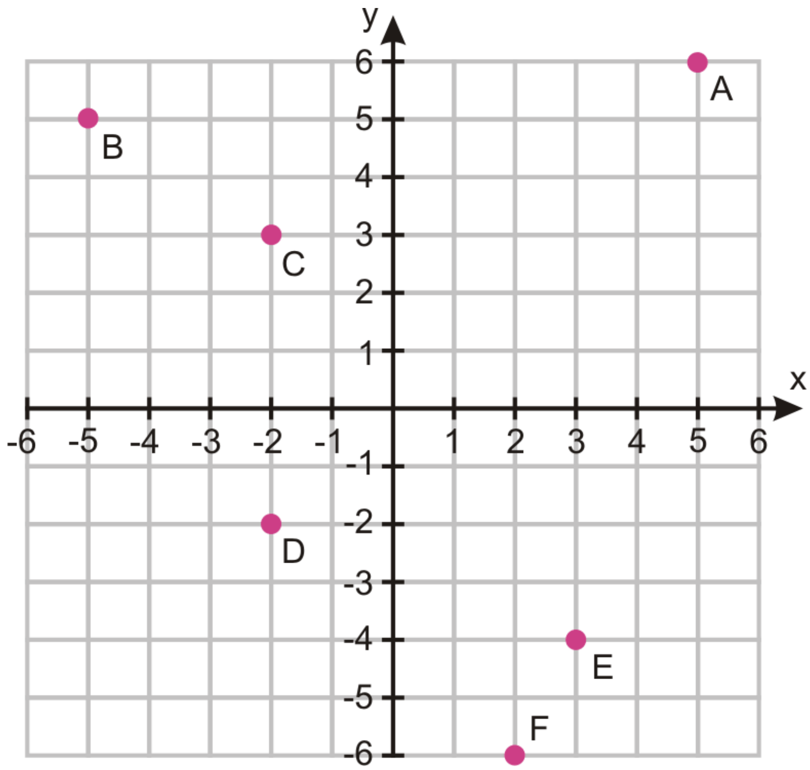
c) (5, 7) and (11, 7)

Solution:

These two points create a horizontal line, so the slope is zero.

Review

1. Identify the coordinates of each point, $A - F$, on the graph below.



2. Draw a line on the above graph connecting point B with the origin. Where does that line intersect the line connecting points C and D?

Identify which quadrant each point lies in:

3. (4, 2)
4. (-3, 5.5)
5. (4, -4)
6. (-3, -5)

Find the slope between the two given points:

7. (-9, 5) and (-6, 2)
8. (-6, 0) and (-1, -10)
9. (1, -2) and (3, 6)
10. (-4, 5) and (-4, -3)
11. (4, 1) and (7, 1)
12. (13, 12) and (-23, 14)
13. (-4, 2) and (-16, 12)

Determine if each pair of lines are parallel, perpendicular, or neither. Then graph each pair on the same set of axes:

14. $y = 4x - 2$ and $y = 4x + 5$
15. $y = -x + 5$ and $y = x + 1$
16. $5x + 2y = -4$ and $5x + 2y = 8$
17. $y = -2x + 3$ and $y = \frac{1}{2}x + 3$
18. $y = -3x + 1$ and $y = 3x - 1$

Determine the equation of the line that is ***parallel*** to the given line, through the given point:

19. $y = -5x + 1;\ (-2, 3)$
20. $y = \frac{2}{3}x - 2;\ (9, 1)$
21. $x - 4y = 12;\ (-16, -2)$

Determine the equation of the line that is ***perpendicular*** to the given line, through the given point:

22. $y = x - 1;\ (-6, 2)$
23. $y = 3x + 4;\ (9, -7)$
24. $5x - 2y = 6;\ (5, 5)$
25. $y = 4;\ (-1, 3)$

Review (Answers)

Please see the Appendix.

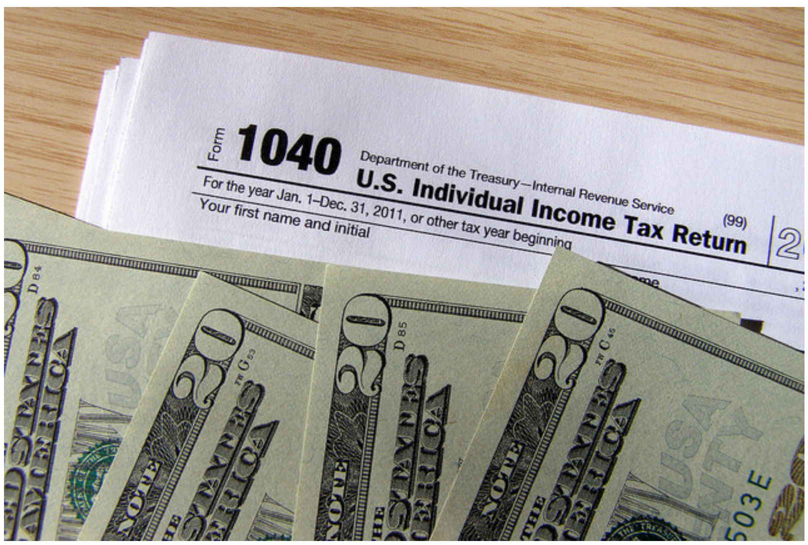
1.13 Linear Equations

Learning Objectives

Learn how to use coordinate pairs and intercepts to graph linear equations.

Introduction

Suppose there is a linear relationship between your annual income and the amount you must pay in state income tax. Could you create a graph that shows how much tax you must pay based on your income?



Linear Equations

You probably recall how to solve equations in one variable. The answers are in the form *variable* = *some number*. In this section, you will learn how to solve equations with two variables. Below are several examples of two-variable equations:

$$
\begin{aligned}
p &= 20(h) \\
3m+4n &= 1 \\
y &= 4x+7
\end{aligned}
$$

You may recognize these equations from another section, and you'll see how to graph them in this section. Their solutions are not one value because there are two variables. The solutions to these equations are pairs of numbers. These pairs of numbers can be graphed in a Cartesian plane.

The solutions to an equation in two variables are sets of ordered pairs.

The solutions to a **linear equation** are the coordinates on the graphed line.

Graphing Using a Table of Values

You can make a table of values for the two variables that are the solutions to the linear equation.

See the examples below or watch the following video for a demonstration on how to graph a linear equation by making a table and then graphing points:

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/79617

Graphing a Line Using Other Methods

In addition to creating a table, you can graph a linear equation by using intercepts or coordinates of points, or an intercept and the slope of the line.

To find the graph of a line using the intercepts, start by setting $y = 0$. This will give you the x-intercept. Then set $x = 0$ to find the y-intercept.

Examples

Example 1

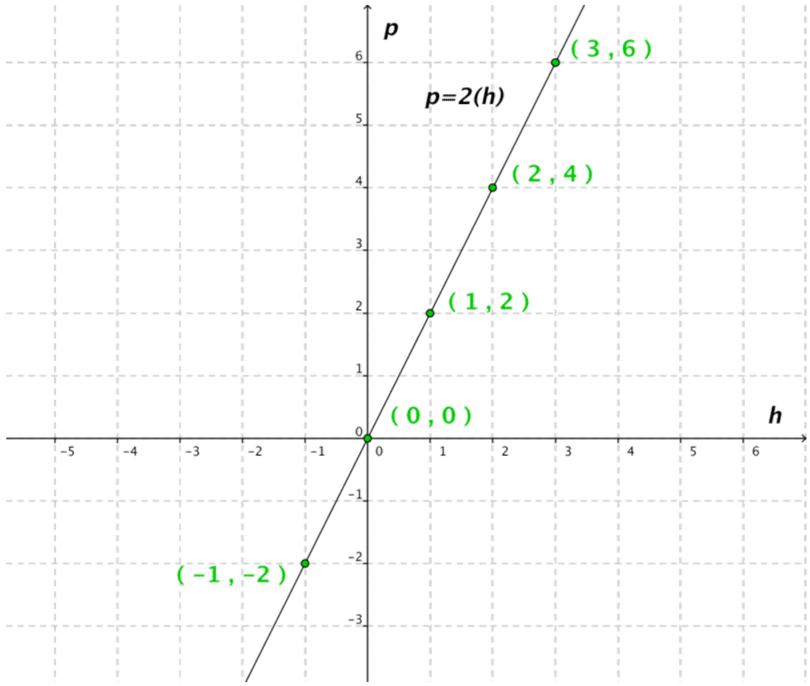
Graph $p = 2(h)$.

Solution:

Make a table and then graph the points:

TABLE 1.4:

h	p
0	0
1	2
2	4
3	6



Example 2

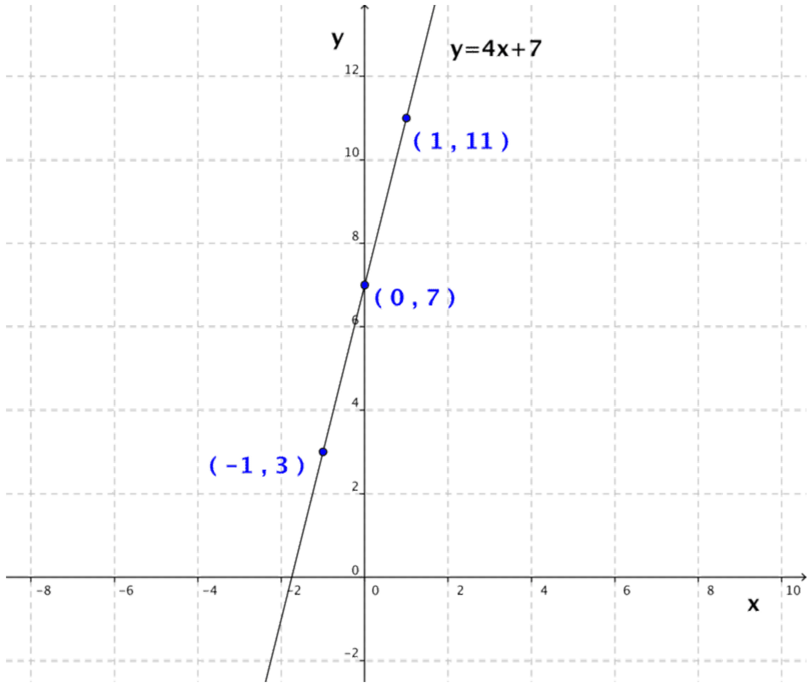
Graph $y = 4x + 7$.

Solution:

Make a table and then graph the points.

TABLE 1.5:

x	y
-1	3
0	7
1	13



Example 3

A taxi fare costs more the further you travel. Taxis usually charge a fee on top of the per-mile charge. In this case, the taxi charges $3 as a set fee, and $0.80 per mile traveled. Find all the possible solutions to this equation.

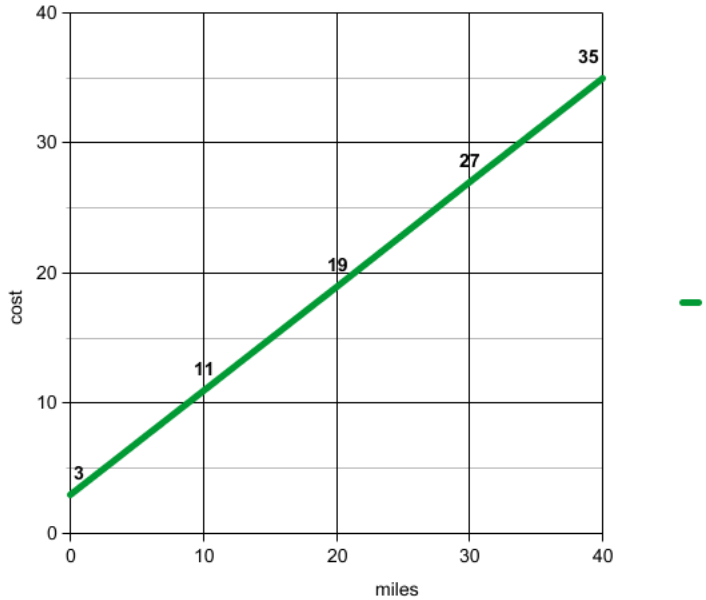
Solution:

Here is the equation in **slope-intercept form**, linking the cost in dollars (y) to hire a taxi and the distance traveled in miles $(x) : y = 0.8x + 3$.

This is an equation in two variables. By creating a table, we can graph these ordered pairs to find the solutions:

TABLE 1.6:

x **(miles)**	y **(cost $)**
0	3
10	11
20	19
30	27
40	35



The solutions to the taxi problem are located on the green line graphed above. To find any cab ride cost, you need to find the y of the desired x.

Note: While the above examples found multiple points, which can be helpful to check your work, a line is defined by any two points on it. Once these are plotted, you can graph the whole line.

Example 4

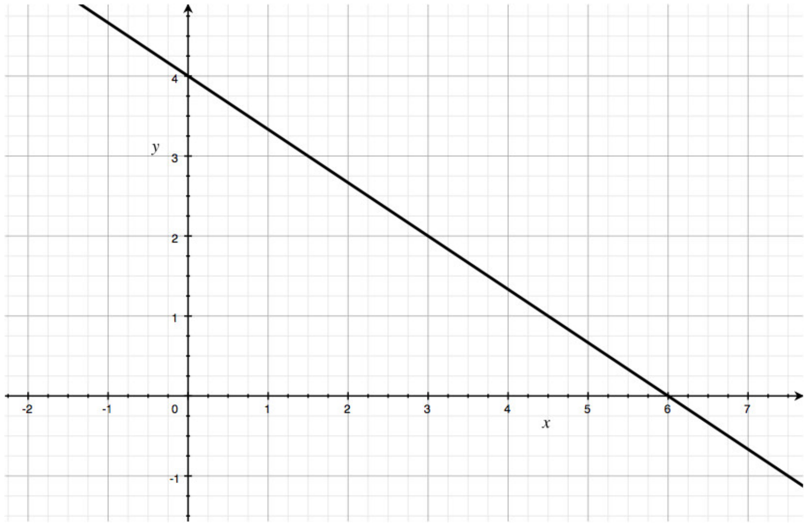
Graph $2x+3y=12$.

Solution:

Find the intercepts by setting each variable to zero and solving for the other:

$$\begin{aligned} 2x+3(0) &= 12 \\ 2x &= 12 \\ x &= 6 \end{aligned} \qquad \begin{aligned} 2(0)+3y &= 12 \\ 3y &= 12 \\ y &= 4 \end{aligned}$$

Plot the intercepts and draw a line to connect them:



To find the graph of a line using the y-intercept, 1st solve for y to get the equation in slope-intercept form, or $y = mx + b$, where m is the slope and b is the y-intercept. Then plot the y-intercept and use the slope to find the next point on the line.

Example 5

Graph $2x + 3y = 21$.

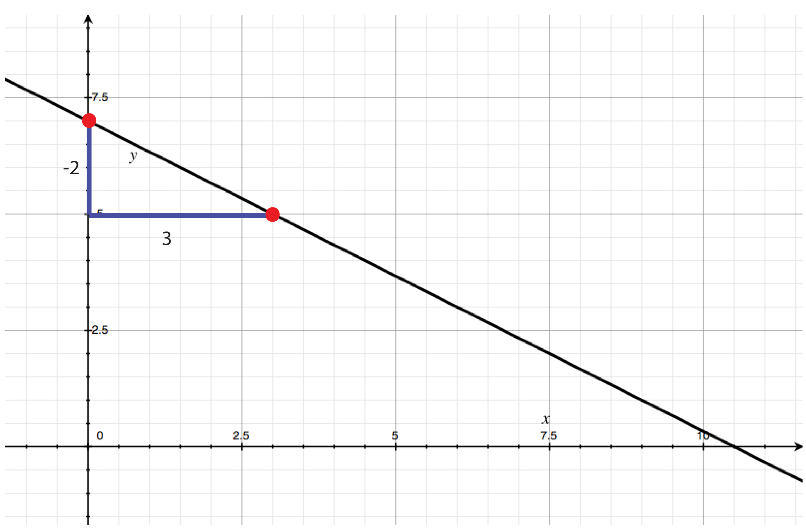
Solution:

Start by rewriting this equation in slope-intercept form:

$$
\begin{aligned}
2x + 3y &= 21 \\
3y &= -2x + 21 \\
y &= -\tfrac{2}{3}x + 7
\end{aligned}
$$

This is now in slope-intercept form, so identify the slope and intercept: $m = -\frac{2}{3}$ and $b = 7$.

Finally, graph the intercept and use the slope to find the next point:



Note that you could use any point on the graph (not just the intercept) along with the slope to plot a line.

Review

1. What are the solutions to an equation in two variables? How is this different from an equation in one variable?
2. ***Think of a number, triple it, and then subtract 7 from your answer***. Make a table of values and plot the function that this sentence represents.

Graph the solutions to each linear equation by making a table and graphing the coordinates.

3. $y = 2x + 7$
4. $y = 0.7x - 4$
5. $y = 6 - 1.25x$
6. $y = \frac{4}{3}x - 6$

Graph the solutions to each equation by using the intercepts.

7. $7x + y = -7$
8. $y + 3 = \frac{3}{2}(x + 4)$
9. $y = x - 6$

Graph the solutions to each equation by using the y-intercept or another point and the slope.

10. $y = -5x - 2$
11. $y - 5 = \text{-}4(x + 2)$
12. $y + 8 = \text{-}\frac{1}{6}x$

Graph the following using any method.

13. $y + 4 = \frac{2}{3}(x - 2)$
14. $y = -3x + 7$
15. $-4x + 5y = 30$

Review (Answers)

Please see the Appendix.

1.14 Intervals and Interval Notation

Learning Objectives

Learn to identify real functions and recognize closed and open intervals.

Learn to interpret and express intervals in interval notation.

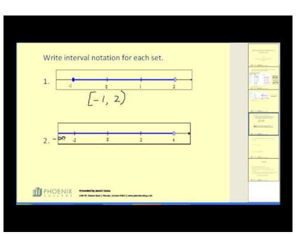
Introduction

Suppose you and two friends are out for lunch and you decide to buy tacos.

Together you have \$15 to spend on lunch, and tacos are \$1.25 each. It is clear that the total cost could be graphed as a discrete function of the number of tacos purchased, but how would you specify that the graph should not include values greater than \$15 or less than \$3.75 (one taco each)?

Real Values and Intervals

This video provides an overview of how to express a set of real numbers in interval notation, in set notation, and as a graph on a number line:

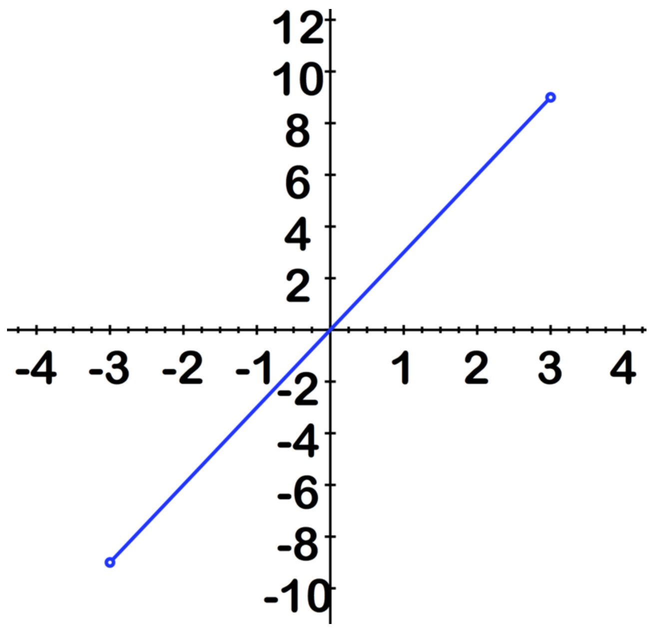


MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/55039

A function is defined as a **real function** if both the domain (the x or other input values) and the range (the y or other output valuare sets of real numbers. Many of the functions you have likely encountered before are real functions, and many of these functions have Domain = $\mathbb{R}$. Consider, for example, the function $y = 3x$. A section of the graph of this function is shown below:



You may already be familiar with the graphs of lines. And you may be in the habit of placing arrows at the ends. We do this to indicate that the line will continue forever in both positive and negative directions, in terms of both the domain and the range. The line above, however, only shows the function $y = 3x$ on the **interval** (-3, 3). The parentheses indicate that the graph does not include the endpoints of the interval, where $x =$ -3 and $x = 3$. We call this an **open interval**. You may have noticed that the open interval notation looks like the notation for a point (x, y) in the plane. It is important to read an example or a homework problem carefully to avoid confusing a point with an interval! The difference is generally quite clear from the context. In contrast, a **closed interval** does contain its endpoints. We indicate a closed interval with square brackets. For example, [-3, 3] indicates the set of numbers between -3 and 3, including -3 and 3.

The table below summarizes the kinds of intervals you may need to consider while studying functions and their domains:

TABLE 1.7:

Interval notation	Inequality notation	Description
$[a, b]$	$a \leq x \leq b$	The value of x is between a and b, including a and b, where a, b are real numbers.
(a, b)	$a < x < b$	The value of x is between a and b, *not* including a and b.
$[a, b)$	$a \leq x < b$	The value of x is between a and b, including a, but not including b.
$(a, b]$	$a < x \leq b$	The value of x is between a and b, including b, but not including a.
(a, ∞)	$x > a$	The value of x is strictly greater than a.

TABLE 1.7: (continued)

Interval notation	Inequality notation	Description
$[a,\infty)$	$x \geq a$	The value of x is greater than or equal to a.
$(-\infty,a)$	x	The value of x is strictly less than a.
$(-\infty,a]$	$x \leq a$	The value of x is less than or equal to a.

Inequalities

In another section, we discussed inequalities and briefly noted the interval notation that could be used.

A **compound inequality** is an inequality that combines two other inequalities. For example, $x <$ -3 or $x > 0$. The "or" indicates that the solution is the union of the graphs. Thus, any number that fits at least one inequality would be valid.

Another example is $x > -5$ and $x < 2$ or, when appropriate, -5 $< x <$ 2. The "and" indicates the solution is the intersection of the two graphs. Thus, a number must make both inequalities valid to be valid.

The same options for expressing inequalities in interval notation can be used to describe the domain and range of functions, as explored in the later examples.

Examples

Example 1

Identify the sets described:

a) $(-3,9]$

Solution:

The set of numbers between -3 and 9, "not including" the actual value of -3, but "including" 9.

b) $[-23,12]$

Solution:

The set of numbers between -23 and 12, "including" the values -23 and 12.

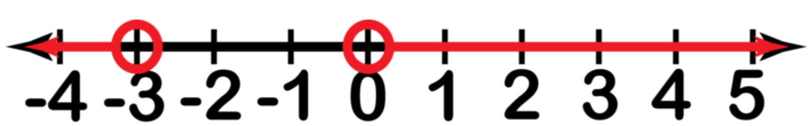
c) $(-\infty,0)$

Solution:

All numbers less than zero, not including zero itself.

Example 2

a) Describe the set shown in the image using interval notation.



Solution:

b) $(-\infty,3),(0,\infty)$

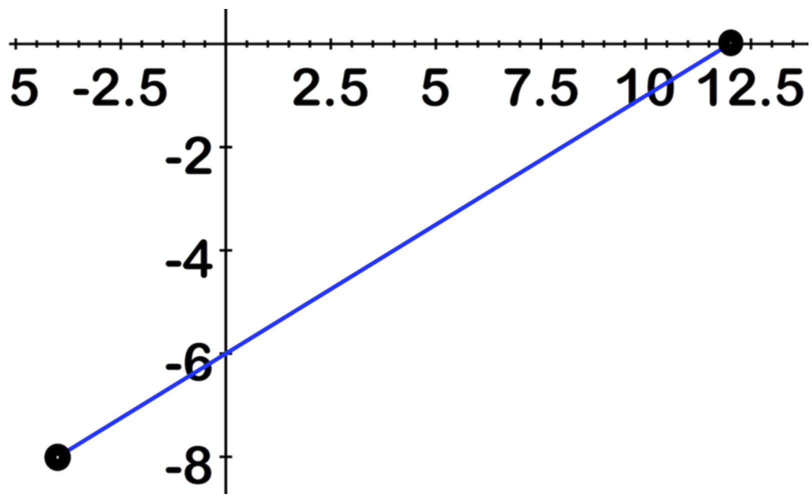
Since negative infinity cannot be reached, the set is opened with "(". It is then closed with ")" because 3 is not included. The set is reopened with "(" since 0 is not included and closed with ")" since positive infinity cannot be reached, either.

Example 3

Sketch the graph of the function $f(x) = \frac{1}{2}x - 6$ on the interval [-4, 12).

Solution:

The figure below shows a graph of $f(x) = \frac{1}{2}x - 6$ on the given interval:



Example 4

Describe the specified intervals, using interval notation:

a) All positive numbers

Solution:

$(0, \infty)$

Zero is neither positive nor negative, so the “(“ is used to specify that zero is ”not” included. Since there is no maximum positive number, we specify that infinity is the upper value, and use “)” because positive infinity cannot be reached.

b) The numbers between -8 and 242, including both

Solution:

$[-8, 242]$

The “[“ is used on both ends because both values are included.

c) All negative numbers, zero, and the positive numbers up to and including 9

Solution:

$(-\infty, 9]$

The “(“ denotes that negative infinity cannot be reached and “]” on the other end specifies that 9 is included in the set.

Example 5

Return to the problem in the Introduction: To specify that the graph of the cost of lunch includes only values between $3.75 and $15, specify the interval of the domain as [3.75, 15].

Example 6

a) Describe the specified intervals, using interval notation:

i) All negative numbers
Solution:

$(-\infty, 0)$

Since there is no maximum negative number, we specify that infinity is the lower value, and use “(” because negative infinity cannot be reached. Zero is neither positive nor negative, so the “)“ is used to specify that zero is ”not” included.

ii) The numbers between 5 and 12, including 5, but not 12
Solution:

$[5, 12)$

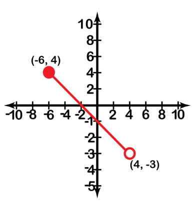
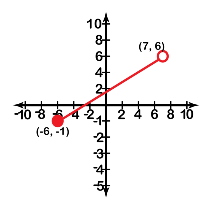
The “[“ is used to open the set because 5 is included. The ")" is used to close because 12 is not included.

iii) Negative numbers down to -6, zero, and all positive numbers
Solution:
$[-6, \infty)$

The “[“ is used because -6 is included, but positive infinity cannot be reached, so ")" is used.

b) Describe the **domain** for the given images using interval notation:

TABLE 1.8:

a)	b)
	

Solutions:

The domain of graph a) is the set of x -values starting with the included -6 and ending at 4, which is not included: [-6, 4). The domain of graph b) is the set of x-values starting with the included -6 and ending at 7, which is not included: [-6, 7).

c) Describe the **range** in the sets in the images above using interval notation.

Solutions:

The range of graph a) is the set of y-values from -3 (not included) to 4 (included): (-3, 4]. The range of graph b) is the set of y-values from -1 (included) to 6 (not included): [-1, 6).

Review

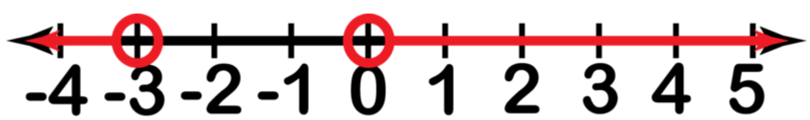
Write the following in interval notation:

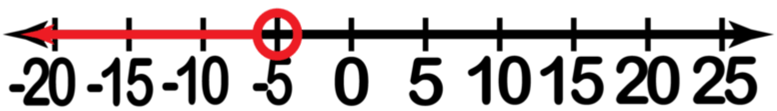
1. $-3 \leq x < 1$
2. $0 < x < 2$
3. $x > -3$
4. $x \leq 2$

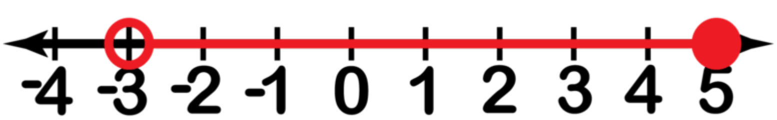
Solve and put your answer in interval notation:

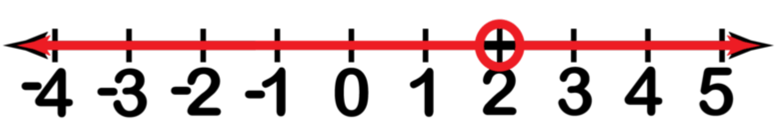
5. $-2x + 3 < 1$
6. $7x + 4 \leq 2x - 6$

For each number line, write the given set of numbers in interval notation:

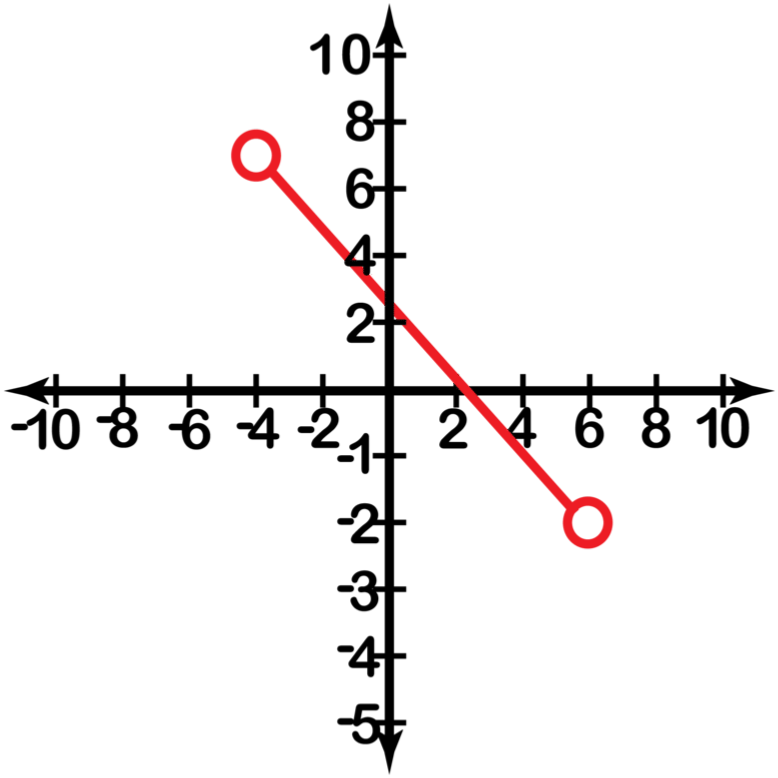
7. 

8. 

9. 

10. 

Name the domain and range for graph, using interval notation:



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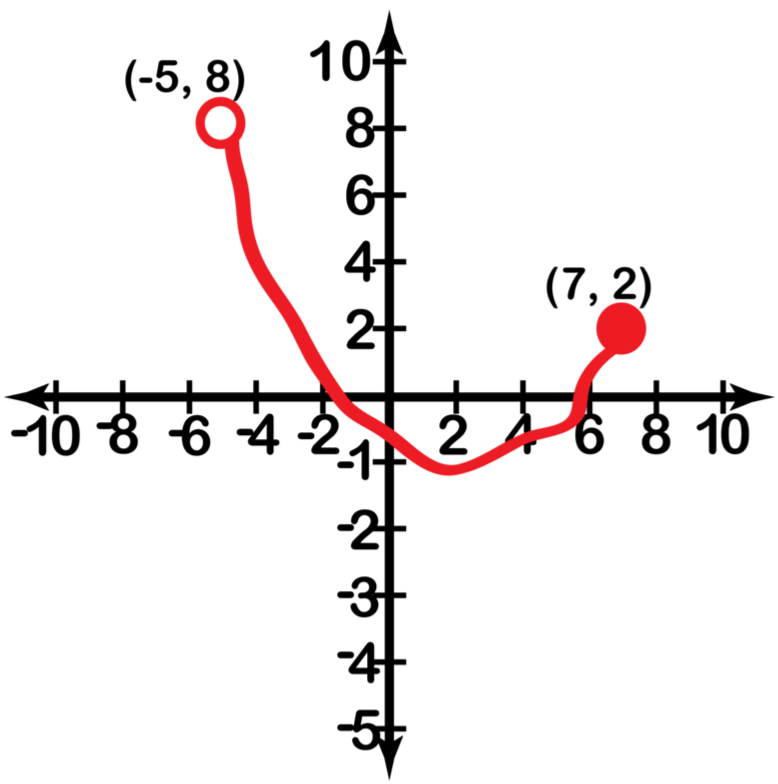


Express the following sets using interval notation, then sketch them on a number line:

13. $\{x : 1 \leq x \leq 3\}$
14. $\{x : 2 \leq x < 1\}$
15. A is the set of all numbers bigger than 2 but less than or equal to 5.

16. $\{x : 3 < x < \infty\}$

Review (Answers)

Please see the Appendix.

1.15 Average Rate of Change

Learning Objectives

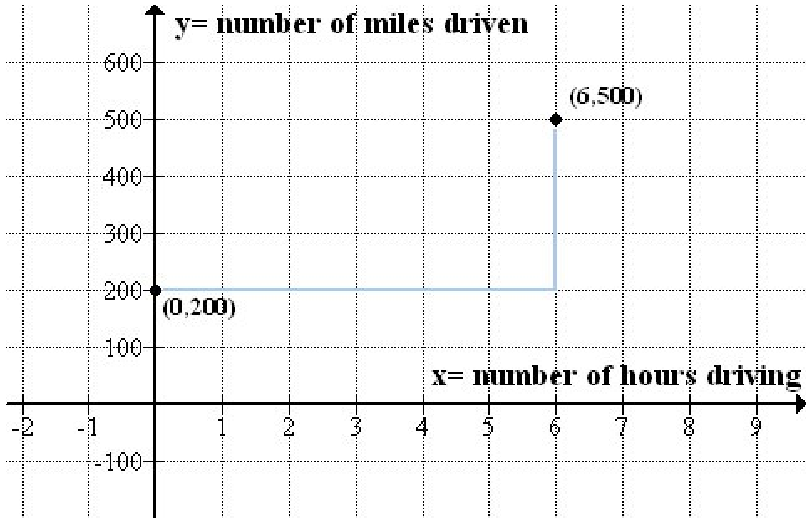
Learn to identify the average slope, or rate of change, for a portion of a graph called an interval.

Introduction

Consider the following situation: You are on a week-long road trip with your friend. When you begin to drive on the 2nd day, you have already driven a total of 200 miles. After 6 hours of driving on the 2nd day, you have driven a total of 500 miles. On average, how many miles did you drive per hour on the 2nd day of the trip?

Average Rate of Change

The graph below shows this situation, with the x-axis representing the number of hours driving (on the 2nd day), and the y-axis representing the number of miles driven. The 1st point on the graph, (0, 200), says that at the beginning of the 2nd day you have already driven 200 miles. The 2nd point on the graph, (6, 500), says that after 6 hours of driving on the 2nd day, you have driven 500 miles total.



Notice that, in total, during your 6 hours of driving, you have driven 300 miles. The rate at which you drove is 300 miles in 6 hours, or 50 miles per hour. We refer to this rate as the **average rate of change** because it is an average across the 6 hours. That is, you did not necessarily drive 50 miles every hour. There could have been one hour in which you drove 70 miles, and another in which you drove only 30 miles.

We can represent the average rate of change on the graph by indicating how much each quantity has changed: The y-values increased by 300, and the x-values increased by 6. The average rate of change is the ratio of these changes in each variable. This is how we can define average rate of change in general:

Average Rate of Change

$$\text{Average rate of change } = \frac{\text{change in y}}{\text{change in x}} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

We can examine the average rate of change of a function, whether it is represented as data, as in the previous example, or by an equation, or graphically.

Notice that the average rate of change of the function $f(x) = 4x$ over any interval is the slope of the line, 4. While a linear function has a constant slope, other functions, such as $f(x) = x^2$, will not. You will explore this idea in greater detail in your study of calculus.

Examples

Example 1

Find the average rate of change on the given interval:

$f(x) = x^2$ on [0, 2]

Solution:

$f(x) = x^2$

The endpoints of the interval are (0, 0) and (2, 4). Therefore, the change in y is 4 and the change in x is 2. The average rate of change is 4/2 = 2.

Example 2

Now let's check if our claim regarding the average rate of change of $f(x) = 4x$ is valid for a specific interval. Find the average rate of change on the given interval

$f(x) = 4x$ on [1, 7]

Solution:

$f(x) = 4x$

The endpoints of the interval are (1, 4) and (7, 28). Therefore, the change in y is 28 - 4 = 24 and the change in x is $7-1 = 6$ The average rate of change is 24/6 = 4.

Example 3

What is the average rate of change of the function $f(x) = 3x^2$ on the interval [2, 5]?

Solution:

The two endpoints are (2, 12) and (5, 75). The average rate of change is 63/3 = 21.

Example 4

a) Find the average rate of change of the function $f(x) = x^2$ as x varies from 1 to 3.

Solution:

The average rate of change is the slope of the line that passes through the two points $(3,9)$ and $(1,1)$ on the graph.

$$\begin{aligned}\frac{f(x_2)-f(x_1)}{x_2-x_1} &= \frac{f(3)-f(1)}{3-1}\\ &= \frac{9-1}{3-1}\\ &= \frac{8}{2}\\ &= 4\end{aligned}$$

b) If the temperature at 1 p.m. was 82°F, and at 9 p.m. was 50°F, what was the average rate of change of temperature over the given time interval?

Solution:

The temperature dropped by 32° over a period of 8 hours: $\frac{-32}{8} = -4$, or 4 degrees per hour.

c) Brian drove to town to get some milk. He left at 9 a.m., drove 8 miles north, 5 miles west, 3 miles north, and 2 miles west. He arrived at the store at 9:45 a.m. What was his average rate of change in miles per minute?

Solution:

Brian drove a total of 18 miles in a total of 45 minutes:

$$\frac{18 \text{ mi}}{45 \text{ min}} = \frac{2}{5} \text{ miles per minute}$$

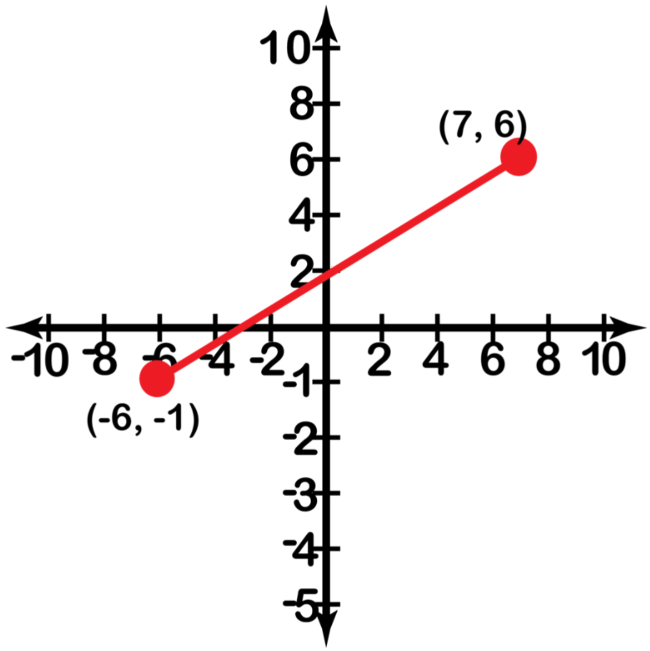
d) If Kelli spends \$5 at 1 p.m., \$7 at 2:30 p.m, \$12 at 4 p.m., and \$2 at 4:30 p.m., what is her average rate of spending in dollars per hour?

Solution:

Kelli spent a total of \$26 over a total of 3.5 or $\frac{7}{2}$ hours. Set up the fraction and then multiply the top and bottom by 2 to remove the fraction from the denominator:

$$\frac{\$26}{7/2 \text{ hrs}} \frac{\times 2}{\times 2} = \frac{\$52}{7 \text{ hrs}} \approx 7.43 \text{ dollars per hour}$$

e) What is the average rate of change shown in the graph below?



Solution:

The two points in the graph are $(-6,-1)$ and $(7,6)$.

The average rate of change is the same as the slope of the line. Recall: $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

$$m = \frac{(6)-(-1)}{(7)-(-6)} = \frac{7}{13}$$

Summary

- Average rate of change formula: Average rate of change $= \frac{\text{change in y}}{\text{change in x}}$.
- We can examine the average rate of change of a function, whether it is represented as data, as in the previous example, or by an equation, or in a graph.

Review

For $y = x^3$, find the average rate of change as:

1. x increases from 1 to 3
2. x increases from -4 to -1
3. Suppose $f(1) = 2$ and the average rate of change of f between 1 and 5 is 3. Find $f(5)$

4. Jamie went on a bicycle trip and stopped regularly at half-hour intervals. At each break, he recorded his total distance since leaving home. What was his average speed in km/h during the 1st half of the trip? During the last half? Jamie hoped to average at least 11.5km/h over the course of the trip. Did he? Explain.

TABLE 1.9:

Stops	Time (h)	Distance (km)
1	0.5	7
2	1.0	15
3	1.5	21
4	2.0	24
5	2.5	28
6	3.0	36

5. On Monday, the price of a gallon of gas was \$3.74. On Saturday, the price had risen to \$4.09. What is the average rate of change of the price of a gallon of gas per day?
6. According to census figures, the population of Clovis was 31,194 in 1980 and 32,511 in 2001. What was the average rate of change of the population over that time interval?
7. Griego and Sons will deliver 12 yd^3 of gravel for \$240 and 30 yd^3 for \$575. What is the average rate of change of the cost as the number of cubic yards varies from 12 to 30?
8. When a load of 5 pounds is placed on a spring, its length is 6 inches, and when a load of 9 pounds is placed on the spring, its length is 8 inches. What is the average rate of change of the length of the spring as the load varies from 5 pounds to 9 pounds?
9. Driving at a speed of 75 mph, a Mini Cooper's fuel efficiency is 29 miles per gallon. If the driver slows to a speed of 60 mph, he will have a fuel efficiency of 34 miles per gallon. What is the average rate of change of the fuel efficiency as the speed drops from 75 mph to 60 mph?
10. Let $y = f(x) = x^2 + x + 2$.

    a) Find the average rate of change of y with respect to x between x = -1 and x = 2.

    b) Draw the graph of $f(x)$ and the graph of the line through $(-1,-2)$ and (2, 4) on the function. c) Find the slope of the line through $(-1,-2)$ and (2, 4). d) What do you notice about the answers in parts b) and c)?
11. Amy takes a trip from Chicago to Milwaukee. Due to road construction, she drives the 1st 10 miles at a constant speed of 20 mph. For the next 30 miles, she maintains a constant speed of 60 mph and then stops at a diner for 10 minutes. She drives the next 45 miles at a constant speed of 45 mph. a) What is Amy's average **driving** speed for the trip? b) What is her average speed for the entire trip, including the stop at the diner?
12. The weight $w(t)$ (in grams) of a tumor t weeks after it forms is expressed by $w(t) = \frac{t^2}{15}$. Find the average rate at which the tumor is growing per day during the 5th week after it formed.
13. The price of a house increases from \$120,000 in 1995 to \$250,000 in 2007. What was the average rate of change to the nearest dollar per year?
14. Find the rate of change for each set of ordered pairs below. What is the average rate of change of all the sets?

    $$(2,8) \text{ and } (3,12)$$
    $$(\text{-}1,14) \text{ and } (3,9)$$
    $$(3,2) \text{ and } (\text{-}1,8)$$

    Write your answer to the average rate of change as a fraction.
15. The balance in Sam's savings account changed from \$1,200 in June to \$1,600 in August. What was the average rate of change per month to the nearest dollar?
16. Every hour a runner is able to run the same number of miles. If on his 3rd hour of running he has gone 15 miles, and on his 5th hour of running he has gone 25 miles, what is the rate at which he runs?

Review (Answers)

Please see the Appendix.

1.16 Relations and Functions

Learning Objectives

Learn the meaning of the term 'function' and how to demonstrate the relationship between relations and functions.

Introduction

Suppose you want to predict how much it would cost to see a movie at the theater. You text a number of friends who've recently been to the movies, and ask how much it cost them. Here are their responses:

"$14.50 :-("

"$8.75 + $3.50 for popcorn"

"five bucks - dollar theater"

"$17.50 :'-(broke now"

"$12.75 loved the 3D!"

Can you accurately predict the cost of going to a movie from these responses? Why or why not?

Relations and Functions

Consider two situations shown in the boxes below:

TABLE 1.10:

Situation 1: You are selling candy bars for a school fundraiser. Each candy bar costs $3.	Situation 2: You collect data from several students in your class about their ages and heights, and get the following information: (18,65"), (17,64"), (18,67"), (18,68"), (17,66").

In the 1st situation, let the variable x represent the number of candy bars you sell, and let y represent the amount of money you make. If you sell x candy bars, you'll make $y = 3x$ dollars. For example, if you sell 25 candy bars, you'll make 3(25) = $75. Notice you can use the number of candy bars you sell to predict how much money you'll make.

Now consider the 2nd situation. Can you similarly use the data you collected to predict specific height, based on age?

No, you cannot make such a prediction in this case. For example, if a student is 18 years old, there are a number of possible heights that student could be.

The 1st situation is an example of a *function,* and the 2nd example is *not* a function.

Definition of a Function

A **function** is a relationship in which each input number corresponds to one and only one output number.

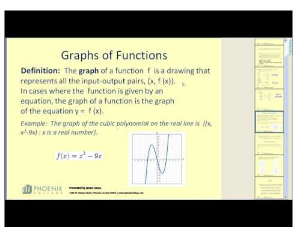
In the 1st situation, for each different number of candy bar sales you input, there is one and only one output number representing your profit.

In the 2nd situation, if you input "18 years," there are multiple outputs, so you can't identify a specific relationship between age and height.

It is important to note that both situations above are relations. A **relation** is a pairwise relationship between two sets of numbers or data. For example, in the 2nd situation, we created a relationship between students' ages and heights by writing each student's information as an ordered pair. In the 1st situation, there is a relationship between the number of candy bars you sell and the amount of money you make. The 1st example is different from the 2nd because it represents a **function**: every x is paired with only one y.

Functions may be represented in many ways. Some of the most common ways to represent functions include sets of ordered pairs, equations, and graphs. The table below in Example 1 shows the same function depicted in three different ways.

This video below provides an overview of how to determine if a relation is a function. It includes vocabulary definitions, the techniques of mapping and the vertical line test to determine if a relation is a function, and examples.



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/189448

To further explore the idea of vertical line testing, see this PLIX: www.ck12.org/a/2586733 .

Examples

Example 1

Determine if each relation is a function:

TABLE 1.11:

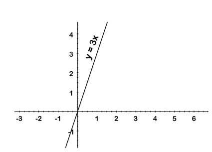
Representation	Example
Set of ordered pairs	(1,3), (2,6), (3,9), (4,12) (a subset of the ordered pairs for this function)
Equation	$y = 3x$
Graph	

TABLE 1.11: (continued)

Representation	**Example**

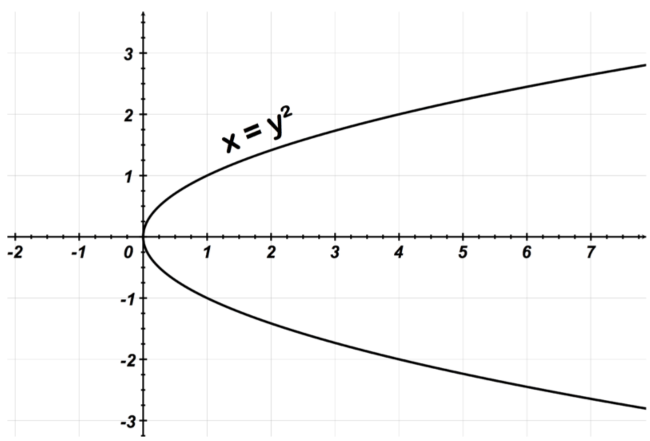
Solution:

In the 1st representation above, we are given a set of ordered pairs. To verify that this is a function, we must ensure that each x-value is associated with a single y-value. In this example, the 1st number in each pair (the x-value) is different, so we can be certain that there are no cases in which a particular x is associated with more than one y.

In the 2nd representation, the equation of a line, it is apparent that any number put in place of x will result in a different y, since the x number is simply being multiplied by 3.

The 3rd representation above is a graph. A good way to determine whether a relation is a function when looking at a graph is by doing a "vertical line test." If a vertical line can be drawn anywhere on the graph such that the line crosses the relation in two places, then the relation is not a function. If all possible vertical lines cross the relation in only one place, then the relation is a function. This works because if a vertical line crosses a relation in more than one place, it means there must be two y values corresponding to one x value in that relation. For example, the graph above of $y = 3x$ shows it is a function because any vertical line that is drawn crosses the relation in only one place.

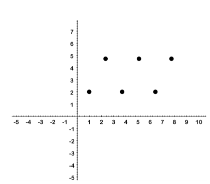
Conversely, the graph below of $x = y^2$ shows it is not a function because a vertical line can be drawn that crosses the relation in two places:



Example 2

Determine if each relation is a function:

TABLE 1.12:

a) (2, 4), (3, 9), (5, 11), (5, 12)	b) Relation defined as: 

Solutions:

a) (2, 4), (3, 9), (5, 11), (5, 12)

This relation is not a function because 5 is paired with 11 and with 12.

b) This relation is a function because every x is paired with only one y. A vertical line through the graph will always only encounter a single point.

Example 3

Recall the problem in the Introduction about movie tickets. Does the data you received from your friends represent a function? Can you use the data to predict the cost of going to a movie?

Solution:

If we were to organize the information we received into ordered pairs, it might look something like $(1,14.5), (1,8.75), (1,5), (1,1$ each x-value represents the number of tickets bought, and each y-value represents the price.

Since there are many different y-values for the only x-value, it is definitely not a function.

It should now be clear that the information received from friends' text messages **cannot** be used to accurately predict the cost of a movie.

Example 4

Determine if each relation is a function:

a) $(-1,4), (0,3), (1,5), (1,7), (2,15)$

Solution:

There are two different "outputs" or y-values for the "input" or x-value of 1. Because we cannot know whether 1 should go with 5 or 7 at any given time, this relation is **not** a function.

b) $y = x$

Solution:

Since $y = x$, any time a number is chosen to represent x, that and only that number becomes y.From this it is apparent that each input has one and only one output, so this relation is a function.

c) $(2,0), (4,-1), (2.1,4), (1,4), (4,-1)$

Solution:

Don't be fooled! This is a function, as there is only one unique output for each input. The fact that both x-values 2.1 and 1 are associated with y value 4 does not mean that 2.1 and 1 don't have a specific associated value. Also, no matter how close two x's (2 and 2.1, for instance) may be, if they are not exactly the same, they don't affect the definition of a function.

d) $y = 4x$

Solution:

This is a function. Any value chosen for x has one and only one associated value for y (four times as big).

e) $x = |y|$

Solution:

This is not a function. This graph looks like a "<" with the vertex on the origin. Any positive value chosen for x will have two associated y -values. For instance, 4 = |-4| *and* 4 = |4|.

Summary

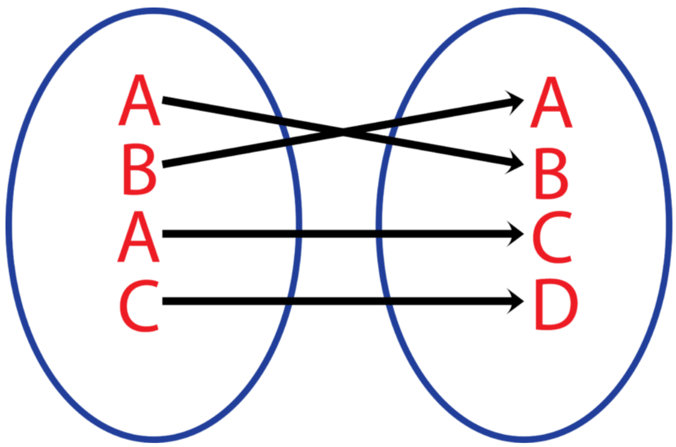
- A **relation** is a relationship between two sets of numbers or data.
- A relation is also a **function** if every x is paired with only one y.

Review

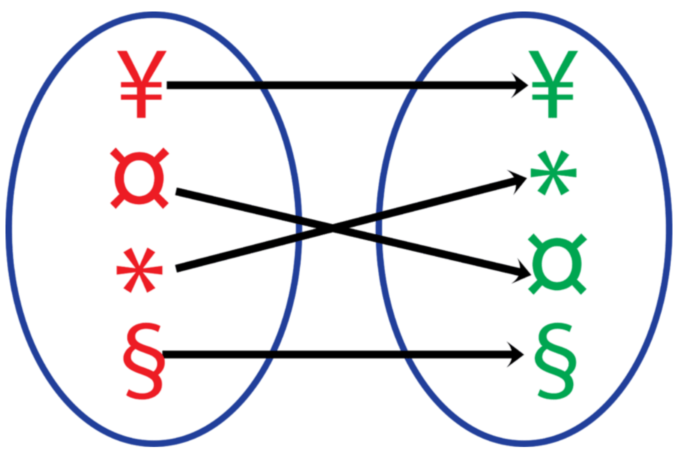
1. What is the definition of a function?
2. Can a function be written in the form $x = 3y$ instead of $y = 3x$?
3. Is it mandatory for a function to have both an input and an output?
4. Can a statement be a function if there is only one input and output?
5. Give an example of a relation that is not a function, and explain why it is not a function.

For Questions 6-14, identify each relation as either a function, or not a function:

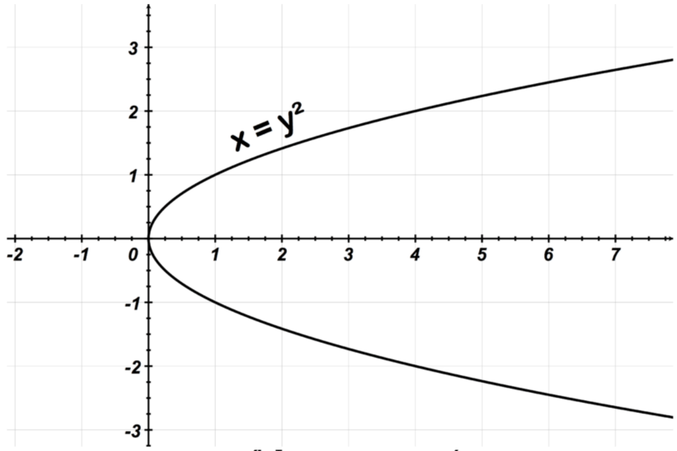
6. (2, 4) (4, 6) (6, 8) (3, 4) (5, 7) (8, 2)
7. (-1, 6) (0, 4) (-4, 0) (-1, -6) (-3, -8)
8. This relation with input values on the left and output on the right:

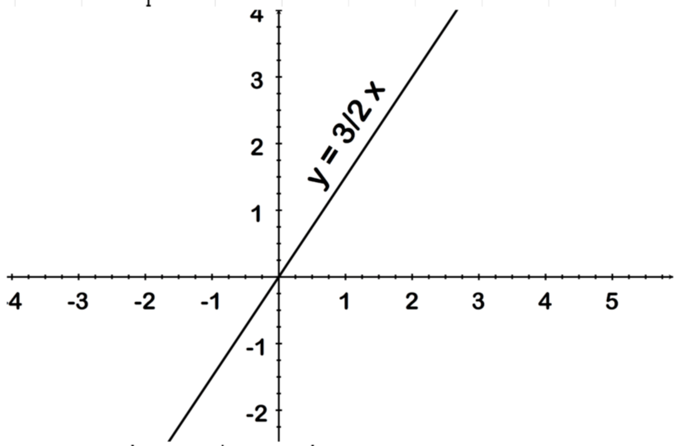


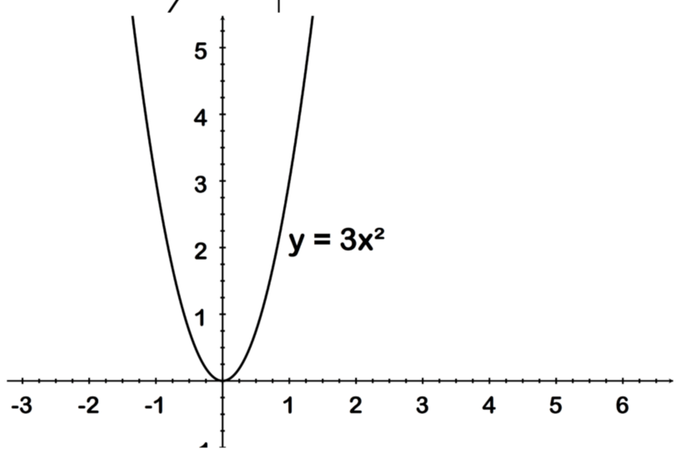
9. This relation with input values on the left and output on the right:



10. (Jim, Kitty) (Joe, Betty) (Brian, Alice) (Jesus, Anissa) (Ken, Kelli)
11. (Jim, Alice) (Joe, Alice) (Brian, Betty) (Jim, Kitty) (Ken, Anissa)

12. 

13. 

14. 

15. At a prom dance, each boy pins a corsage on his date. Is this an example of a function?
16. Later, at the same dance, Cory shows up with two dates. Does this change the answer?

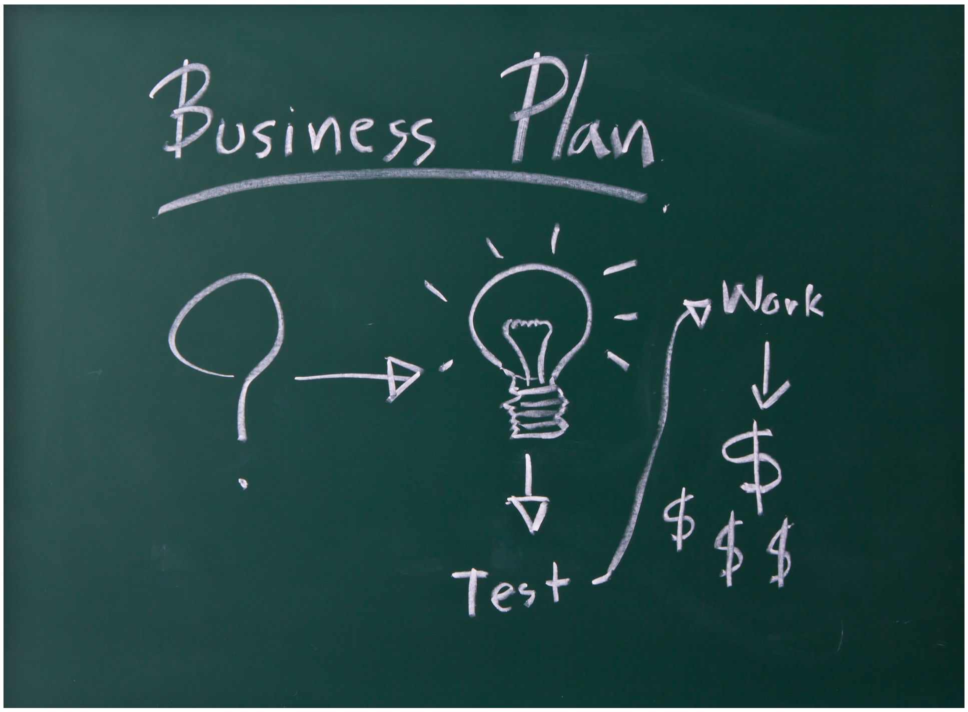
Review (Answers)

Please see the Appendix.

1.17 Project: Prerequisites

To practice a number of different topics covered in this chapter, from plotting points to determining slope, explore the following situation:

You have designed a new invention, and you're doing some analysis of cost and expenses. You're currently making about $65,000 per year, but you are hoping your new invention will allow you to double your salary.

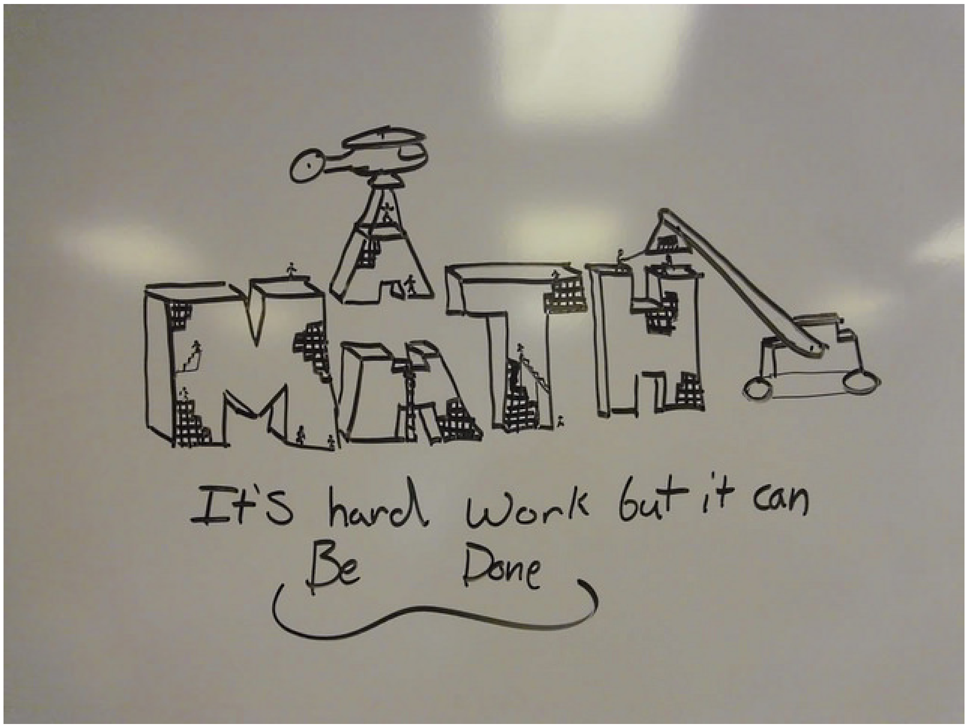


You have found that you have to make an initial investment in supplies, plus a certain amount per unit produced. The cost to make 40 units is $10,000, and the cost to make 100 units is $20,500. Your market research also tells you that your price point should be $250 per unit.

Directions:

1. Plot the expense and the sales costs on the coordinate plane.
2. Determine the slope of the cost function. Determine the slope of the price function.
3. What was the cost of your initial investment?
4. After how many units will you break even?
5. How many units will you need to sell each year to replace your current salary?

1.18 Summary: Prerequisites



Chapter Summary

This chapter has revisited the fundamental concepts from algebra that are necessary to build and manipulate the mathematics to be learned in precalculus. The rules of numbers and number systems, as well as the properties of operations, are essential to the material ahead, including working with triangles and functions. With the ideas and steps of algebra under your belt, you're ready for the next level. You might find that lessons that once made you ask yourself, "When am I going to need this?" are now super-helpful in solving applications of real-life mathematics!

Review

Try the following cumulative review problems to practice the concepts in this chapter:

1.19 References

1. Petteri Sulonen. https://flic.kr/p/4Qwkbi .
2. CK-12 Foundation. CK-12 Foundation .
3. NASA (mod by Jcpag2012). https://commons.wikimedia.org/wiki/File:Neptune,_Earth_size_comparison_2.jpg .
4. Andrea Schafthuizen. http://www.publicdomainpictures.net/view-image.php?image=3699&picture=&jazyk=NL .
5. Petr Kratochvil. http://www.publicdomainpictures.net/view-image.php?image=1591&picture=christmas-decoration .
6. FutUndBeidl. https://flic.kr/p/diGSsZ .
7. Liana_Kyle. https://flic.kr/p/btkbos .
8. Unknown. http://pdpics.com/photo/4881-coffee-beans/ .
9. Chris Potter. https://www.flickr.com/photos/86530412@N02/8266136492/ .
10. CK-12 Foundation. .
11. CK-12 Foundation. .
12. Karl-Martin Skontorp. https://flic.kr/p/7Thh9c .
13. Marcin Wichary. https://commons.wikimedia.org/wiki/File:NM_124_and_US_66_WB_near_Budville_NM.jpg .
14. Unknown. http://clipart-library.com/clipart/kcMK9ga5i.htm .
15. Lauren Coolman. https://flic.kr/p/9VG85f .

CHAPTER 2 Functions and Graphs

Chapter Outline

2.1 Introduction: Functions and Graphs

"Ever since Isaac Newton used math to describe gravity, applied mathematicians have been inventing new mathematics or using existing forms to describe natural phenomena." —Gino Biondini, professor

Mathematicians create functions, or models, to describe observations and phenomena in the world. These tools can be used to predict the outcome of events, and that is the value of creating and using mathematical functions. Mathematician Gino Biondini made recent advances in the study of waves. His team's goal was to improve the wave equations first created in the 1700s, which predicted the behavior of waves, but broke down when waves became irregular. Biondini's team showed mathematically that many different kinds of disturbances evolve to produce wave forms belonging to a single class. Their discovery provides a simpler set of categories for waves of all types, so that scientists can make better predictions.

Mathematicians develop models that offer a type of map into the world of mathematical research and application. These methods allow exploration of often unseen worlds that can be used to predict, change, and improve our relationship with the world in which we live.

In this chapter, we review and explore the primary concepts used to analyze functions and their graphs. The topics we cover, which will establish a foundation for further study, include function domain, range, extrema, symmetry, intercepts, asymptotes, continuity, transformations, composition, and inverses.

2.2 Domain and Range

Learning Objectives

Learn about the domain and range of a function.

Introduction

Mathematical functions represent a rule of correspondence and are used to produce information. For example, the height of a rounded bridge over a brook is expressed by $h(x) = \sqrt{12x - x^2}$, where x is the horizontal distance traveled from one shore to the other. Because the brook is 12 feet wide, this function is defined with the boundary $0 \leq x \leq 12$. The domain (which we will define in this lesson) of the function is this set of values for x. Similarly, the bridge is 6 feet high so the values for the height are limited. The range (which we will also define in this lesson) of the function is $0 \leq h(x) \leq 6$.

The following interactive PLIX example also illustrates these concepts with the scope of sunrise to sunset: www.ck12.org/a/1824032 .

Domain

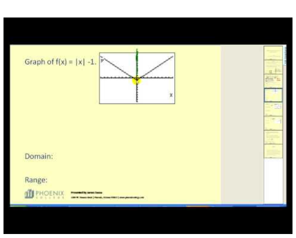
The **domain** of a function is the set of all x-values for which a function is defined. In the above exploration of the path of the sun, the domain is the set of time values past dawn, and range from 0 to 12 hours. For the function $y = 3x$, the domain is the set of all real numbers, often written as $\mathbb{R}$. The domain of the function $y = \sqrt{x}$ is the set of all real numbers greater than or equal to 0, because the square root of a negative number is not defined in the real number system.

The variable x is often referred to as the **independent** variable, while the variable y is referred to as the **dependent** variable, because the y-values of a function depend on the corresponding x-values.

Range

The **range** of a function is defined as the set of all y-values for which a function is defined. Just as with the domain, the range can be described. Consider for example the function $y = x^2$. The domain of this function is all real numbers $\mathbb{R}$, but what about the range?

The range of this function is the set of all real numbers greater than or equal to 0, since every y-value is the square of an x-value.



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Interval Notation

Domain and range are often described in interval notation. Interval notation is used to describe sets of numbers, such as when describing domain and range. Intervals are either open or closed or both. Open intervals use parentheses ( ) and refer to intervals that do not include the endpoints. Closed intervals use square or box brackets [ ] and refer to intervals that do include the endpoints. Braces or curly brackets { } are used when the domain or range consists of discrete numbers and not an interval of values.

If the domain or range of a function is all numbers, the notation includes negative and positive infinity $(-\infty, \infty)$. If the domain is all positive numbers plus 0, the domain would be written as $[0, \infty)$. If the range of a function is every number between 5 and 6 but not including 5 or 6, the notation would be (5, 6).

In addition, $\in$ is a symbol that means "is an element of," and $\cup$ is a symbol that means "union." This latter symbol is used to connect two groups and is associated with the logical term OR.

Examples

Example 1

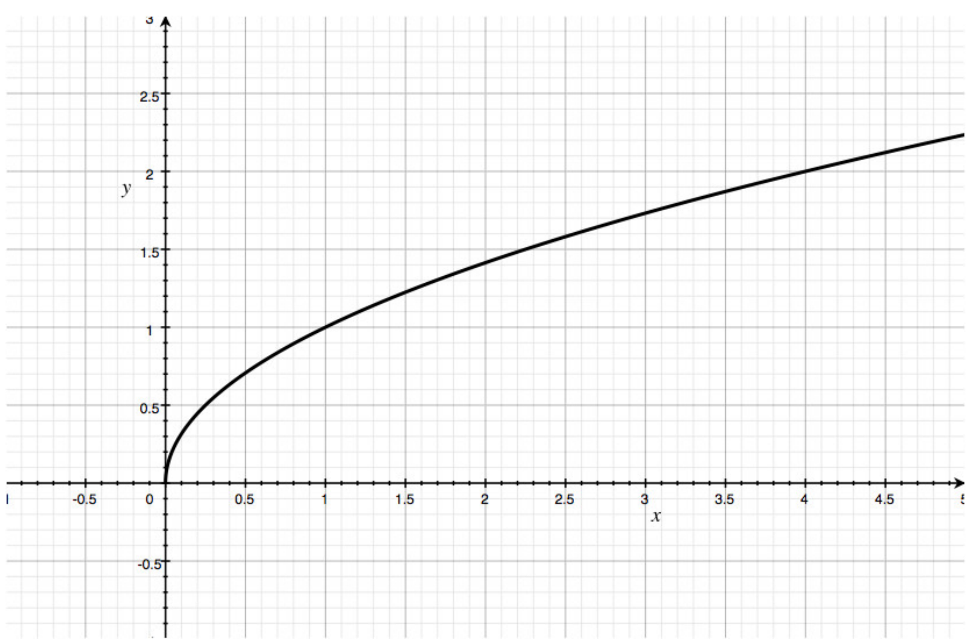
Explain the domain and range of the square root function $y = \sqrt{x}$.

Solution:

Domain: $x \geq 0$

Range: $y \geq 0$

The square root of a negative number is not a real number, so the domain is restricted. The domain and range can be observed with the graph of a function, because the curve is only defined where x and y are nonnegative.



Example 2

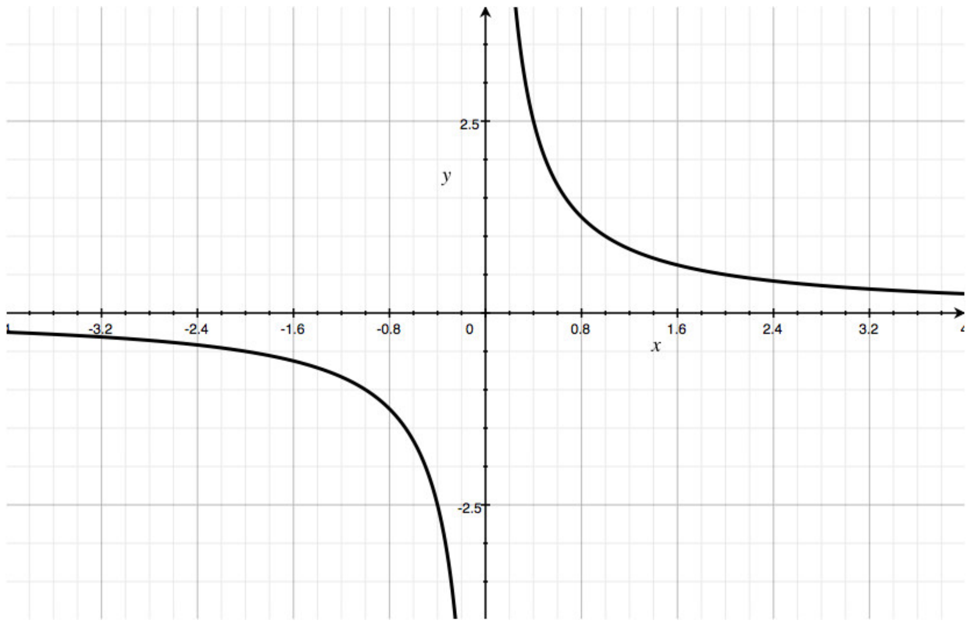
Explain the domain and range of the reciprocal function $y = \frac{1}{x}$.

Solution:

Domain: $x \neq 0$

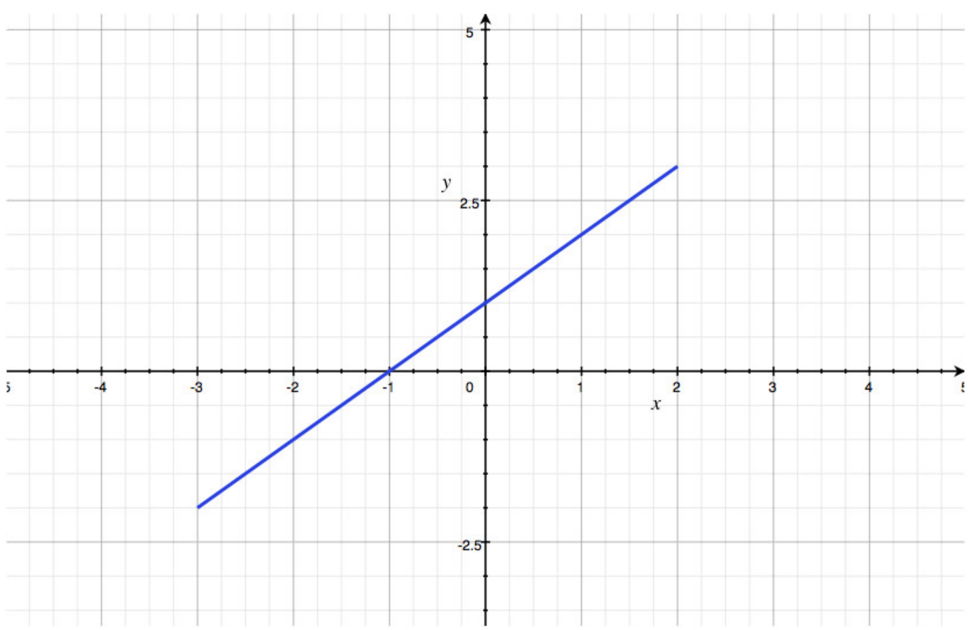
Range: $y \neq 0$

The reciprocal function is restricted because division by 0 is not defined. This restriction can be observed in the graph of the function by the fact that y approaches both $-\infty$ and ∞, as x approaches 0 from both the left and the right. Also, y approaches 0 as x approaches $-\infty$ and ∞.



Example 3

Identify the domain and range for the following function:



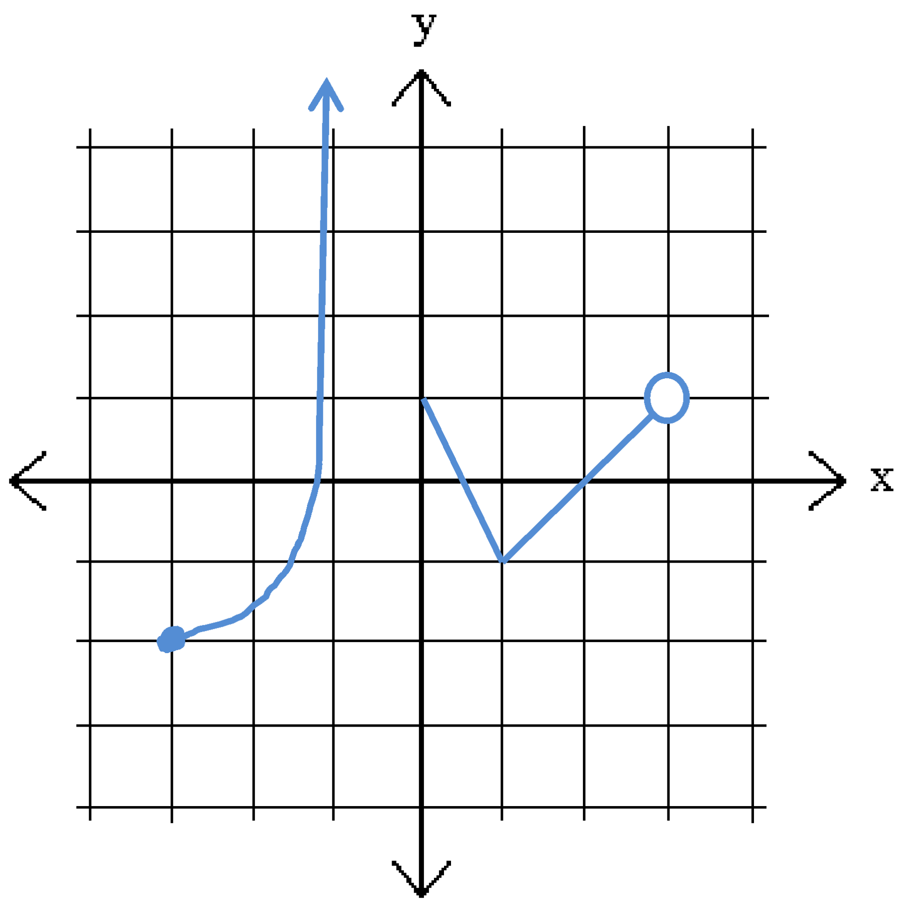
Solution:

Domain: $x \in [-3, 2]$

Range: $y \in [-2, 3]$

Example 4

Identify the domain and range of the following function:



Solution:

Domain: $x \in [-3, -1) \cup [0, 3)$

Range: $y \in [-2, \infty)$

Note that the function seems to approach the vertical line $x = -1$ without actually reaching it. Also note the hole at the point (3, 1). This symbolizes that the domain excludes the x-value of 3.

Example 5

Identify the domain and range of the following function written in a table:

x	y
0	5
1	6
2	7
$\frac{1}{2}$	6
π	$\frac{\pi}{2}$

Solution:

The equation of the function may be hidden, but from the table you can determine the domain and range directly from the x- and y-values. It may be tempting to guess that other values could potentially work in the table, especially if the pattern is obvious. But this question doesn't ask what the function might or could be. It asks what is the stated domain and range.

Domain: $x \in \{0, 1, 2, \frac{1}{2}, \pi\}$

Range: $y \in \{5, 6, 7, \frac{\pi}{2}\}$

Note that the 2 sixes that appear in the table do not need to be written twice in the range.

Example 6

Identify the domain of the following transformed functions:

a. $y = 10\sqrt{2-x} - 3$

Solution:

The expression under the square root must be greater than or equal to 0:

$$\begin{aligned} 2 - x &\geq 0 \\ -x &\geq -2 \\ x &\leq 2 \end{aligned}$$

Domain: $x \in (-\infty, 2]$

b. $y = \frac{3x}{x^2+7x+12}$

Solution:

The denominator cannot be equal to 0. First, find what values of x would make it equal to 0 and then you can exclude those values:

$$\begin{aligned} x^2 + 7x + 12 &= 0 \\ (x+4)(x+3) &= 0 \\ x &= -4, -3 \end{aligned}$$

Thus, the domain is all real numbers except -4 and -3.

Domain: $x \in (-\infty, -4) \cup (-4, -3) \cup (-3, \infty)$

c. $y = -4\log(3x-9) + 11$

Solution:

The expression in the logarithm must be strictly greater than 0:

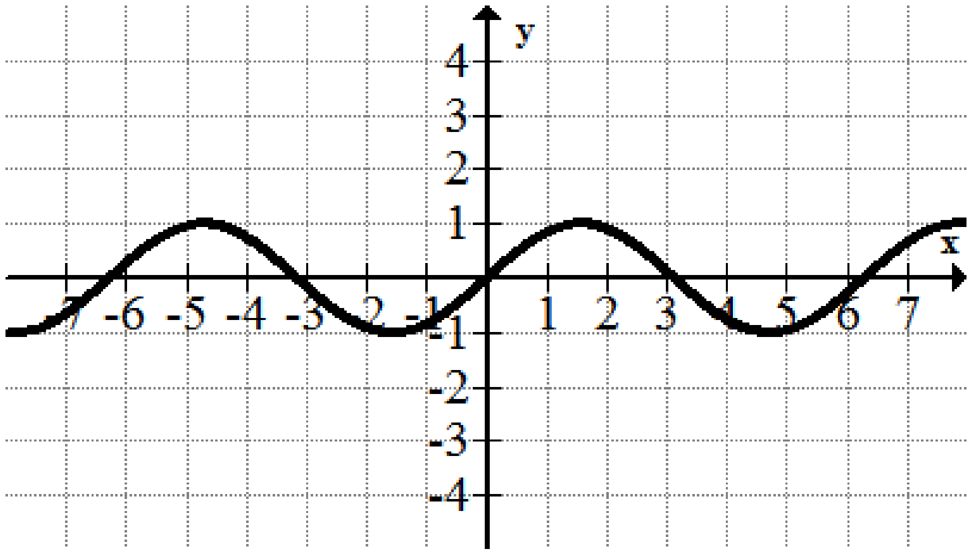
$$\begin{aligned} 3x - 9 &> 0 \\ 3x &> 9 \\ x &> 3 \end{aligned}$$

Domain: $x \in (3, \infty)$

Example 7

What is the domain and range of the sine wave?

Solution:



Domain: $x \in (-\infty, \infty)$

Range: $y \in [-1, 1]$

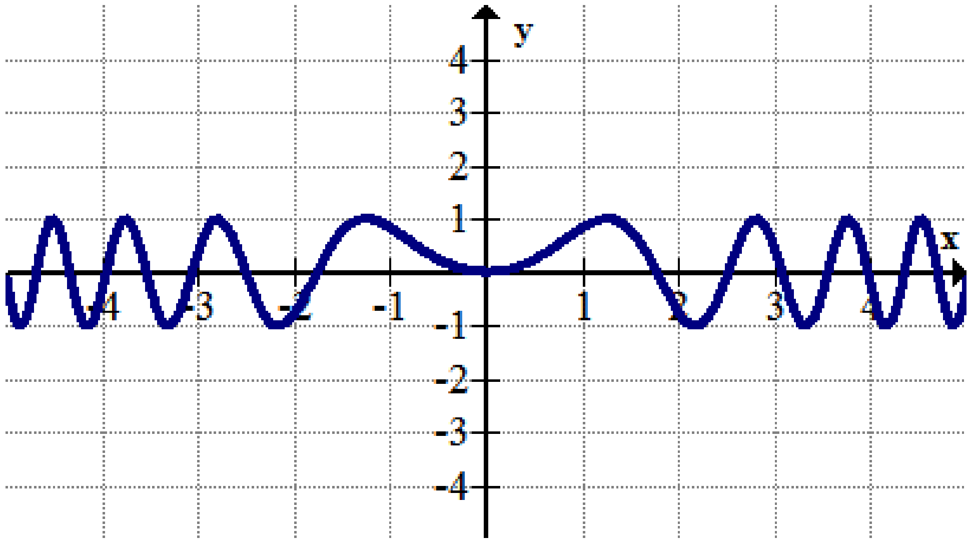
Summary

- **Domain** is the set of inputs for a function.
- **Range** is the set of outputs for a function.
- **Interval notation** is used to describe groups of numbers, such as when describing domain and range. Intervals are either open or closed or both.
- **Open intervals** use parentheses ( ) and refer to intervals that do not include the endpoints.
- **Closed intervals** use square or box brackets [ ] and refer to intervals that do include the endpoints.
- **Braces** or **curly brackets** { } are used when the domain or range consists of discrete numbers and not an interval of values.
- $\in$ is a symbol that means "**is an element of**."
- $\cup$ is a symbol that means **union** and is used to connect two groups. It is associated with the logical term OR.

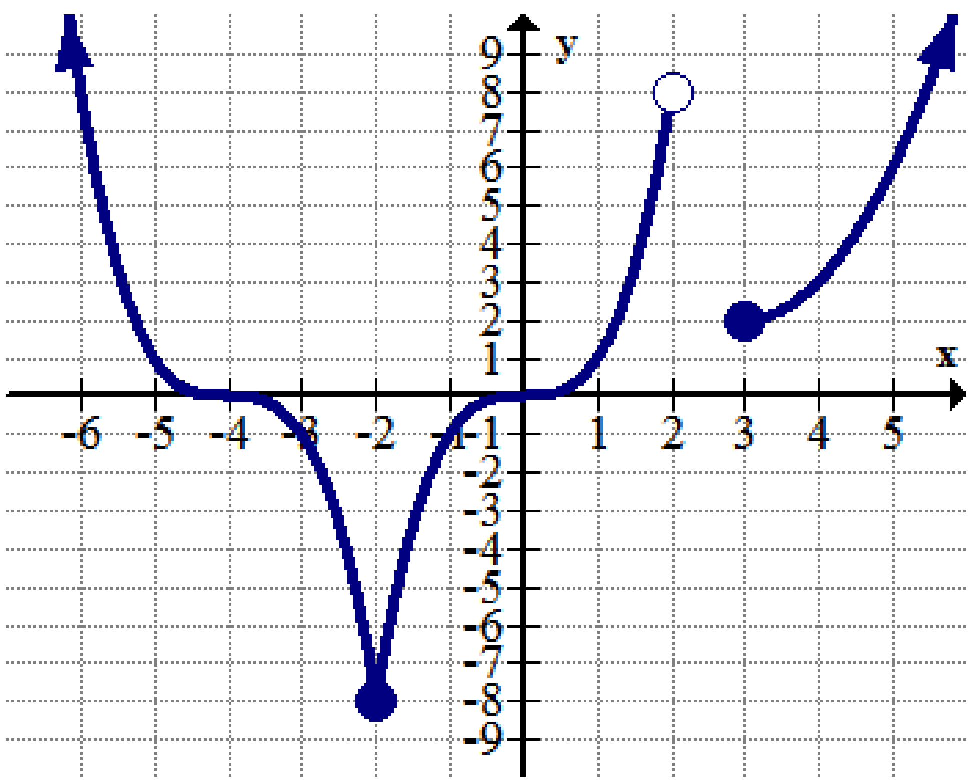
Review

Find the domain and range of each graph:

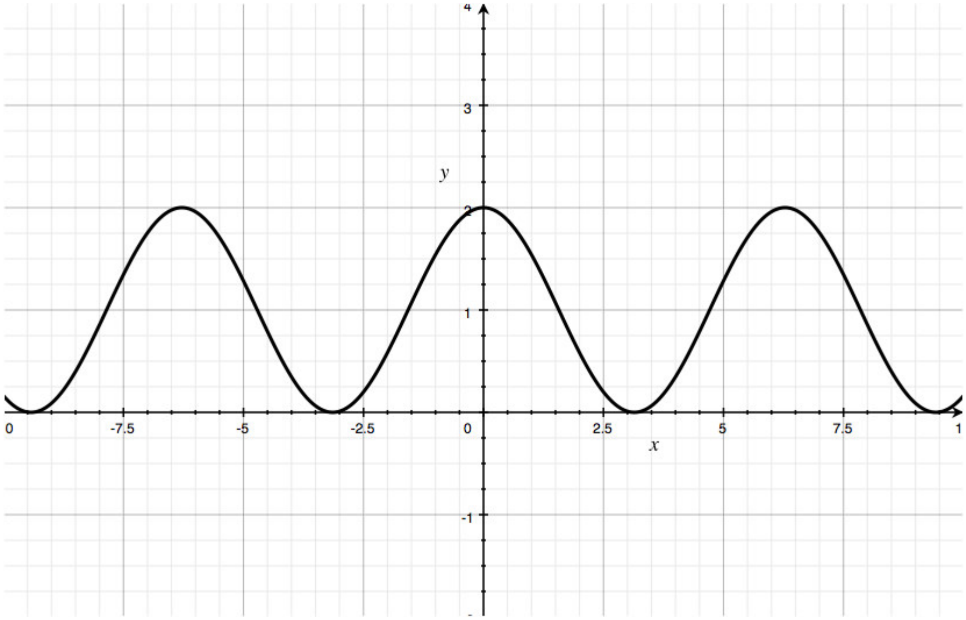
1.



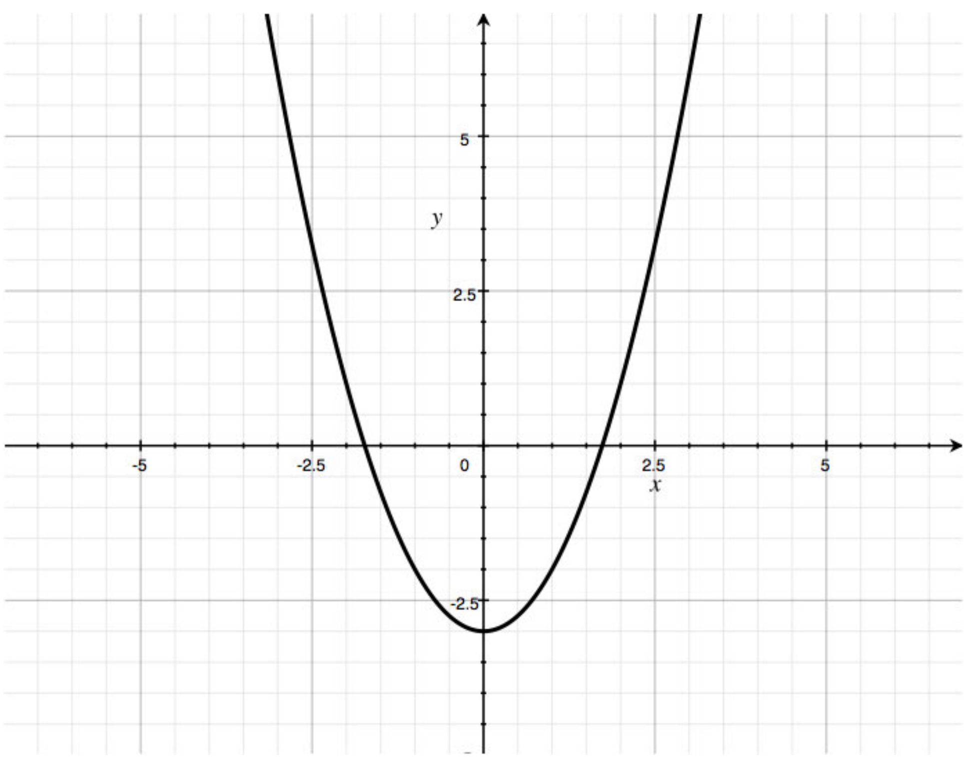
2.



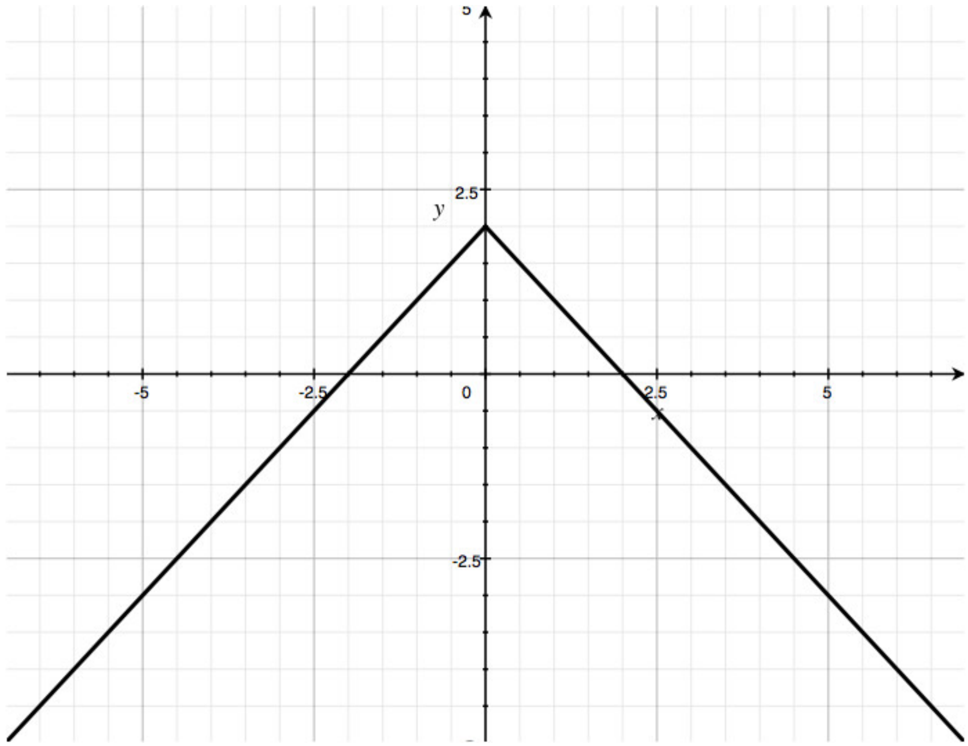
3.



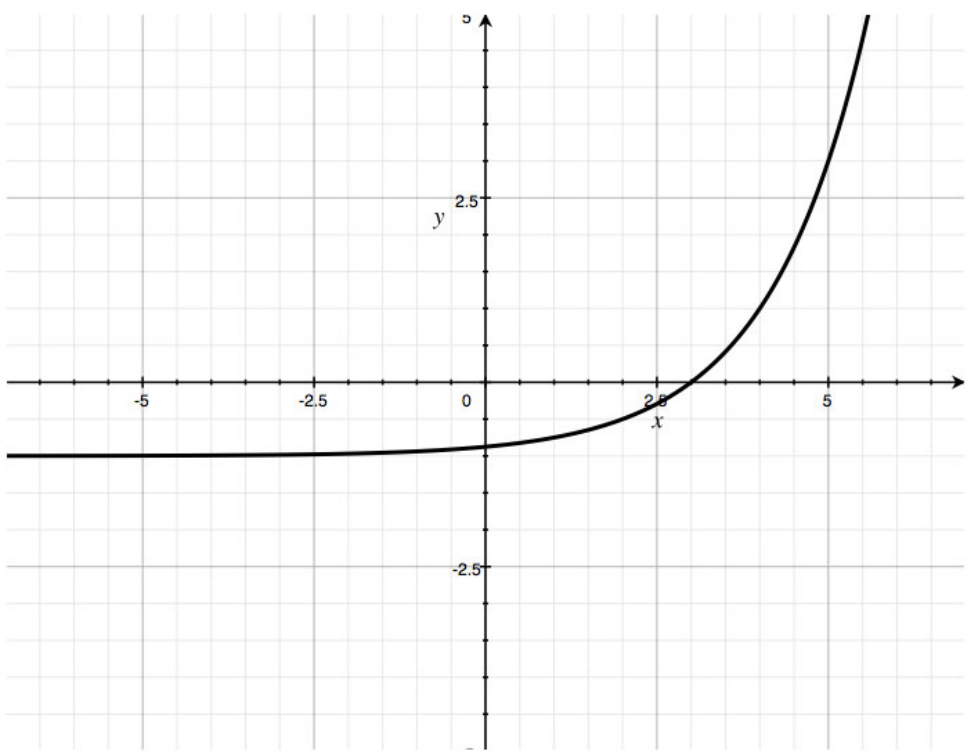
4.



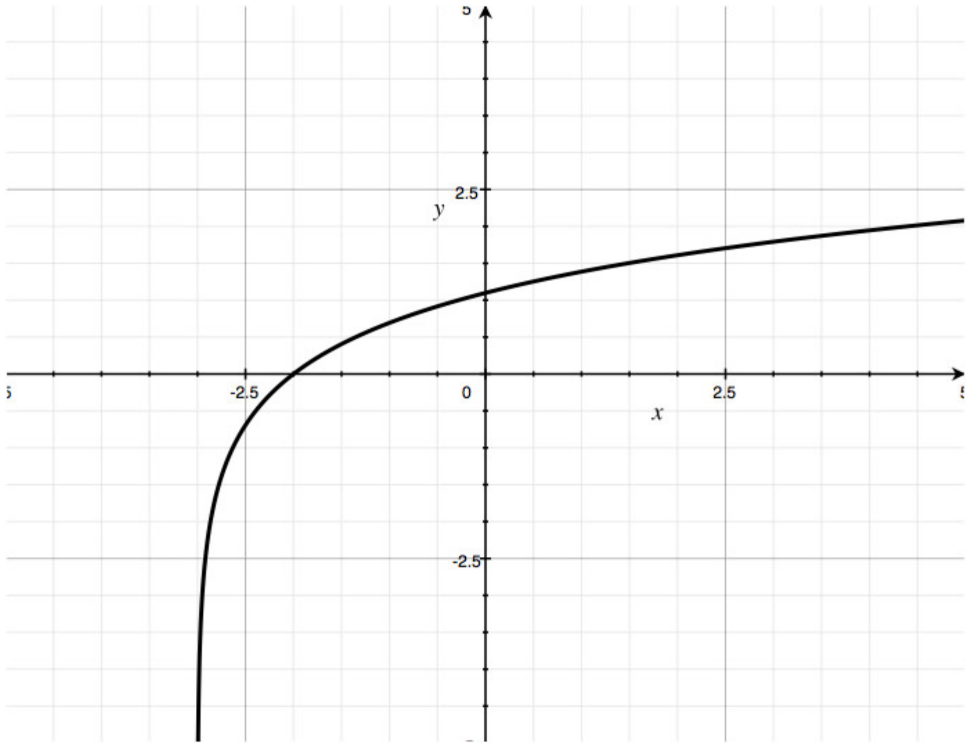
5.



6.



7.



Given the table, find the domain and range:

8.

x	y
-2	7
3	7
2	1
$\frac{3}{4}$	5
$\frac{\pi}{2}$	π

Find the domain and range for the following functions:

9. $y=-(6x+1)^2+4$

10. $y=\log(8x-4)$

11. $y=\frac{5}{x^3-1}$

12. $y=-3\sqrt{x+4}-1$

13. $y=\frac{7}{x+6}-1$

14. $y=5\log(x^2-1)+4$

15. $y=\sqrt{4x^2-9}+6$

16. Bob had a summer job that paid $10 per hour, and he worked 20-25 hours every week. His weekly salary can be modeled by the equation $S=10h$, where S is his weekly salary and h is the number of hours he worked per week. What is the independent variable for this problem? Describe the domain and range.

17. Tina's car travels about 30 miles on one gallon of gas. She has between 10 and 12 gallons of gas in the tank. Find the domain and range of the function to calculate how far she can drive.

18. Joe and his three friends plan to go bowling and to bowl one or two games each. Each game costs \$2.75. Find the domain and range of the function, calculating the cost of the trip.

Review (Answers)

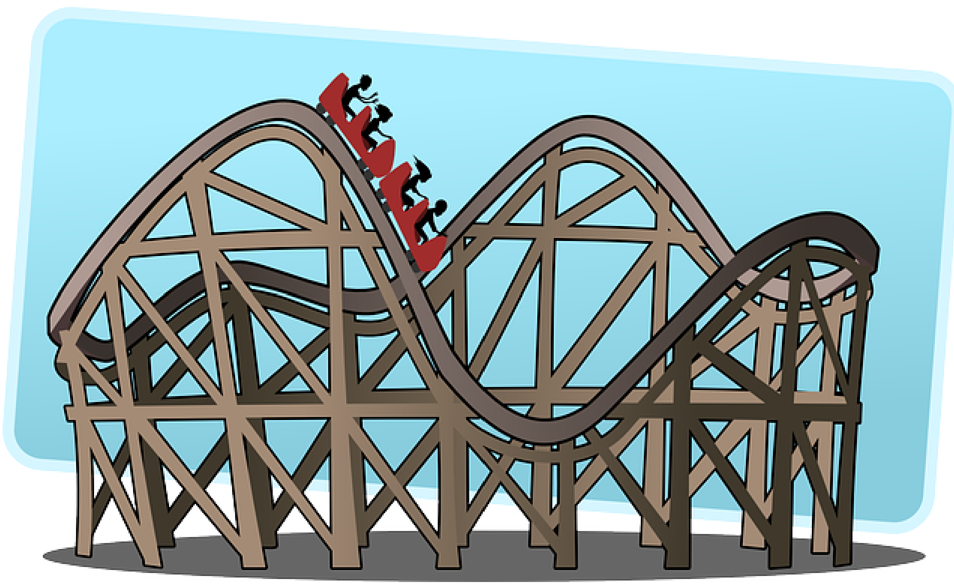
Please see the Appendix.

2.3 Maximums and Minimums

Learning Objectives

Learn about the local or relative extrema and global or absolute extrema of a function.

Introduction



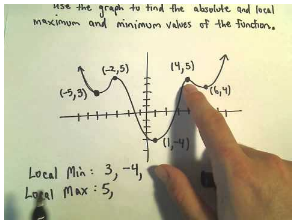
When riding a roller coaster, you will always reach one point that is the absolute highest off the ground. There are usually many other places that reach fairly high—just not as high as that one. How do you identify and distinguish between these different peaks in a precise way?

Local and Absolute Maximum and Minimum

Local extrema and **relative extrema** are synonyms that refer to the points with the y-values that are the highest or lowest y-values on a local neighborhood of the domain of a function.

Global extrema and **absolute extrema** are synonyms that refer to the points with the y-values that are either the highest or the lowest y-values on the entire domain of the function.

One way to identify local extrema is by looking at a graph. The local minimum values are all the y-values on the graph that appear as a valley. The local maximum values are all the y-values on the graph that appear as the top of a hill. The absolute minimum is the smallest y-value on the graph. The absolute maximum is the largest y-value on the graph. These ideas are further illustrated in the following video:



MEDIA

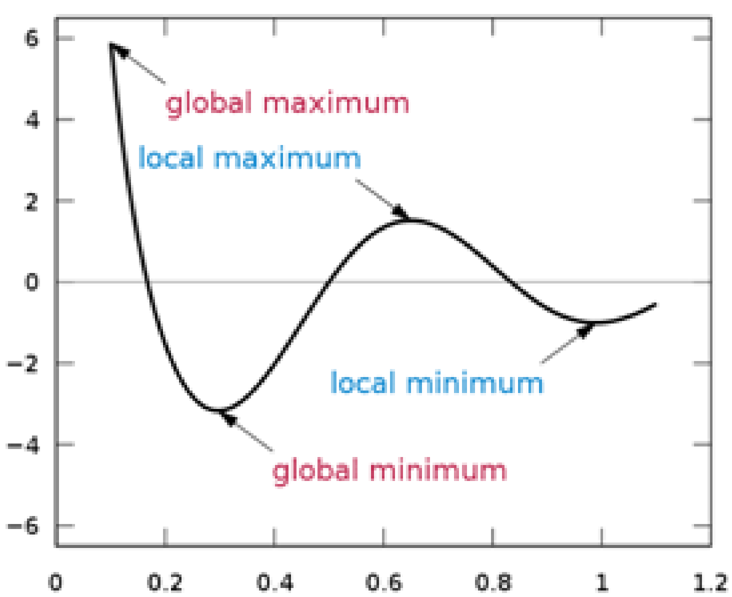
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/57938

Global or Absolute Extrema and Local or Relative Extrema

A global maximum refers to the point with the largest y-value on the graph of a function when a largest y-value exists. A global minimum refers to the point with the smallest y-value. Together these two values are referred to as global extrema. Global refers to the entire domain of the function. Global extrema are also called absolute extrema. There can be only one global maximum value and only one global minimum value. For example, $f(x) = \sin x$ has infinitely many global maximums and global minimums, but only one global maximum value and only one global minimum value.

In addition to global maximums and global minimums, there are also local extrema or relative maximums and relative minimums. The word "relative" is used because, in relation to some neighborhood, these values stand out as being the highest or the lowest. The graph of the function below is defined for a closed interval and illustrates the global maximum, global minimum, local maximum, and local minimum values.

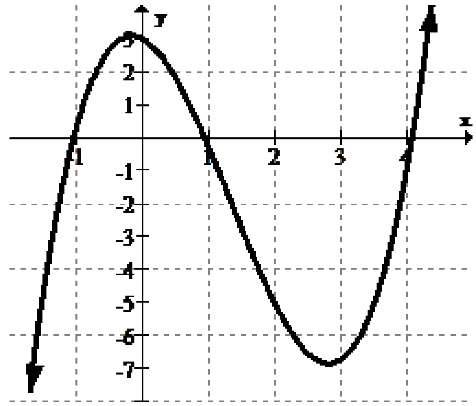


Locating these values is one goal of calculus. In this course, extreme values will be determined by inspection or using technology.

Examples

Example 1

Identify and categorize all extrema:

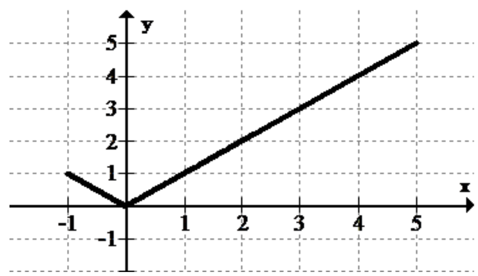


Solution:

Since the function increases and decreases without bound, there are no global extrema. There is a local maximum at approximately $(-0.25, 3)$ (0, 3) and a local minimum at approximately (2.8, -7).

Example 2

Identify and categorize all extrema for the given function defined on the closed interval:



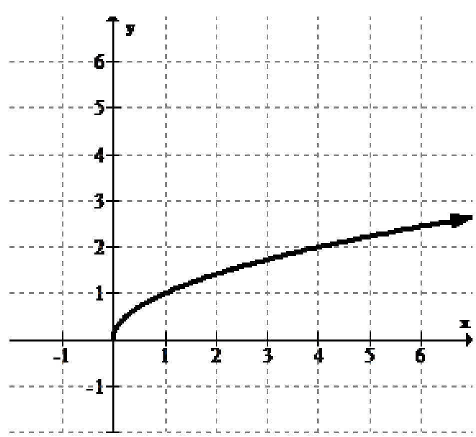
Solution:

The endpoints of the graph of the function are important. These are included in the set of possible extreme values.

There is a global minimum and a local minimum at (0, 0). There is a local maximum and the global maximum are both located at the endpoints, so the local maximum is at (-1, 1) and the global maximum is at (5, 5).

Example 3

Identify and categorize all extrema:

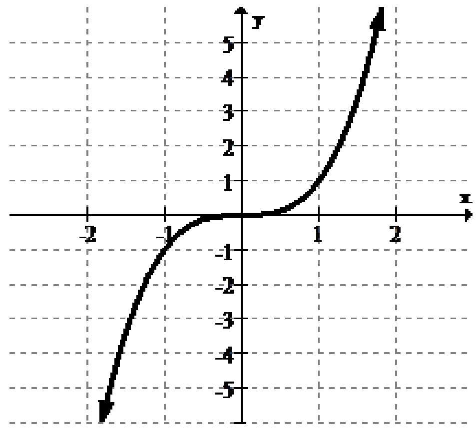


Solution:

Since this function increases to ∞ as x increases, there is no global maximum and no local maximum. At the only endpoint, there is a global minimum at (0, 0).

Example 4

Identify and categorize all extrema:

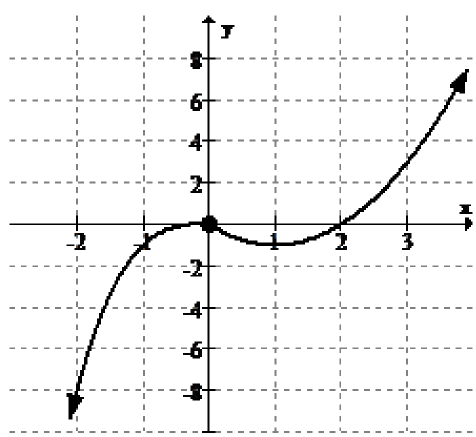


Solution:

There are no global or local maximums or minimums. The function flattens near $x = 0$, but does not have a minimum or maximum value.

Example 5

Identify and categorize the extrema:

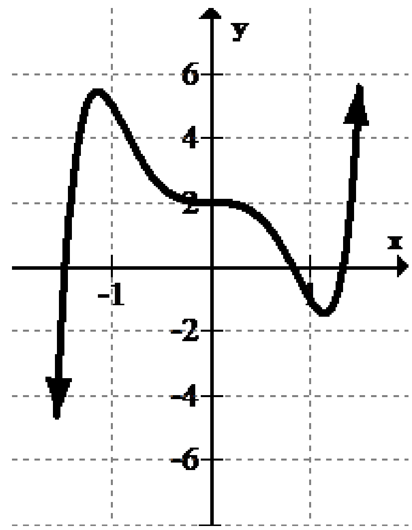


Solution:

There are no global extrema. There appears to be a local maximum at (0, 0) and a local minimum at (1, -1)

Example 6

Identify and categorize the extrema:



Solution:

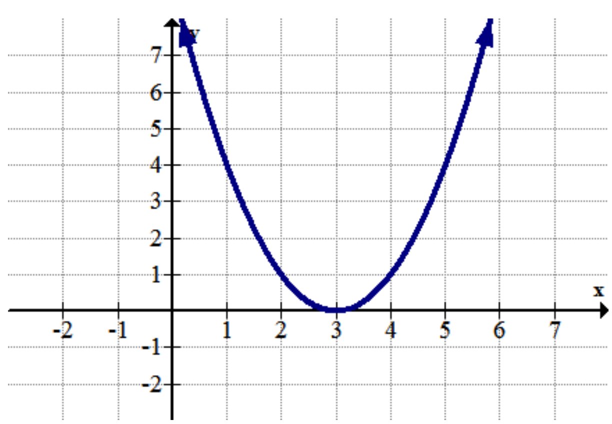
There are no global extrema. There appears to be a local maximum at (-1.2, 5.3) and a local minimum at (1.2, -1.8). These values are approximated.

Summary

- Maximums and minimums identify the highest points and the lowest points, or the peaks and the valleys, in a graph.
- The **absolute extreme values** are the highest or lowest function values across the entire domain of the function.
- The **relative extreme values** are the highest or lowest function values within an interval.

Review

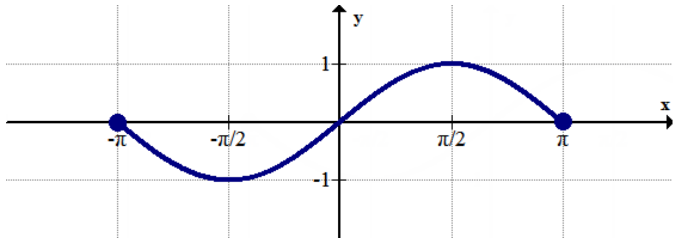
Use the graph below for 1-2:



1. Identify any global extrema.

2. Identify any local extrema.

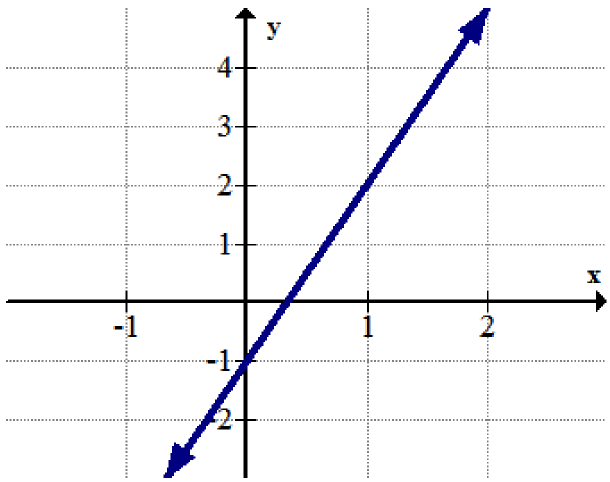
Use the graph below for 3-4:



3. Identify any global extrema.

4. Identify any local extrema.

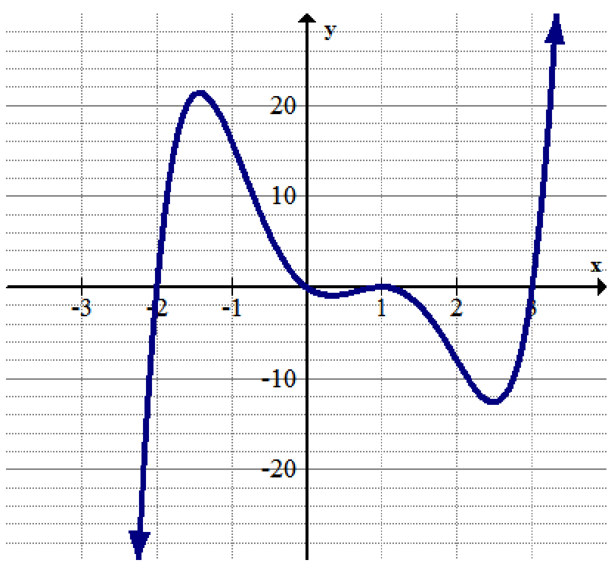
Use the graph below for 5-6:



5. Identify any global extrema.

6. Identify any local extrema.

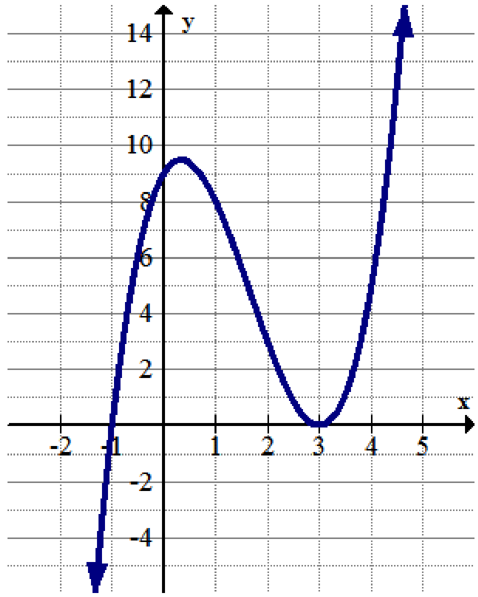
Use the graph below for 7-8:



7. Identify any global extrema.

8. Identify any local extrema.

Use the graph below for 9-10:



9. Identify any global extrema.

10. Identify any local extrema.

11. Explain the difference between a global maximum and a local maximum.

12. Draw an example of a graph with a global minimum and a local maximum, but no global maximum.

13. Draw an example of a graph with local maximums and minimums, but no global extrema.

14. Use a graphing device to identify and categorize the extrema of

$f(x) = \frac{1}{2}x^4 + 2x^3 - 6.5x^2 - 20x + 24.$

15. Use a graphing device to identify and categorize the extrema of

$g(x) = -x^4 + 2x^3 + 4x^2 - 2x - 3.$

16. A rectangle has area 20 in2. Write an expression for the perimeter of the rectangle as a function of its width, x. What dimensions of the rectangle will minimize its perimeter? What is the minimum perimeter?

17. A rectangle has perimeter P. Write a function for the area of the rectangle as a function of P and x, the width of the rectangle. What do you think will be the rectangle with maximum area?

18. A rectangular lot beside a river is fenced on the other 3 sides with 80 feet of fencing. What is the largest possible size of the lot?

Review (Answers)

Please see the Appendix.

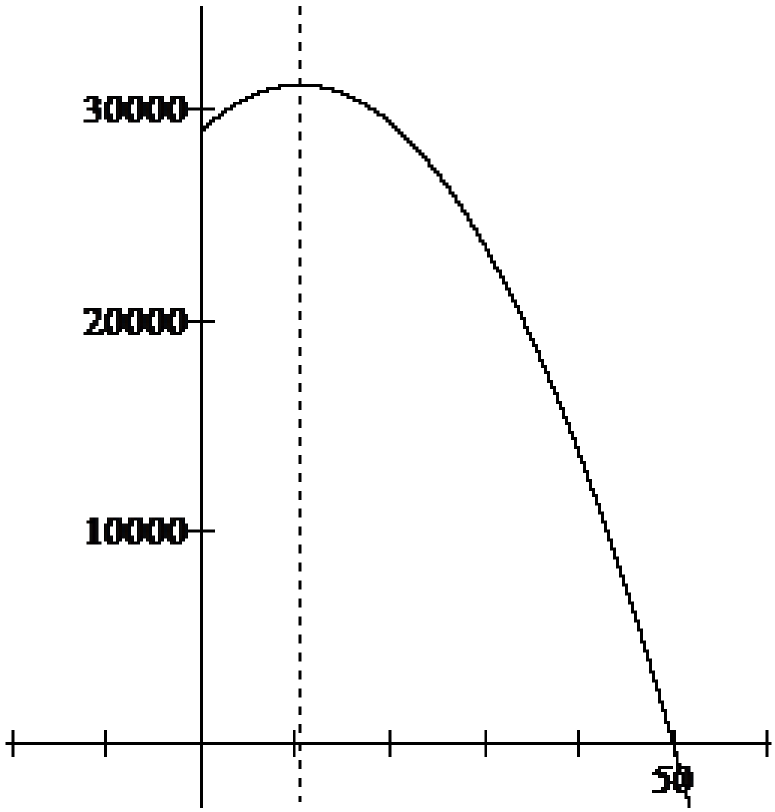
2.4 Symmetry

Learning Objectives

Learn about odd and even functions and the symmetry seen in the graphs of these types of functions.

Introduction

Function symmetry is useful when analyzing functions because it provides powerful information about a mathematical model's behavior. Consider this example of an apartment building with 50 rental units, each unit initially costing \$580 a month to rent. The apartment manager knows that every time the rent is increased by \$20, one tenant moves out. The function $R(x) = (580+20x)(50-x)$ can be used to calculate the amount of rent that is collected based on the number of \$20 rent increases. The domain of this function is $0 \le x \le 50$, because there are 50 apartments in the building. The graph of $R(x)$ has an absolute maximum value of about \$31,000 rent near 10 rent increases. After that point, the total amount of rent collected decreases as each tenant's rent increases. The manager can use the symmetry about the line near $x = 10$ to inform her decisions when making increases near there.



Even and odd function analysis formalizes the algebraic definitions of symmetry. Even functions exhibit symmetry with respect to the y-axis or reflection symmetry. Odd functions have symmetry with respect to the origin or rotational symmetry about the origin. "With respect to the origin" means the graph of the function is reflected across both the $x-$ and y-axes.

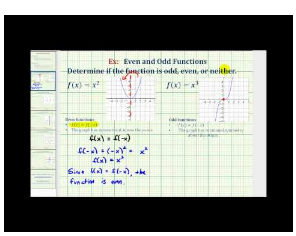
Even and Odd Functions

Functions that display symmetry with respect to $x = 0$ (the y-axis) are called **even functions**.

$f(x)$ is even only if $f(-x) = f(x)$ for every x in the domain

Functions that have rotational symmetry about the origin are called **odd functions**.

$f(x)$ is odd only if $f(-x) = -f(x)$ for every x in the domain.



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URL: http://www.ck12.org/flx/render/embeddedobject/189488

Play, Learn, and Explore rotational symmetry with odd functions: www.ck12.org/a/1823947

Examples

Example 1

Show that $f(x) = 3x^4 - 5x^2 + 1$ is even.

Solution:

Step 1: Replace x with $-x$:

$$\begin{aligned} f(-x) &= 3(-x)^4 - 5(-x)^2 + 1 \\ &= 3x^4 - 5x^2 + 1 \\ &= f(x) \end{aligned}$$

Step 2: Compare with the original function. Since $f(-x) = f(x)$, $f(x)$ is an even function.

Example 2

Show that $f(x) = 4x^3 - x$ is odd.

Solution:

Step 1: Replace x with $-x$:

$$\begin{aligned} f(-x) &= 4(-x)^3 - (-x) \\ &= -4x^3 + x \\ &= -(4x^3 - x) \\ &= -f(x) \end{aligned}$$

Step 2: Compare with the original function. Since $f(-x) = -f(x)$, $f(x)$ is an odd function.

Example 3

Identify whether the function is even, odd or neither and explain why:

$f(x) = 4x^3 - |x|$

Solution

Step 1: Replace x with $-x$:

$$\begin{aligned} f(-x) &= 4(-x)^3 - |-x| \\ &= -4x^3 - |x| \end{aligned}$$

Step 2: Compare with the original function. Since $f(-x) \neq f(x)$ the function is not even. Since $f(-x) \neq -f(x) = -4x^3 + |x|$, the function is not odd. Therefore, this function is neither even nor odd.

Example 4

Which of the following basic functions are even, which are odd, and which are neither?

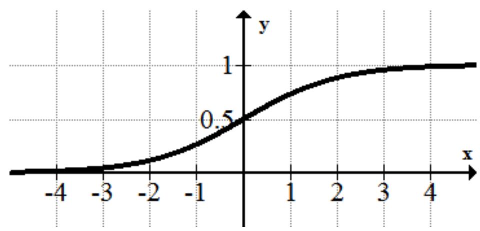
$f(x) = x,\ f(x) = x^2,\ f(x) = x^3,\ f(x) = |x|,\ f(x) = \frac{1}{x},\ f(x) = \sqrt{x} f(x) = \sin x,\ f(x) = \cos x,\ f(x) = ab^x$, and $f(x) = log_b x$

Solution

Even Functions: The quadratic function $f(x) = x^2$, the absolute value function $f(x) = |\ x\ |$, and the cosine function $f(x) = \cos x$ are even.

Odd Functions: The identity or linear function $f(x) = x$, the cubic function $f(x) = x^3$, the reciprocal function $f(x) = \frac{1}{x}$, and the sine function $f(x) = \sin x$ are odd.

Neither: The square root function $f(x) = \sqrt{x}$, the exponential function $f(x) = ab^x$, and the log function $f(x) = \log_b x$ are neither. As a note, the logistic function is also neither because it is rotationally symmetric about the point $\left(0, \frac{1}{2}\right)$ as opposed to the origin.



Example 5

The mechanics of combining functions is covered in another section, but let's begin by thinking about how operations might affect the symmetry of the resulting function.

Suppose $h(x) \neq 0$ is an even function and $g(x) \neq 0$ is an odd function. Let $f(x) = h(x) + g(x)$. Is $f(x)$ even or odd?

Solution:

If $h(x)$ is even, then $h(-x) = h(x)$. If $g(x)$ is odd, then $g(-x) = -g(x)$. Therefore, $f(-x) = h(-x) + g(-x) = h(x) - g(x)$.

This does not match $f(x) = h(x) + g(x)$, nor does it match $-f(x) = -h(x) - g(x)$. So the sum of an even function and an odd function is neither even nor odd.

Example 6

Determine whether the following function is even, odd, or neither:

$f(x) = x(x^2 - 1)(x^4 + 1)$

Solution:

Step 1: Replace x with $-x$:

$$\begin{aligned} f(x) &= x(x^2 - 1)(x^4 + 1) \\ f(-x) &= (-x)((-x)^2 - 1)((-x)^4 + 1) \\ &= -x(x^2 - 1)(x^4 + 1) \\ &= -f(x) \end{aligned}$$

Step 2: Compare with the original function. The function is odd.

Summary

- A function is **even** if $f(-x) = f(x)$ for every x in the domain. Even functions have reflection symmetry about the vertical line $x = 0$ (the y-axis).
- A function is **odd** if $f(-x) = -f(x)$ for every x in the domain. Odd functions have rotation symmetry about the origin, which means if you rotate the graph 180 degrees around the point (0, 0), you end up with the same graph.
- Even and odd functions describe different types of symmetry, but both derive their name from the properties of exponents. A negative number raised to an even number will always be positive. A negative number raised to an odd number will always be negative.

Review

Determine whether the following functions are even, odd, or neither:

1. $f(x) = -4x^2 + 1$
2. $g(x) = 5x^3 - 3x$
3. $h(x) = 2x^2 - x$
4. $j(x) = (x-4)(x-3)^3$
5. $k(x) = x(x^2-1)^2$
6. $f(x) = 2x^3 - 5x^2 - 2x + 1$
7. $g(x) = 2x^2 - 4x + 2$
8. $h(x) = -5x^4 + x^2 + 2$
9. Suppose $h(x)$ is even and $g(x)$ is odd. Show that $f(x) = h(x) - g(x)$ is neither even nor odd.
10. Suppose $h(x)$ is even and $g(x)$ is odd. Show that $f(x) = \frac{h(x)}{g(x)}$ is odd.
11. Suppose $h(x)$ is even and $g(x)$ is odd. Show that $f(x) = h(x) \cdot g(x)$ is odd.
12. Is the sum of two even functions always an even function? Explain.
13. Is the sum of two odd functions always an odd function? Explain.
14. Why are some functions neither even nor odd?
15. If you know that a function is even or odd, what does that tell you about the symmetry of the function?

Review (Answers)

Please see the Appendix.

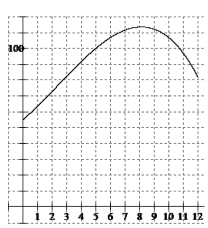
2.5 Increasing and Decreasing

Learning Objectives

Learn about increasing and decreasing intervals of functions.

Introduction

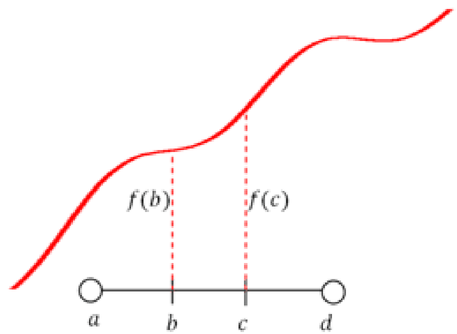
Analyzing a function is a careful process that begins with classification by family, determining the domain, range, symmetry, and extrema. Determining when functions increase and decrease is the next step in this process. Consider this graph of the average temperature per month during 2015 in Las Vegas:



Moving from left to right along the graph, the average monthly temperatures increase from the beginning of the year until the 8th month. Then the temperature decreases until the end of the year. For those who live in the Northern Hemisphere, this behavior makes sense, since August is usually the hottest month in that region of the Earth. Indeed, we all have a graph like this in our minds when we make decisions relating to seasonal events.

Increasing and Decreasing Functions

A function $f(x)$ is **increasing** on an interval (a,d) if $f(b) \leq f(c)$ for any b and c in the interval where b.

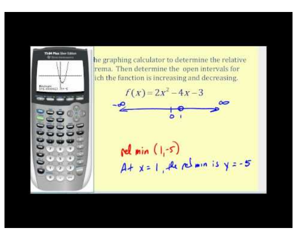


An interval is said to be **strictly increasing** if $f(b) < f(c)$ for any b and c in the interval where b.

The formal definition of **decreasing** and strictly decreasing is identical to the definition of increasing with one of the inequality signs reversed.

A function is **monotonic** on an interval if it is *either* nonincreasing or nondecreasing on the interval. A function that is nonincreasing is either decreasing or constant, and a function that is nondecreasing is either increasing or constant.

Graphing devices can be used to determine if a function is increasing or decreasing by visually analyzing the graph. A function is increasing if the graph moves up when we look at the graph from left to right. A function is decreasing if the graph moves down when we look at the graph from left to right. These ideas are further explained in the following video:



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/57943

Examples

Example 1

Identify which of the parent functions are monotonically increasing on their domains.

Solution:

Of the basic function families, the monotonically increasing functions are:

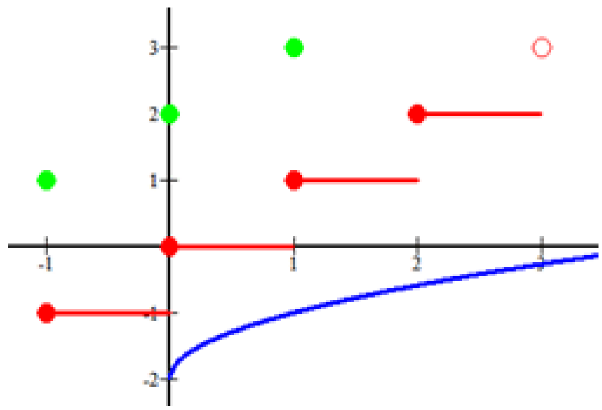
$f(x) = x$, $g(x) = x^3$, $h(x) = \sqrt{x}$, $k(x) = e^x$, $m(x) = \ln x$, and $n(x) = \frac{1}{1+e^{-x}}$.

The only basic functions that are not monotonically increasing are:

$f(x) = x^2$, $g(x) = |x|$, $h(x) = \frac{1}{x}$, and $k(x) = \sin x$.

Example 2

Identify whether the green, red, or blue function is monotonically increasing, and explain why:



Solution:

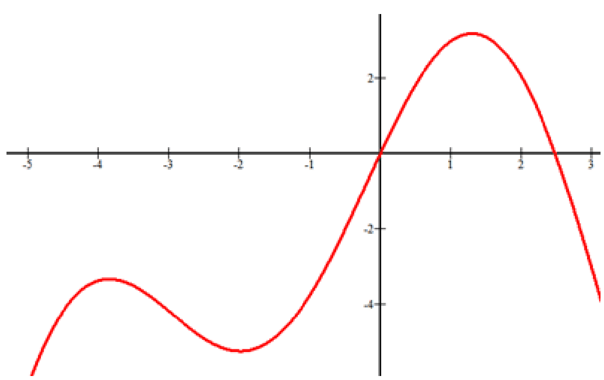
The green function is defined on a discrete domain along the line $y = x + 2$. While the discrete values clearly increase, the definition requires that any two points can be chosen in an interval. This is impossible, so the green function is not monotonically increasing.

The red function is nondecreasing on the interval $(-1, 3)$, so it is monotonic.

The blue function seems to be $y = \sqrt{x} - 2$ is increasing everywhere for the region displayed on the graph. This function is monotonic for $x \in [0, \infty)$.

Example 3

Estimate the intervals where the function is increasing and decreasing:



Solution:

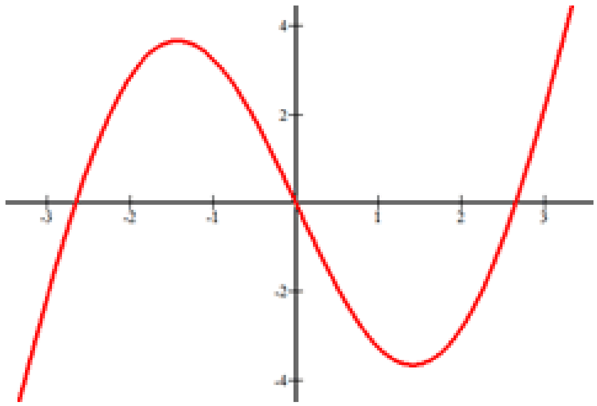
Increasing: $x \in (-\infty, -4) \cup (-2, 1.5)$

Decreasing: $x \in (-4, -2) \cup (1.5, \infty)$

Note that open intervals are used because at $x = -4, -2, 1.5$, the function is neither increasing nor decreasing. This is the point where the function changes its monotonicity from increasing to decreasing.

Example 4

Estimate the intervals where the following function is increasing and decreasing:

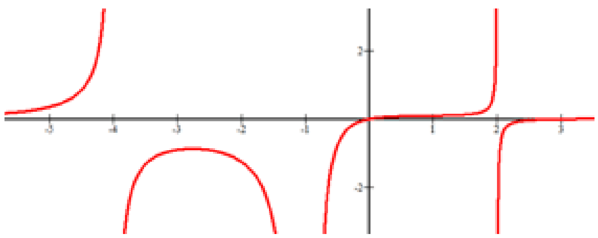


Solution:

Increasing: $x \in (-\infty, -1.5) \cup (1.5, \infty)$. Decreasing: $x \in (-1.5, 1.5)$

Example 5

Estimate where the following function is increasing and decreasing:



Solution:

Increasing: $x \in (-\infty, -4) \cup (-4, -2.7) \cup (-1, 2) \cup (2, \infty)$. Decreasing: $x \in (-2.7, -1)$

Example 6

A function defined for all real numbers has a global maximum at the point (3, 2), a global minimum at the point (5, -12), and has no relative extrema. What are the intervals where the function is increasing and decreasing?

Solution:

Increasing: $x \in (-\infty, 3) \cup (5, \infty)$. Decreasing: $x \in (3, 5)$

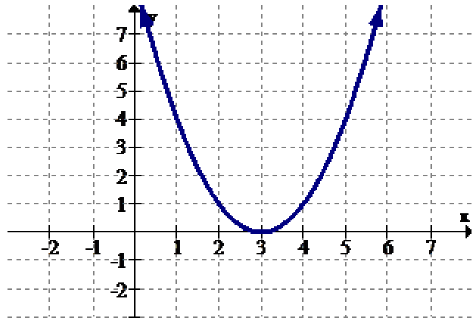
Notice that the intervals define the portion of the domain where this behavior occurs.

Summary

- A function $f(x)$ is **increasing** if $f(b) \leq f(c)$ for any b and c in the interval where b.
- A function $f(x)$ is **decreasing** if $f(b) \geq f(c)$ for any b and c in the interval where b.
- A function is **strictly increasing** or **strictly decreasing** when the function inequalities are strictly less than.
- A function is **monotonic** on an interval if it is either nonincreasing or nondecreasing on the interval.

Review

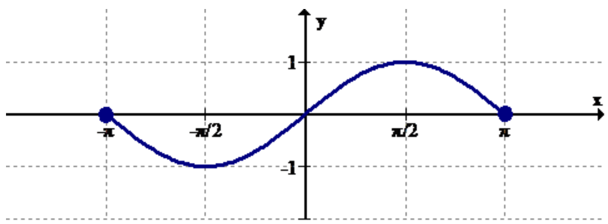
Use the graph below for 1-2:



1. Identify the intervals (if any) where the function is increasing.

2. Identify the intervals (if any) where the function is decreasing.

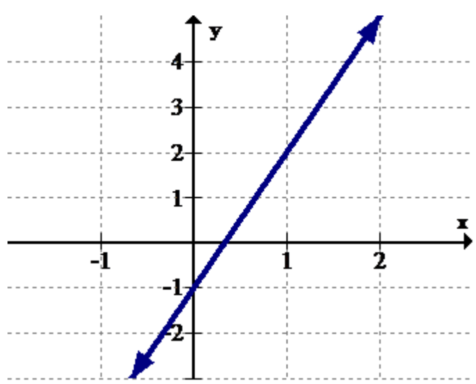
Use the graph below for 3-4:



3. Identify the intervals (if any) where the function is increasing.

4. Identify the intervals (if any) where the function is decreasing.

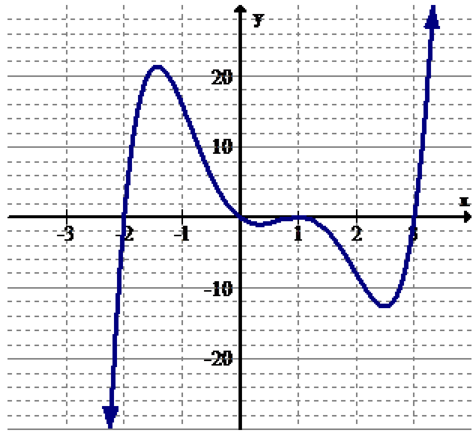
Use the graph below for 5-6:



5. Identify the intervals (if any) where the function is increasing.

6. Identify the intervals (if any) where the function is decreasing.

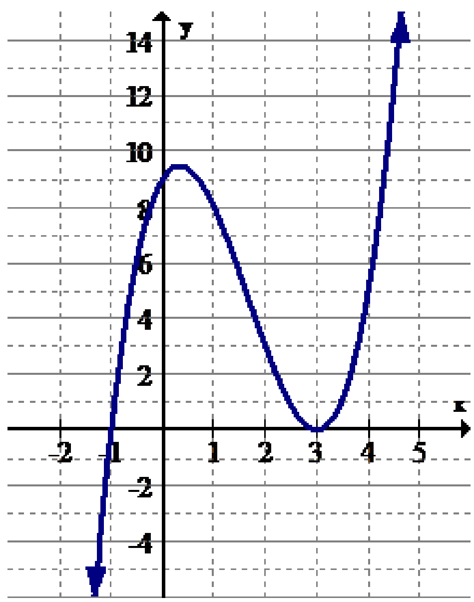
Use the graph below for 7-8:



7. Identify the intervals (if any) where the function is increasing.

8. Identify the intervals (if any) where the function is decreasing.

Use the graph below for 9-10:



9. Identify the intervals (if any) where the function is increasing.

10. Identify the intervals (if any) where the function is decreasing.

11. Give an example of a monotonically increasing function.

12. Give an example of a monotonically decreasing function.

13. A function defined for all real numbers has a global maximum at the point (1, 4) and a global minimum at (3, -6), and has no relative extrema or other places with a slope of 0. What are the increasing and decreasing intervals for this function?

14. A function defined for all real numbers has a global maximum at the point (1, 1) and has no other extrema or places with a slope of 0. What are the increasing and decreasing intervals for this function?

15. A function defined for all real numbers has a global minimum at the point (5, -15) and has no other extrema or places with a slope of 0. What are the increasing and decreasing intervals for this function?

For questions 16-18, use the following information: Sam makes $7.50 per hour at work, and he works anywhere from 15 to 40 hours per week. Occasionally, the store gets very busy, and Sam's boss allows him to work up to 15 hours of overtime. Sam loves the busy weeks because he makes "time-and-a-half," or $11.25, for overtime hours.

16. Sketch a graph showing Sam's **regular** income range. Where are the local extrema?
17. What happens to the local extrema if Sam is working a week with overtime? Where are they now?
18. Sketch another graph including Sam's regular and overtime income, identifying all local extrema.

Review (Answers)

Please see the Appendix.

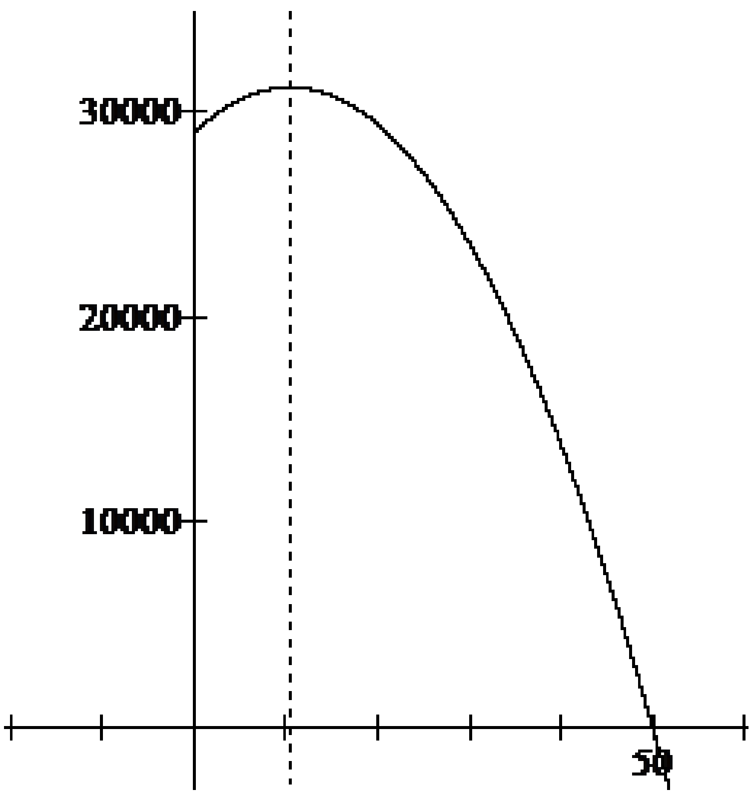
2.6 Intercepts of Graphs of Functions

Learning Objectives

Learn about x- and y-intercepts of a function.

Introduction

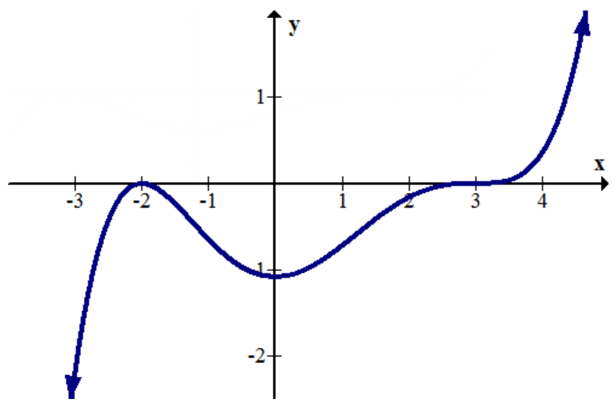
The revenue collected by an apartment manager is given by $R(x) = (580+20x)(50-x)$, where x is the number of \$20 rent increases:



In another section we used this example to discuss symmetry. Now, notice the points $(0, 29000)$ and $(50, 0)$ on the graph. These points convey special information. The 1st represents the revenue from rent of \$29,000 per month before increasing the rent. The 2nd point says that after fifty \$20 increases (or an increase of \$1000), all tenants will move and the rent revenue will be \$0 per month. These special points are called intercepts.

Intercepts

An **intercept** in mathematics is the point where a function crosses the x- or y-axis. For example, identify the intercepts for the following graph:



The x-intercepts are $(-2,0)$ and $(3,0)$. The y-intercept is approximately $(0,-1.1)$.

Zeros, roots, and solutions are synonyms for the x-values where a function crosses or touches the x-axis. In order for a relation to pass the vertical line test and thus be a function, it must have only one y-intercept. However, it may have multiple x-intercepts.

x-intercept

The x-intercept is the point where the function crosses or touches the x-axis. The x-value of these points are also called roots, solutions to an equation, and zeros of a function. They are found algebraically by setting $y=0$ and solving for x.

y-intercept

The y-intercept is the point where the function crosses or touches the y-axis. This point is found by setting x to zero and solving for y.

Play, Learn, Interact, and Explore Intercepts: www.ck12.org/a/1817921

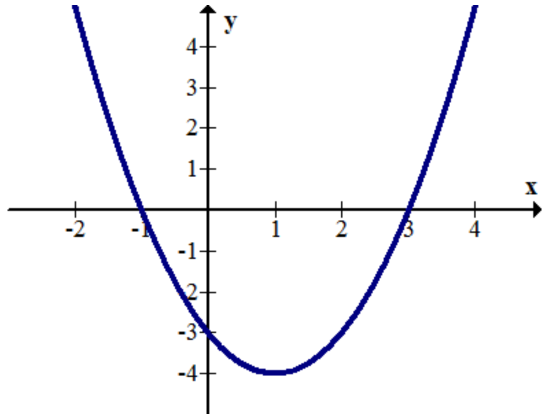
Examples

Example 1

What are the zeros and y-intercept for the parabola $y=x^2-2x-3$?

Solution:

Method 1: Solution using a graph:



The zeros are at $x = -1, 3$. The y-intercept is at (0, -3).

Method 2: Solution using algebra:

When $y = 0$, substitute 0 for y to find zeros.

$0 = x^2 - 2x - 3 = (x - 3)(x + 1)$

$y = 0, x = 3, -1$

The zeros are at (-1, 0) and (3, 0).

When $x = 0$, substitute 0 for x to find the y-intercept.

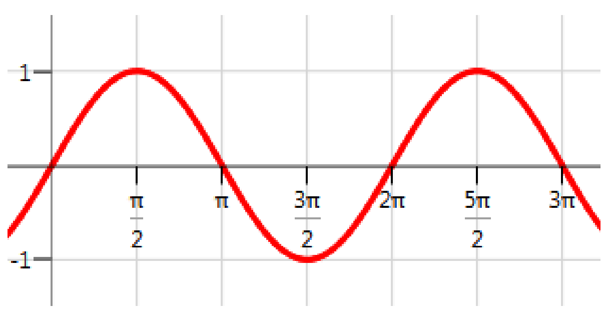
$y = (0)^2 - 2(0) - 3 = -3$

$x = 0, y = -3$

The y-intercept is at (0, -3).

Example 2

Identify the zeros and y-intercepts for the sine function:



Solution:

The y-intercept is (0, 0). There are four zeros visible on this portion of the graph. The sine graph is periodic and repeats in both directions. In order to capture every x-intercept, a pattern for the zeros should be identified.

The visible x-intercepts are 0, π, 2π, and 3π. The pattern is that there is an x-intercept every multiple of π, including negative multiples. These values can be described in this way:

The x-intercepts are $n\pi$, where n is an integer $\{0, \pm 1, \pm 2, \ldots\}$.

Example 3

Identify the intercepts and zeros of the function $f(x) = \frac{1}{100}(x-3)^3(x+2)^2$.

Solution:

To find the y-intercept, substitute 0 for x:

$y = \frac{1}{100}(0-3)^3(0+2)^2 = \frac{1}{100}(-27)(4) = -\frac{108}{100} = -1.08$

To find the x-intercepts, substitute 0 for y:

$0 = \frac{1}{100}(x-3)^3(x+2)^2$

By the zero product property, which states that if the product of a set of factors equals zero then one or more of the factors must equal zero, $x-3=0$ or $x+2=0$.

$x = 3, -2$

Thus, the y-intercept is (0, -1.08), and the x-intercepts are (3, 0) and (-2, 0).

Example 4

Determine the zeros and y-intercept of the following function, using algebra: $f(x) = (x+3)^2(x-2)$.

Solution:

To find the y-intercept, substitute 0 for x:

$y = (0+3)^2(0-2) = 3^2 \cdot (-2) = 9 \cdot (-2) = -18$

The y-intercept is (0, -18).

To find the x-intercepts, substitute 0 for y:

$0 = (x+3)^2(x-2)$

By the zero product property, $x+3=0$ or $x-2=0$.

$x = -3, 2$

The zeros (or x-intercepts) are (-3, 0) and (2, 0).

Example 5

Determine the roots and y-intercept of the following function, using algebra or a graph:

$f(x) = x^4 + 3x^3 - 7x^2 - 15x + 18$

Solution:

To find the y-intercept, substitute 0 for x:

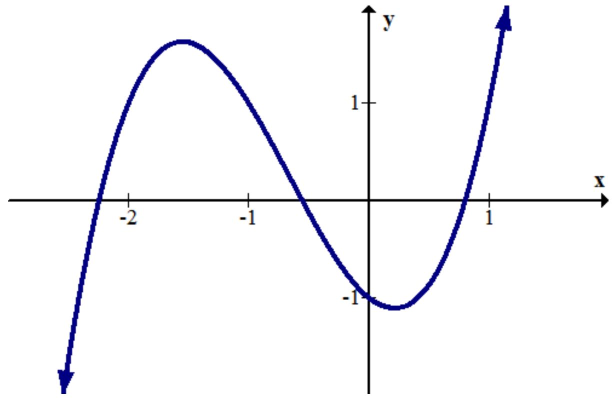
$y = 0^4 + 3 \cdot 0^3 - 7 \cdot 0^2 - 15 \cdot 0 + 18 = 18$

The y-intercept is (0, 18).

By graphing $f(x) = x^4 + 3x^3 - 7x^2 - 15x + 18$,the roots (or x-intercepts) are (2, 0), (1, 0) and (-3, 0).

Example 6

Determine the intercepts of the following function graphically:



Solution:,

The y-intercept is approximately (0, -1). The x-intercepts are approximately (-2.3, 0), (-0.6, 0) and (0.8, 0). When finding values graphically, answers are always approximate. Exact answers need to be found analytically.

Summary

- ***Zeros, roots,*** and ***solutions*** are synonyms for the x-values where a function crosses the x-axis.
- An ***x*-intercept** is the point where the graph of a function crosses or touches the x-axis.
- A ***y-intercept*** is the point where the graph of a function crosses or touches the y-axis.
- Note that in order for a function to pass the vertical line test and thus be a function, it should only have one y-intercept. However, it may have multiple x-intercepts.

Review

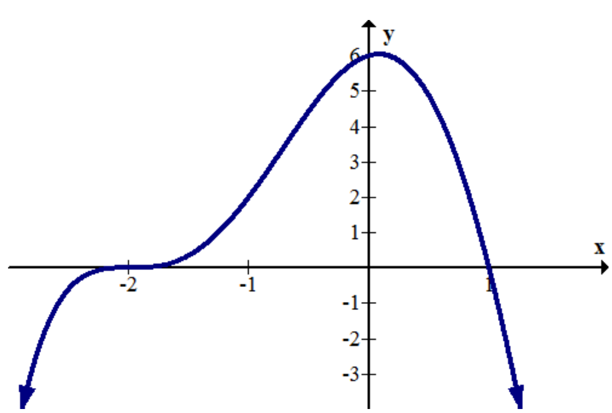
1. Determine the zeros and y-intercept of the following function algebraically:

   $f(x) = (x+1)^3(x-4)$

2. Determine the roots and y-intercept of the following function using algebra or a graph:

   $g(x) = x^4 - 2x^3 - 7x^2 + 20x - 12$

3. Determine the intercepts of the following function graphically:



Find the intercepts for each of the following functions:

4. $y = x^2$

5. $y = x^3$

6. $y = \log x$

7. $y = \frac{1}{x}$

8. $y = 2^x$

9. $y = \sqrt{x}$

10. Are there any functions without a y-intercept? Explain.

11. Are there any functions without an x-intercept? Explain.

12. Explain why it makes sense that an x-intercept of a function is also called a "zero" of the function.

Determine the intercepts of the following functions using algebra or a graph:

13. $h(x) = x^3 - 6x^2 + 3x + 10$

14. $j(x) = x^2 - 6x - 7$

15. $k(x) = 4x^4 - 20x^3 - 3x^2 + 14x + 5$

Review (Answers)

Please see the Appendix.

2.7 Function Families

Learning Objectives

Learn about the parent functions for basic function families.

Introduction

Functions are used in mathematics to illustrate patterns. For example, when a ball is thrown into the air, it rises and then comes back to earth. Its height at any time can be determined by the mathematical function $h(t) = -at^2 + bt + c$. The rule defined by this function provides a way to calculate the height of the ball (in feet) at any time (in seconds) after the ball is thrown.

Suppose the height function of a thrown ball is given by $h(t) = -16t^2 + 80t + 6$. The table of heights $h(t)$ for some particular values of time t is seen below. Notice that the heights repeat, due to the ball traveling up and then falling back to the ground.

Height of a Ball as a Function of Time

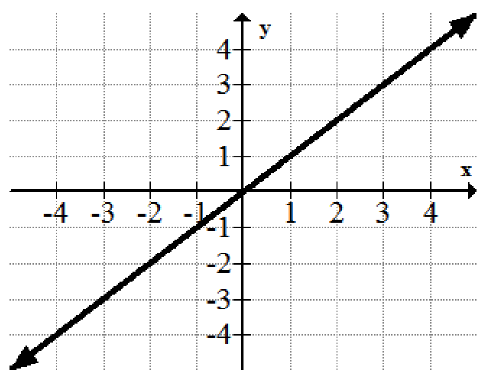
time	height
0	6
2	102
4	70
5	6

In our example, the height is calculated as a function of time, because the height of the ball depends on the amount of time that has passed since it was released. This function is a member of a family of functions called quadratic functions.

Basic Families of Functions

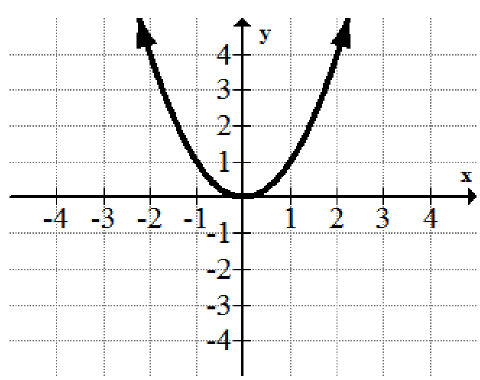
Mathematicians classify algebraic functions into several families, which have special characteristics. Analytical techniques use these characteristics to describe and explain situations in our world. Below are the graphs of the parent functions for some typical families of functions. The given characteristics of each of these parent functions are helpful to consider when thinking about transformations:

The Identity Function: $f(x) = x$

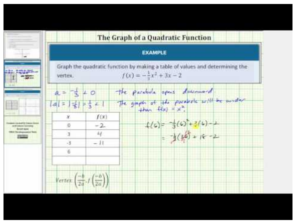


The identity function is a simple function, and yet all straight lines can be derived from it. It is called the identity function because each output value is the same as the value that was used for the input. In this sense, the identity function preserves the identity of an input.

The Quadratic Function: $f(x) = x^2$



The quadratic function's graph is called a parabola. This particular quadratic is symmetric in respect to the y-axis and has a minimum y-value. In general, quadratic functions are symmetric in respect to a vertical line. This parent function is symmetric in respect to the particular vertical line, the y-axis. Also, quadratic functions have either a minimum or maximum value. The vertex of a quadratic function occurs at the minimum or maximum value. These ideas are also discussed in the following video:

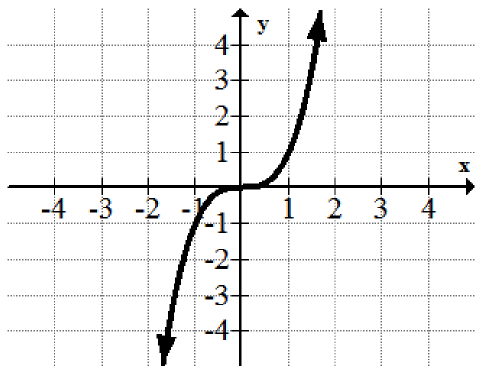


MEDIA

Click image to the left or use the URL below.

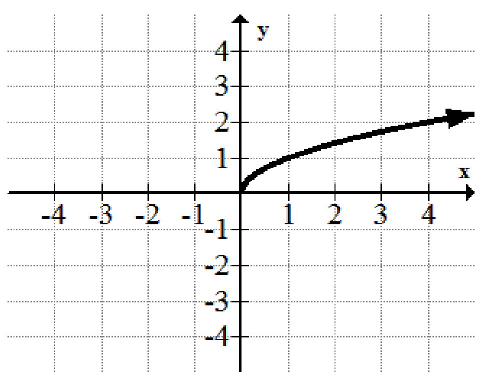
URL: http://www.ck12.org/flx/render/embeddedobject/189474

The Cubic Function: $f(x) = x^3$



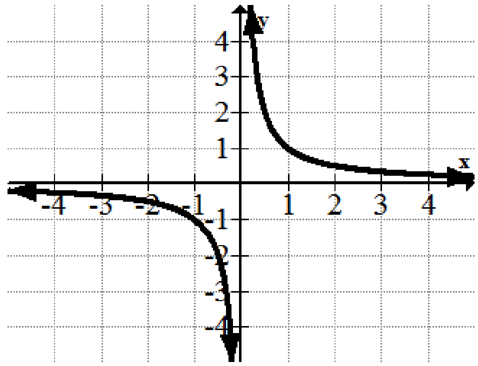
The cubic function exhibits a different kind of symmetry than the quadratic function. In this parent function, the point of symmetry occurs at (0, 0) and the point (1,1) has a mirror image in the origin as the point (-1,-1). This is true for every point on the graph. In general, cubic functions will have a point of symmetry where the two other parts of the graph are mirror images of each other.

The Square Root Function: $f(x) = \sqrt{x} = x^{\frac{1}{2}}$



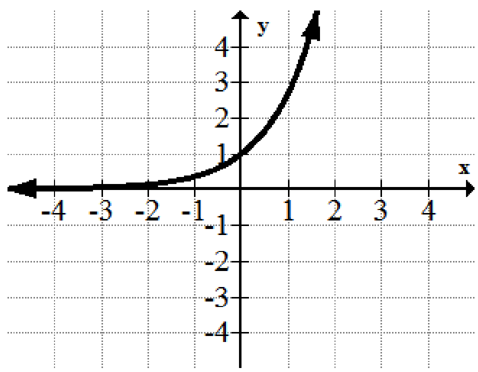
The square root function is half of a side-opening parabola. The parent function has a vertex at (0, 0) and is only defined for $x \geq 0$ in the real number system. In general, a square root function with a vertex point at $\left(a, \sqrt{a}\right)$ is only defined for $x \geq a$ in the real number system.

The Reciprocal Function: $f(x) = \frac{1}{x} = x^{-1}$



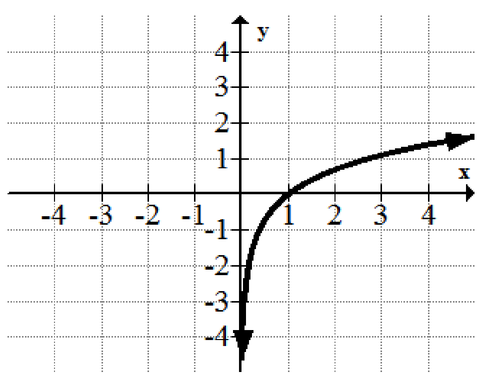
The graph of the reciprocal function is a hyperbola. This parent function is the basis for some rational functions. Notice the way the graph gets close to the x-axis without touching it as the absolute value of x gets larger and as x gets smaller. This is called asymptotic behavior as $x$approaches $\pm\infty$.

The Exponential Function: $f(x) = b^x,\ b > 0,\ b \neq 1$



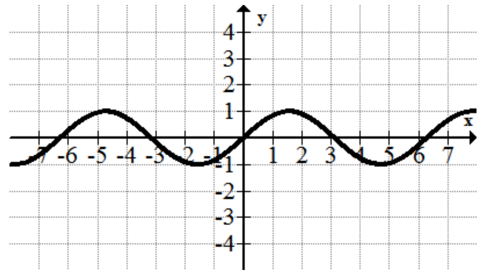
The exponential function is used to model growth. In this graph, $b = 3$, so $f(x) = 3^x$. Notice the way the graph gets close to the x-axis without touching it, so the exponential function also exhibits asymptotic behavior as x gets smaller. As x gets larger, this function grows very quickly.

The Logarithm Function: $f(x) = \log_b x,\ b > 0,\ b \neq 1$



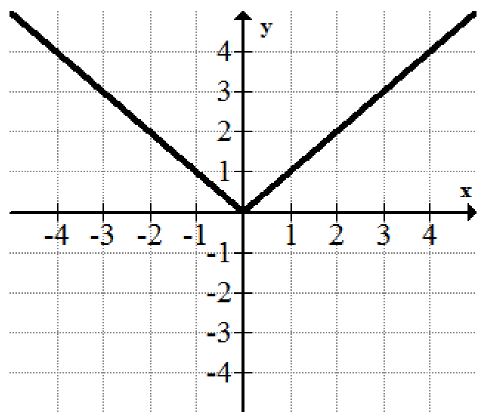
The logarithmic function is the inverse to the exponential function. In this graph, $b = 3$, so $f(x) = \log_3 x$. Notice the way the graph gets close to the y-axis without touching it, so the exponential function also exhibits asymptotic behavior as x approaches 0 from the right.

The Sine Function: $f(x) = \sin x$



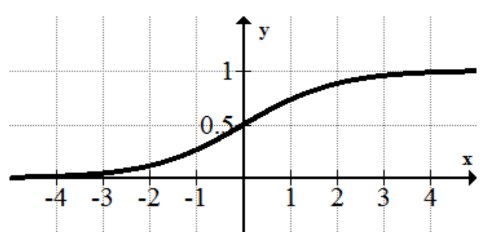
The sine function is one of six basic periodic functions. In this graph, the sine function is graphed using radian mode. Periodic refers to the fact that the sine exhibits cycles that repeat at regular intervals. The maximum y-value for this parent function is 1 and the minimum y-value is -1.

The Absolute Value Function: $f(x) = |x|$



The absolute value function is similar to the quadratic function, except the lines are straight rather than curved and meet at a point. The vertex of the absolute value is this point. Similar to the quadratic, this parent function is symmetric in respect to the y-axis and has a minimum y-value. In general, absolute value functions are symmetric in respect to a line and have either a minimum or maximum value.

The Logistic Function: $f(x) = \frac{1}{1+b^{-x}}$



The logistic function is a combination of the exponential function and the reciprocal function. In this graph, $b = e$, so $f(x) = \frac{1}{1+e^{-x}}$. This function models environmental populations, which cannot grow in an unrestricted fashion. At a certain point, a population will outgrow its food source and space, so its growth will diminish. The level of population that can be sustained by an environment is called the carrying capacity. The graph approaches the carrying capacity level as x increases.

Play, Learn, and Explore Function Families: www.ck12.org/a/1904355

Examples

Example 1

Compare and contrast the graphs of the two functions $f(x) = \log_b x$ and $h(x) = \sqrt{x}$.

Solution:

Similarities: Both functions increase without bound as x gets larger. Both functions are not defined for negative numbers.

Differences: The graph of the logarithmic function exhibits asymptotic behavior as x approaches 0. Notice that the y-values approach negative infinity as x approaches 0. The square root function is defined at $x = 0$.

Example 2

Describe the symmetry displayed by the parent functions discussed in this section.

Solution:

The graphs of these functions are symmetric with respect to the y-axis.

$f(x) = x^2, g(x) = |x|$

The graph of $f(x) = x^3$ has a point of symmetry at (0, 0).

Example 3

Which parent function families have a minimum y-value?

Solution:

The families with a minimum y-value are:

$f(x) = x^2, g(x) = |x|, h(x) = \sqrt{x}, k(x) = \sin x$

Example 4

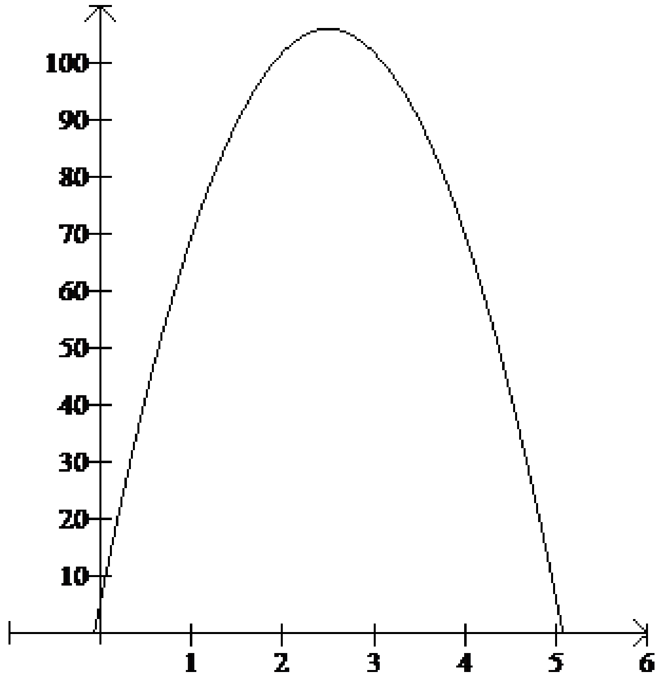
Which parent functions are defined for all values of x ?

Solution:

$y = x,\ y = x^2,\ y = x^3,\ y = |x|,\ y = ab^x, y = \frac{1}{1+ab^{-x}},\ y = \sin x$

Example 5

Below is the graph of the function $h(t) = -16t^2 + 80t + 6$, which models the height of a ball:



Determine the following:

a) the times t that the height is 70 feet

Solution:

The two points on the graph that have a y -value of 70 are (1,70) and (4,70). This means that the ball reaches a height of 70 feet at $t = 1$ and 4 seconds.

b) the maximum height and the time that the ball reaches the maximum height

Solution:

The maximum height occurs at the vertex. The graph is symmetrical about the vertical line through the vertex $x = 2.5$. This means that the maximum height is reached at $t = 2.5$ seconds. The height is 106 feet.

$$h(2.5) = -162.5^2 + 80(2.5) + 6$$
$$h(2.5) = 106$$

Summary

- The basic families of functions are used as a starting point to study techniques used to analyze functions.
- Important features of a graph include maximum or minimum values, symmetry, and asymptotic and periodic behavior.
- A graph is ***symmetrical*** about the y -axis if, when (x, y) is on the graph, then $(-x, y)$ is also on the graph.
- A graph is ***symmetrical*** about the origin if, when (x, y) is on the graph, then $(-x, -y)$ is also on the graph.
- ***Asymptotic behavior*** occurs when a graph approaches a line or axis as x approaches a specific value.
- The **vertex** is the point on a parabola where either the minimum or maximum value occurs.

Review

For 1-10, sketch a graph of the function from memory:

1. $y = b^x$
2. $y = \log_b x$
3. $y = \sin(x)$
4. $y = x^2$
5. $y = |x|$
6. $y = \frac{1}{x}$
7. $y = \frac{1}{1+b^{-x}}$
8. $y = \sqrt{x}$
9. $y = x^3$
10. $y = x$
11. Which parent function is not defined at 0? Why?
12. Name four basic functions that do not have a maximum y-value.
13. What are the differences between $y = x^2$ and $y = x^3$?
14. What symmetry can be seen with $y = b^x$ and $y = \log_b x$?
15. Explain why $y = \sqrt{x}$ is not defined for all values of x.

Review (Answers)

Please see the Appendix.

2.8 Graphical Transformations

Learning Objectives

Learn about several types of transformations that can be applied to basic parent functions.

Introduction

The basic families of functions provide a good starting point to analyze functions, and greater depth can be added to their study by adapting the functions to specific situations. In this section, we'll explore methods of adapting functions, which help us to understand scenarios like this:

Consider the average annual cost of a new refrigerator. Suppose the cost of the refrigerator is spread out over its lifespan. If the price of the refrigerator is $900 and the operating cost is $175 per year, the average annual cost is reduced each year that we keep the refrigerator.

This function assumes that repair expenses are included in the operating cost.

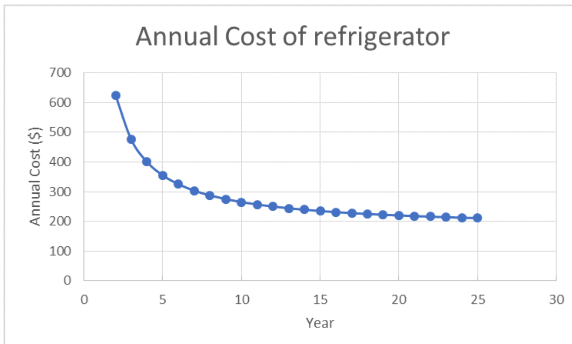
TABLE 2.1: Average Annual Cost of a New Refrigerator

t (years)	Cost: $C(t) = \frac{900}{t} + 175$ ($)

TABLE 2.1: (continued)

t (years)	Cost: $C(t) = \frac{900}{t} + 175$ (\$)
1	1,075
5	355
15	235
20	220

The graph of the cost resembles one branch of the reciprocal graph. Notice that as the years $t$increase, the cost approaches a horizontal line rather than the axis, as it did in the parent graph. In the 25th year, the average annual cost of the purchase price is so small that our operating cost comprises most of the expense.



The techniques we will explore to adapt the parent functions to a wider variety of functions are:

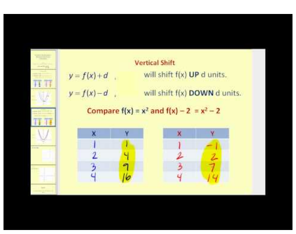
- Shifting (vertically and horizontally)
- Stretching (vertically and horizontally)
- Reflecting (in the x- or y-axis)

Vertical and Horizontal Shifting

Suppose that $c, d > 0$. If the graph of $f(x)$ is known, to graph:

1. $y = f(x) + d$, shift the graph of $f(x)$ up d units.
2. $y = f(x) - d$, shift the graph of $f(x)$ down d units.
3. $y = f(x + c)$, shift the graph of $f(x)$ left c units.
4. $y = f(x - c)$, shift the graph of $f(x)$ right c units.

This process is demonstrated in the following video:



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Vertical and Horizontal Stretching and Shrinking

If the graph of *f(x)* is known, to graph:

1. $y = af(x)$:

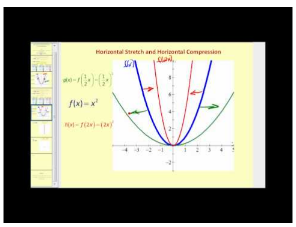
    a. If $a > 1$, stretch the graph vertically by a factor of a.
    b. If $0 < 1$, shrink the graph vertically by a factor of a.

2. $y = f(bx)$

    a. If $b > 1$, shrink the graph horizontally by a factor of b.
    b. If $0 < 1$, stretch the graph horizontally by a factor of $\frac{1}{b}$.

Note that all horizontal transformations are found inside the function $f(bx \pm c)$, and all vertical transformations are outside the function $af(x) \pm d$.

This process is demonstrated in the following video:



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Reflecting in the $x-$ or $y-$axis

If the graph of $f(x)$ is known, to graph:

1. $y = -f(x)$, reflect the graph across the x-axis.
2. $y = f(-x)$, reflect the graph across the y-axis.

Order of Transformations

The order of each transformation is important. When a function involves more than one transformation, complete the transformations in the following order:

1. Horizontal shift
2. Vertical and horizontal stretching or shrinking
3. Reflecting over an axis
4. Vertical shift

Practice these transformations using the following interactive activities:

a. Vertical and Horizontal Shift Activity: www.ck12.org/a/1824117
b. Stretching and Shrinking Activity: www.ck12.org/a/1824170

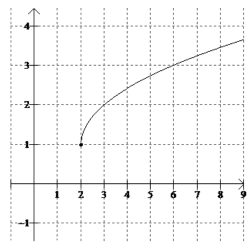
Examples

Example 1

Graph $y = 1 + \sqrt{x-2}$.

Solution:

The basic graph is $y = \sqrt{x}$. To create the requested graph, this graph is 1st shifted to the right 2 units and up 1 unit:

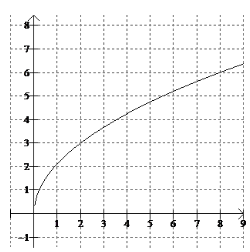


Example 2

Graph $y = 3\sqrt{\frac{1}{2}x}$.

Solution:

The basic graph is $y = \sqrt{x}$. To create the requested graph, this graph is stretched vertically 3 units and horizontally 2 units:

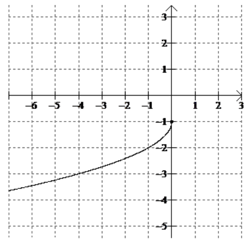


Example 3

Graph $y = -1 - \sqrt{-x}$.

Solution

The basic graph is $y = \sqrt{x}$. For this example, the order that each transformation is applied is important. Starting on the innermost level, reflect the graph in the $y-$axis, then reflect in the $x-$axis, then shift down 1 unit:

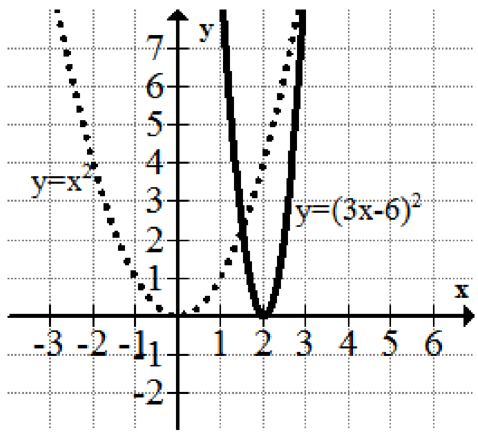


Example 4

Describe the transformation and graph: $f(x) = (3x - 6)^2$

Solution:

The basic graph is $y = x^2$. First, separate the horizontal shrink from the horizontal shift by factoring $f(x) = (3(x-2))^2$. The transformations are (in order) shift horizontally right 2 units, and shrink horizontally by a factor of $\frac{1}{3}$:

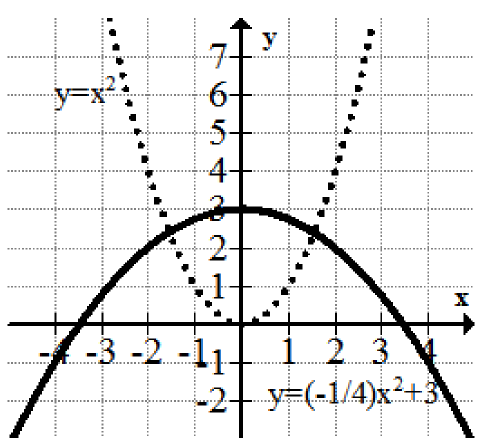


Example 5

Describe the following transformation and graph: $y = -\frac{1}{4}x^2 + 3$

Solution:

The basic graph is $y = x^2$. Using the quadratic function, the graph shrinks vertically by $\frac{1}{4}$, then it is reflected in the x-axis and shifted up by 3 units:



Example 6

Recall the question from the Introduction. The function $C(t) = \frac{900}{t} + 175$ is related to the basic family of the reciprocal function $y = \frac{1}{x}$. It is adapted to model the annual cost of the refrigerator with a vertical stretch of 900 units (the cost of the refrigerator), and then shifted vertically by 175 units (the annual operating cost). Since the basic function approached the x-axis as x increased, the cost function approaches the horizontal line, y = 175 as t increases. In other words, the graph shows that as the refrigerator's age increases, the annual cost of purchasing the refrigerator decreases.

Example 7

a) Describe the following transformation in words: $g(x)$ is transformed to $2g(-x)$: $g(x) \to 2g(-x)$.

Solution:

Vertical stretch of $g(x)$ by a factor of 2 and a reflection across the y-axis.

b) Write the expression for the transformation that would change $h(x)$ in the following ways:

- Vertical compression (shrink) by a factor of 3
- Vertical shift down 4 units
- Horizontal shift right 5 units

Solution:

$\frac{1}{3}h(x-5)-4$

c) Write the equation for the transformation that would change $f(x)$ in the following ways:

- Horizontal stretch by a factor of 4 and a horizontal shift 3 units to the right
- Vertical reflection across the x axis and a shift down 2 units

Solution:

$-f\left(\frac{1}{4}(x-3)\right)-2$ or $-f\left(\frac{1}{4}x-\frac{3}{4}\right)-2$

Review

Describe the following transformations in words:

1. $g(x) \to -g(-x)$
2. $f(x) \to -f(x+3)$
3. $h(x) \to h(x+1)-2$
4. $j(x) \to j(-x+3)$
5. $k(x) \to -k(2x)$
6. $f(x) \to 4f\left(\frac{1}{2}x+1\right)$
7. $g(x) \to -3g(x-2)-2$
8. $h(x) \to 5h(x+1)$
9. Write the equation for the transformation that would change $h(x)$ in the following ways:

- Vertical stretch by a factor of 2
- Vertical shift up 3 units
- Horizontal shift right 2 units

10. Write the equation for the transformation that would change $f(x)$ in the following ways:

- Vertical reflection across the x axis
- Vertical shift down 1 unit
- Horizontal shift left 2 units

11. Write the equation for the transformation that would change $g(x)$ in the following ways:

- Vertical compression by a factor of 4
- Reflection across the y axis

12. Write the equation for the transformation that would change $j(x)$ in the following ways:

- Horizontal compression by a factor of 3
- Vertical shift up 3 units
- Horizontal shift right 2 units

13. Write the equation for the transformation that would change $k(x)$ in the following ways:

- Horizontal stretch by a factor of 4
- Vertical shift up 3 units
- Horizontal shift left 1 unit

14. Write the equation for the transformation that would change $h(x)$ in the following ways:

- Vertical compression by a factor of 2
- Horizontal shift right 3 units
- Reflection across the y axis

15. Write the equation for the transformation that would change $f(x)$ in the following ways:

- Vertical stretch by a factor of 5
- Reflection across the x axis

Review (Answers)

Please see the Appendix.

2.9 Transforming Functions Defined by Data

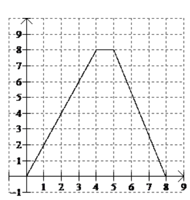
Learning Objectives

Learn to apply transformation techniques to models defined by data.

Introduction

Basic function families are useful guidelines to analyze mathematical models. However, some models can only be defined either graphically or with data points. Transformation techniques can still be applied to such models.

Consider the following graph, which represents the distance that a delivery truck is from a warehouse during one 8-hour trip. This will be used to define the function $D(t)$.

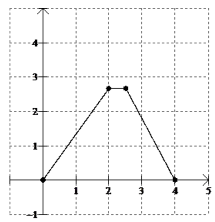


How does this function change if the driver completes the trip twice as fast, but only travels $\frac{1}{3}$the distance away from the warehouse? Using function notation, the transformation is represented as $D(t) \to \frac{1}{3}D(2t)$. From the previous section, we know this represents a horizontal shrink by a factor of 2, and a vertical shrink by a factor of 3. The clearest way to represent the transformation is by moving points easily read from the 1st graph.

TABLE 2.2: Transformation Table

$(t, D(t))$	$\left(\frac{1}{2}t, \frac{1}{3}D(t)\right)$
$(0,0)$	$(0,0)$
$(4,8)$	$\left(2, \frac{8}{3}\right)$
$(5,8)$	$\left(\frac{5}{2}, \frac{8}{3}\right)$
$(8,0)$	$(4,0)$

The transformed graph of the trip is:



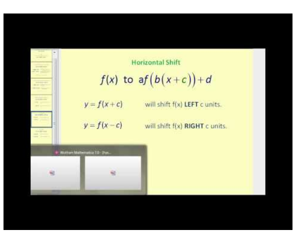
The changes required to create a new distance function are clearly satisfied. The trip took half the time because the driver completed the trip twice as fast, and the distance traveled away from the warehouse is $\frac{1}{3}$ the original distance.

Point Notation for Function Transformation

A transformation can be written using function notation or point notation, which simply illustrates the transformation of each key point. This method works well with tabular data or a well-labeled graph. The process assigns a pairing between (x, y) and new coordinates based on the transformation. The form for point notation is:

$$(x, y) \rightarrow (new\ x, new\ y)$$

Note that in function notation, $y = f(x)$. The new x -coordinate is the value you get when you solve the transformation using the old x input value.



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Example 1

Write the following transformation using function notation and point notation. Then apply the transformation to the 3 points in the table. Transformation: Horizontal shift right 3 units, vertical shift up 4 units.

x	y
1	3
2	5
8	-11

Solution:

$f(x) \to f(x-3)+4$

$(x,y) \to (x+3, y+4)$

x	y
1	3
2	5
8	-11

$\longrightarrow$

x	y
4	7
5	9
11	-7

Example 2

Convert the following function notation into point notation:

$f(x) \to \frac{1}{4} f(-(x+3)) - 1$

x	y
0	0
1	4
2	8

Solution:

Point notation:

$(x,y) \to \left(-x-3, \frac{1}{4}y - 1\right)$

x	y
0	0
1	4
2	8

$\longrightarrow$

x	y
-3	-1
-4	0
-5	1

Example 3

Convert the following point notation to words and to function notation. Then apply the transformation to the tabular data:

$(x,y) \to (2x, -y-1)$

x	y
10	8
12	7
14	6

Solution:

Transformation: Horizontal stretch by a factor of 2, vertical reflection across the x-axis, and vertical shift down 1 unit.

The point transformation in function notation is:

$f(x) \rightarrow -\left(f\left(\frac{x}{2}\right)\right) - 1$

x	y
10	8
12	7
14	6

$\longrightarrow$

x	y
20	-9
24	-8
28	-7

Example 4

Describe the following function notation and rewrite it using point notation. Apply the transformation to 3 points.

$f(x) \rightarrow -2f(x-1)+4$

Solution:

$f(x) \rightarrow -2f(x-1)+4$

Horizontal shift right 1 unit. Vertical stretch by a factor of 2. Vertical reflection across the x axis. Vertical shift 4 units.

$(x,y) \rightarrow (x+1,-2y+4)$

x	y
0	5
1	6
2	7

$\longrightarrow$

x	y
1	-6
2	-8
3	-10

Example 5

Convert the following point notation to function notation. Describe the transformation:

$(x,y) \rightarrow (3x+1,-y+7)$

Solution:

$(x,y) \rightarrow (3x+1,-y+7)$

Horizontal stretch by a factor of 3 and then a horizontal shift right 1 unit.

Vertical reflection over the x axis and then a vertical shift 7 units up.

$f(x) \rightarrow -f\left(\frac{1}{3}x - \frac{1}{3}\right) + 7$

Example 6

Convert the following function notation to point notation and describe the changes:

$f(x) \rightarrow -\frac{1}{2}f(x-1)+3$

Solution:

$f(x) \rightarrow -\frac{1}{2}f(x-1)+3$

Vertical reflection across the x axis, vertical shrink by a factor of $\frac{1}{2}$, and shift up 3. Horizontal shift right 1 unit.

$(x,y) \rightarrow \left(x+1,-\frac{1}{2}y+3\right)$

Summary

- Transformation techniques can still be applied to such models.
- Point notation is used to translate key points based on a transformation of a function. It can be used either in place of or with functional notation.

Review

Convert the following function notation into words and then point notation. Finally, apply the transformation to three example points:

x	y
0	5
1	6
2	7

1. $f(x) \rightarrow -\frac{1}{2}f(x+1)$
2. $g(x) \rightarrow 2g(3x)+2$
3. $h(x) \rightarrow -h(x-4)-3$
4. $j(x) \rightarrow 3j(2x-4)+1$
5. $k(x) \rightarrow -k(x-3)$

Convert the following functions in point notation to function notation:

6. $(x,y) \rightarrow \left(\frac{1}{2}x+3, y-4\right)$
7. $(x,y) \rightarrow (2x+4, -y+1)$
8. $(x,y) \rightarrow (4x, 3y-5)$
9. $(2x,y) \rightarrow (4x, -y+1)$
10. $(x+1, y-2) \rightarrow (3x+3, -y+3)$

Convert the following functions in function notation to point notation:

11. $f(x) \rightarrow 3f(x-2)+1$
12. $g(x) \rightarrow -4g(x-1)+3$
13. $h(x) \rightarrow \frac{1}{2}h(2x+2)-5$
14. $j(x) \rightarrow 5j\left(\frac{1}{2}x-2\right)-1$
15. $k(x) \rightarrow \frac{1}{4}k(2x-4)$

Review (Answers)

Please see the Appendix.

2.10 Asymptotes and End Behavior

Learning Objectives

Learn about the end behavior and asymptotes of a function.

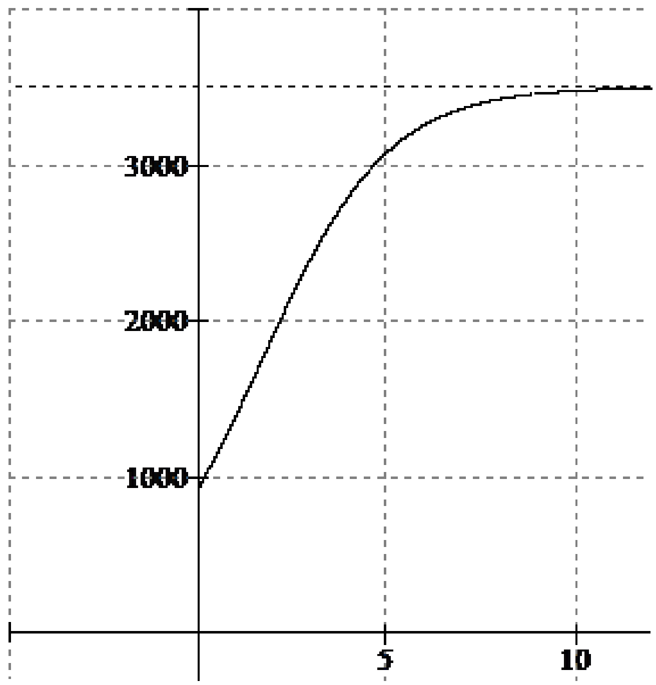
Introduction

To understand the behavior of a function, it is helpful to view its graph. When a graph is displayed, typically the portion near the origin is visible. But there are times when an analyst needs to determine the behavior of the graph outside this region. To show this long-term behavior within a narrow viewing area, several techniques have been developed. In this section, you will study these techniques in order to determine the long-term behavior of a function.

Consider this example: A law enforcement agency gathered data about the number of crime incidents per year in a specific region. The number of crime incidents can be modeled using the given function, where t represents the time since 2005:

$$N(t) = 3,500\left(\frac{1}{1+1.025e^{-0.6t}}\right)$$

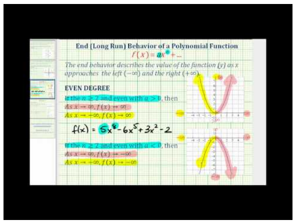
The graph of this function will allow law enforcement to understand long-term behavior and predict the number of crimes in the future. Notice that as the year approaches 2015, the number of crimes in this region seems to be approaching 3,500 crimes per year in the area, but will not exceed that number. This means the function is approaching $y = 3,500$ asymptotically, and the horizontal line is the function's horizontal asymptote.



End Behavior of Polynomials

The **end behavior** of a function describes the long-term behavior of a function as x approaches negative infinity or positive infinity. When the function is a polynomial, then the end behavior can be determined by considering the sign on the leading coefficient and whether the degree of the function is odd or even.

- When the degree of the polynomial is even and the leading coefficient is positive, then the function approaches positive infinity as x approaches negative infinity and positive infinity.
- When the degree of the polynomial is even and the leading coefficient is negative, then the function approaches negative infinity as x approaches negative infinity and positive infinity.
- When the degree of the polynomial is odd and the leading coefficient is positive, then the function approaches positive infinity as x approaches positive infinity, and the function approaches negative infinity as x approaches negative infinity.
- When the degree of the polynomial is odd and the leading coefficient is negative, then the function approaches negative infinity as x approaches positive infinity, and the function approaches positive infinity as x approaches negative infinity.



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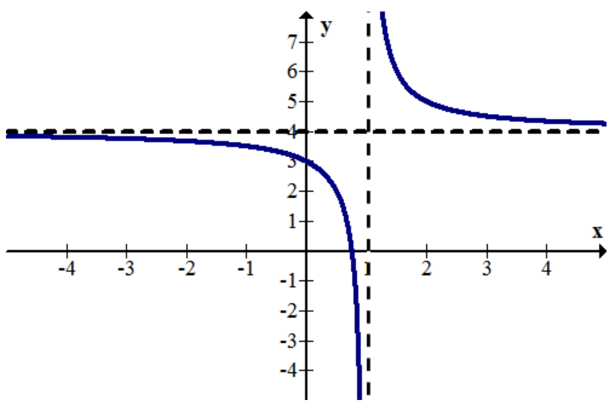
Asymptotes

A **vertical asymptote** is a vertical line, such as $x = 1$, marking a specific value toward which the graph of a function may approach but will never reach.

A **horizontal asymptote** is a horizontal line such as $y = 4$. It indicates a range value that the function approaches as x approaches positive infinity or negative infinity. A function may touch or pass through a horizontal asymptote.

The reciprocal function has two asymptotes, one vertical and one horizontal.

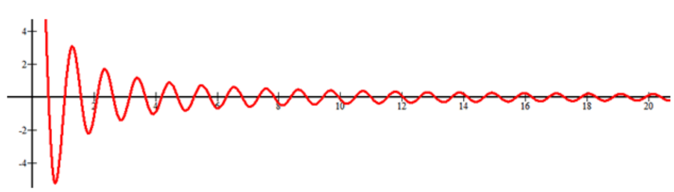


Play, Learn and Explore Asymptotes and End Behavior: www.ck12.org/a/2175885

Examples

Example 1

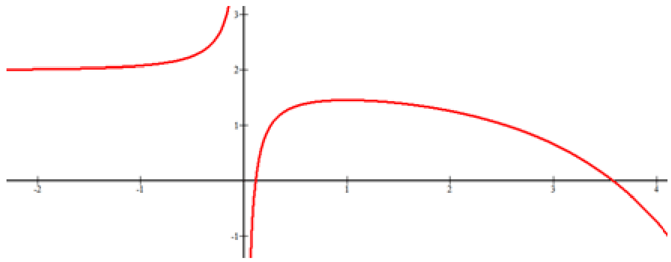
Identify the horizontal asymptote for the following function:



Solution: The graph appears to flatten as x approaches positive infinity. The horizontal asymptote is $y = 0$, even though the function clearly passes through this line an infinite number of times.

Example 2

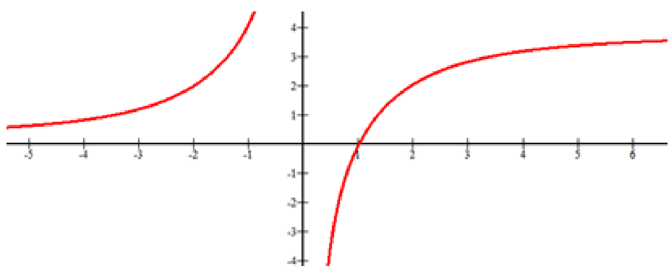
Identify the asymptotes and end behavior of the following function:



Solution: The function has a horizontal asymptote $y = 2$ as x approaches negative infinity. There is a vertical asymptote at $x = 0$.

Example 3

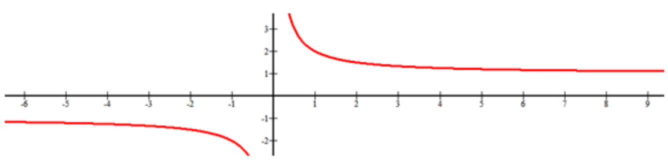
Identify the asymptotes and end behavior of the following function:



Solution: There is a vertical asymptote, $x = 0$. The horizontal asymptote as x approaches negative infinity is $y = 0$ and the horizontal asymptote as x approaches positive infinity is $y = 4$.

Example 4

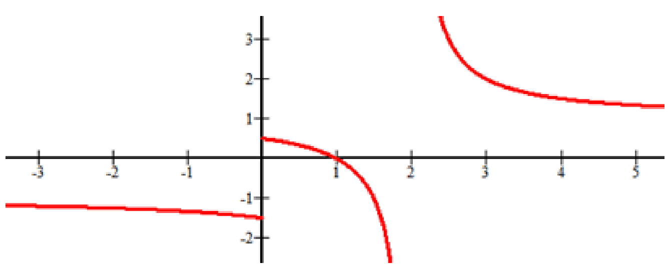
Identify the horizontal and vertical asymptotes of the following function:



Solution: There is a vertical asymptote, $x = 0$. As x approaches negative infinity, there is a horizontal asymptote, $y = -1$. As x approaches positive infinity, there is another horizontal asymptote, $y = 1$.

Example 5

Identify the horizontal and vertical asymptotes of the following function:

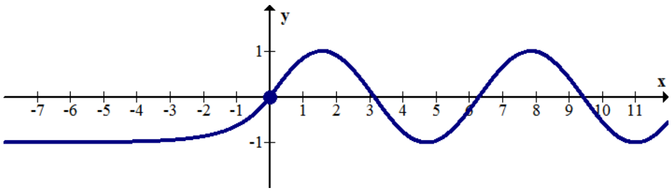


Solution: There is a vertical asymptote, $x = 2$. At $x = 0$, there is a break in the graph but not a vertical asymptote. As x approaches negative infinity, there is a horizontal asymptote, $y = -1$. As x approaches positive infinity, there is another horizontal asymptote, $y = 1$.

Example 6

Identify the horizontal and vertical asymptotes based on the graph of the following piecewise function:

$$f(x) = \begin{cases} e^x - 1 & x \leq 0 \\ \sin x & 0 < x \end{cases}$$



Solution: There is a horizontal asymptote, $y = -1$, as x approaches negative infinity.

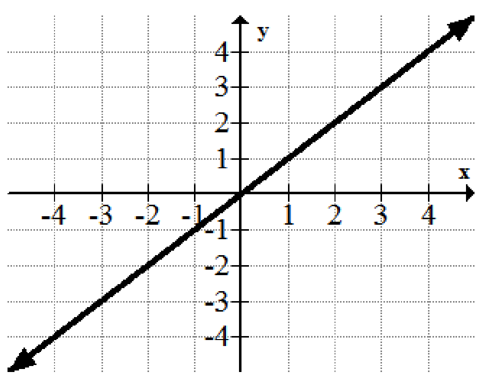
Summary

- The **end behavior** of a function describes the long-term behavior of a function as x approaches negative infinity and positive infinity.
- A **vertical asymptote** is a vertical line that marks a specific value toward which the graph of a function may approach but will never reach.
- A **horizontal asymptote** is a horizontal line that the graph of a function approaches as x approaches positive infinity or negative infinity.
- Graphing asymptotes is a method to illustrate the long-term behavior of a function. They are usually identified with dotted lines on the graph of the function, and indicate how the function will behave outside the viewing window.

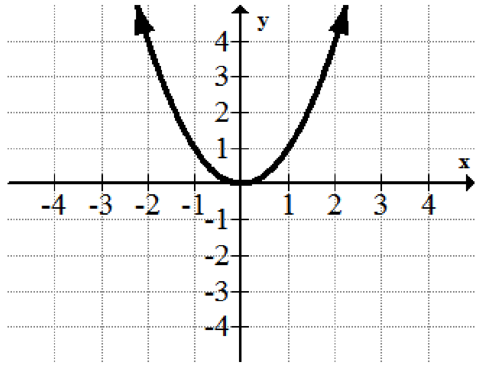
Review

Identify the asymptotes and end behavior of the following functions:

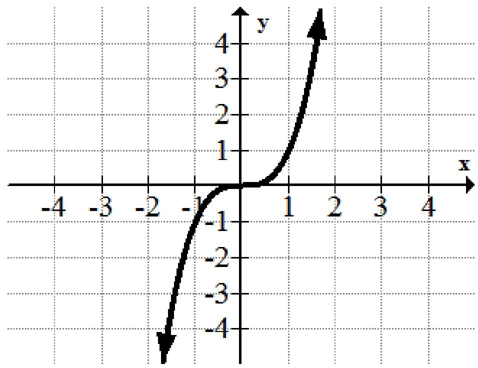
1. $y = x$



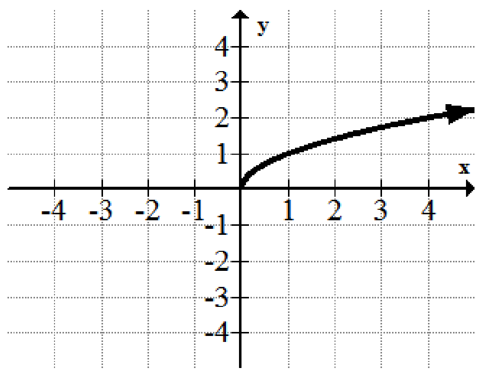
2. $y = x^2$



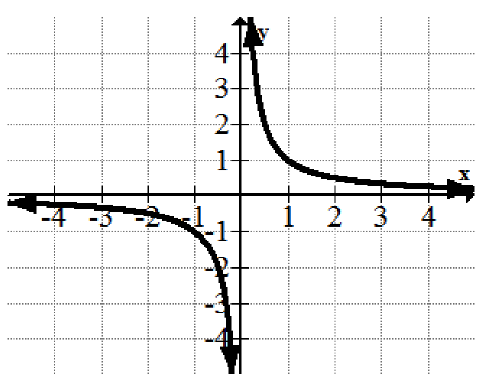
3. $y = x^3$



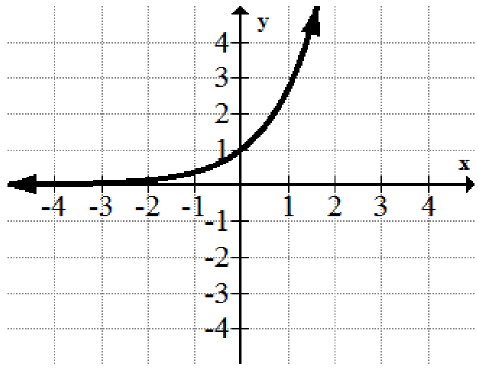
4. $y = \sqrt{x}$



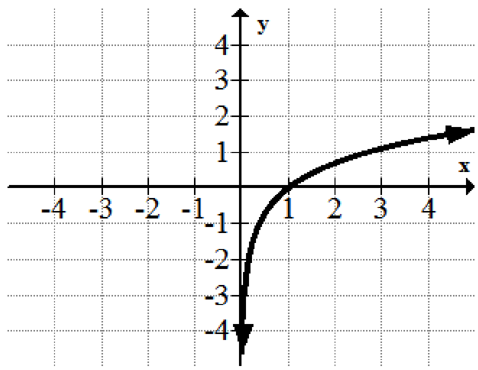
5. $y = \frac{1}{x}$



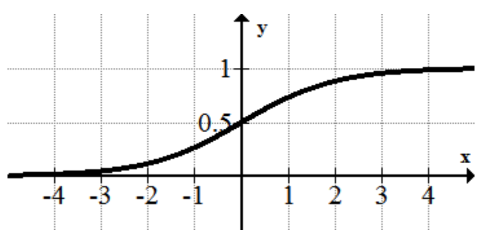
6. $y = e^x$



7. $y = \ln(x)$



8. $y = \frac{1}{1+e^{-x}}$



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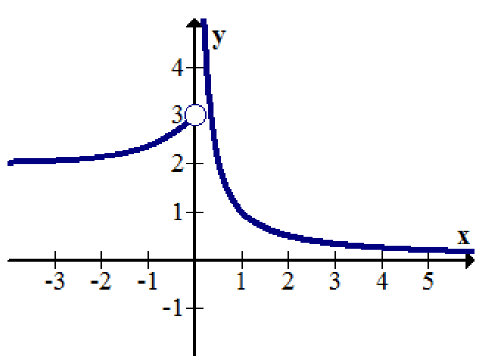

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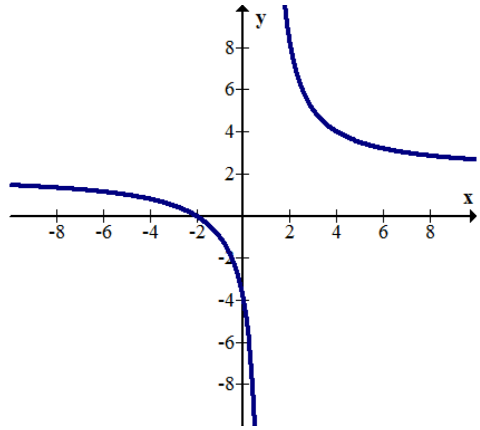

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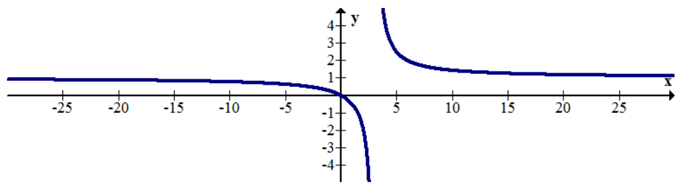


12. Explain why it makes sense that $y = \frac{1}{x}$ has a vertical asymptote at $x = 0$.
13. Explain why it makes sense that $y = \frac{1}{x+3}$ has a vertical asymptote at $x = -3$.
14. Use the technique from the previous problem to determine the vertical asymptote for the function $y = \frac{1}{x-2}$.
15. Use the technique from problem #13 to determine the vertical asymptote for the function $y = \frac{2}{x+4}$.

Review (Answers)

Please see the Appendix.

2.11 Continuity and Discontinuity

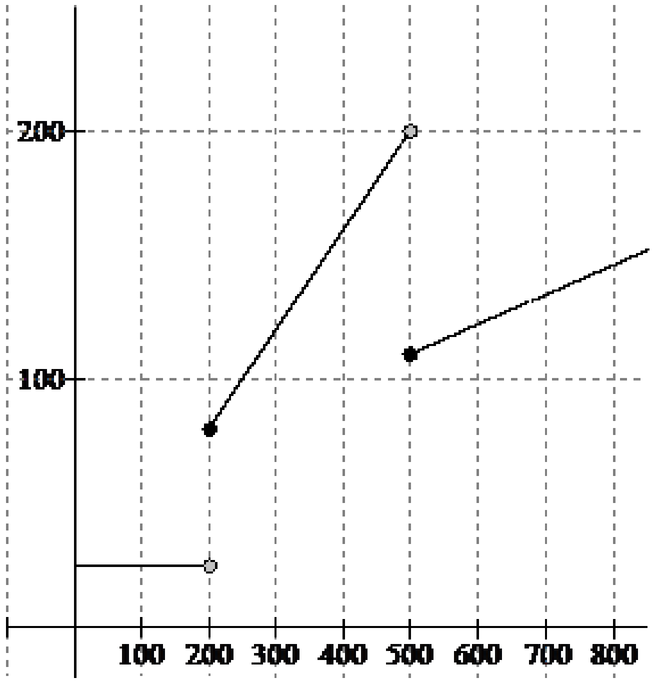
Learning Objectives

Learn about the two types of discontinuities: removable and non-removable.

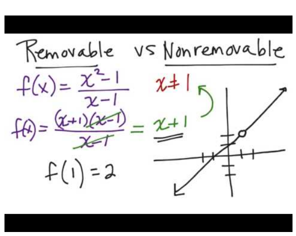
Introduction

An important part of mathematical analysis is to uncover surprises. Mathematical functions with predictable behavior reflect unsurprising situations. Consider the following function, which describes a text-messaging pricing plan. The cost $P(x)$ is based on the number of texts, x, and changes based on the domain. This style of function definition is called **piecewise**.

$$P(x) = \begin{cases} 25 & \text{if } 0 < 200 \\ 0.4x & \text{if } 200 \leq x < 500 \\ 50 + .12x & \text{if } x \geq 500 \end{cases}$$



This function certainly has some unexpected changes in pricing. These breaks in the graph are called **discontinuities**. The function is discontinuous at $x = 200$ and at $x = 500$. A function without discontinuities is called **continuous**. Informally, a function is continuous if its graph can be drawn without lifting the pencil. Functions that cannot be drawn without lifting the pencil are discontinuous.



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Types of Discontinuities

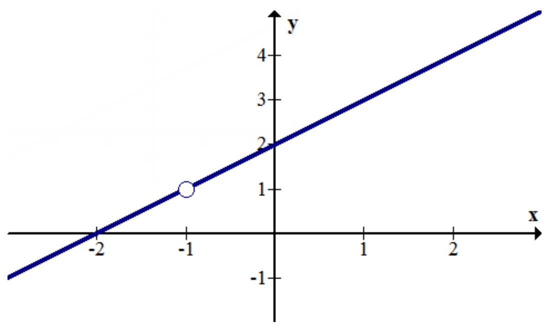
As seen in the video, there are two types of discontinuities: removable and non-removable discontinuities. And there are two types of non-removable discontinuities: jump and infinite discontinuities.

A **removable discontinuity** occurs when the graph of a function has a hole.

For example, consider the following function:

$$f(x) = \frac{(x+2)(x+1)}{x+1}$$

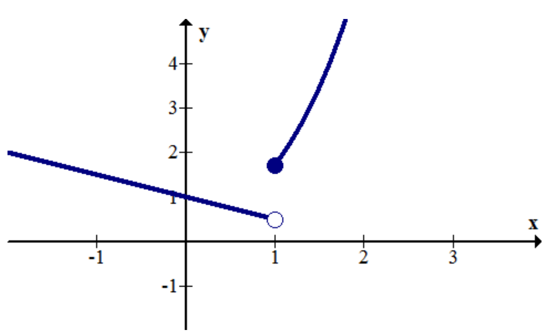
Notice that the set of factors $x+1$can be removed or canceled to get the function $f(x) = x+2$. The graph will then resemble $y = x+2$, except there will be a hole at $x = -1$ to account for the removed factor $x+1$. The reason for the hole is that although the original function can be simplified, -1 must be excluded from the domain of the function. Thus, graph the line $y = x+2$ as usual, but remove the point at $x = -1$:



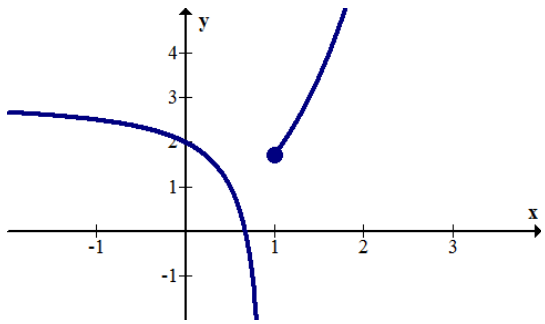
Removable discontinuities can be "filled in" if you make the function a piecewise function and define a part of the function at the point where the hole is. In the example above, to make $f(x)$continuous, you could redefine it as:

$$f(x) = \begin{cases} \frac{(x+2)(x+1)}{x+1}, & x \neq -1 \\ 1, & x = -1 \end{cases}$$

A **jump discontinuity** occurs when a function has two ends that don't meet, even if the hole is filled in at one of the ends. In order to satisfy the vertical line test and make sure the graph is truly that of a function, only one of the end points may be filled. Below is an example of a function with a jump discontinuity:



An **infinite discontinuity** occurs when a function has a vertical asymptote on one or both sides. This is shown in the graph of the function below at $x = 1$:



Examples

Example 1

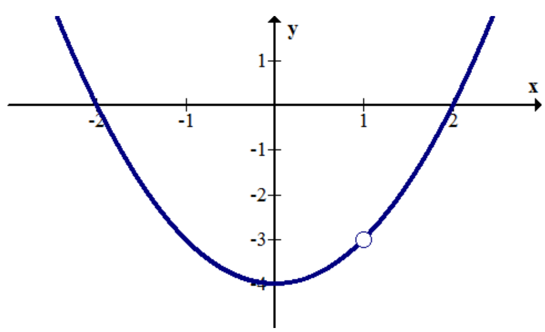
Identify the discontinuity of the function algebraically and then graph the function:

$f(x) = \frac{(x-2)(x+2)(x-1)}{(x-1)}$

Solution:

The factor $x-1$ can be removed or canceled in both the numerator and the denominator to result in $f(x) = (x-2)(x+2)$. Because $x-1$ was canceled, there is a removable discontinuity or hole at $x = 1$.

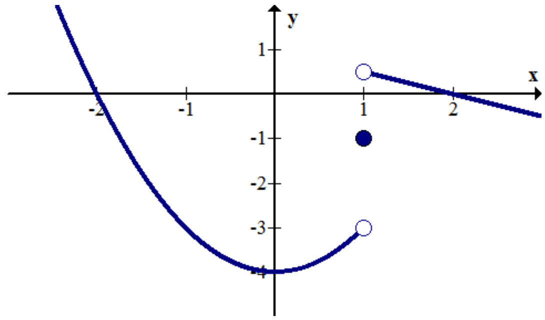
The graph will resemble $y = (x-2)(x+2)$ with a hole at $x = 1$, so graph $y = (x-2)(x+2)$ as usual and then insert a hole in the appropriate spot at the end:



Example 2

Identify the discontinuity of the piecewise function graphically:

$$f(x) = \begin{cases} x^2 - 4 & x < 1 \\ -1 & x = 1 \\ -\frac{1}{2}x + 1 & x > 1 \end{cases}$$

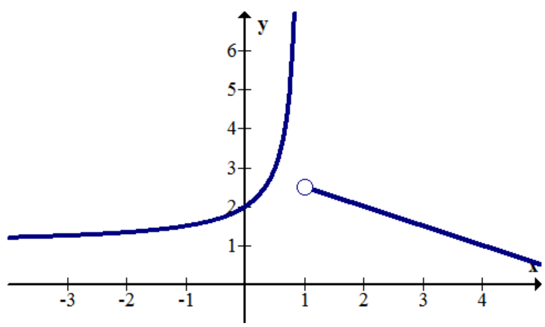


Solution:

There is a jump discontinuity at $x = 1$. The piecewise function describes a function in three parts; a portion of a parabola on the left, a single point in the middle, and a portion of a line on the right.

Example 3

Identify the discontinuity of the function below:

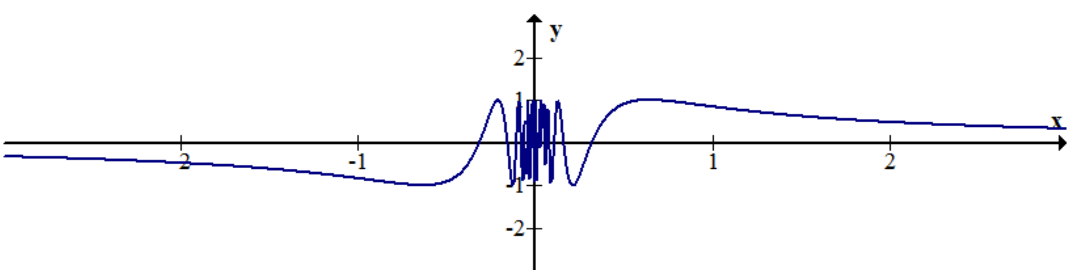


Solution:

Since there is a vertical asymptote at $x = 1$, this is an infinite discontinuity.

Example 4

Describe the continuity or discontinuity of the function $f(x) = \sin\left(\frac{1}{x}\right)$.

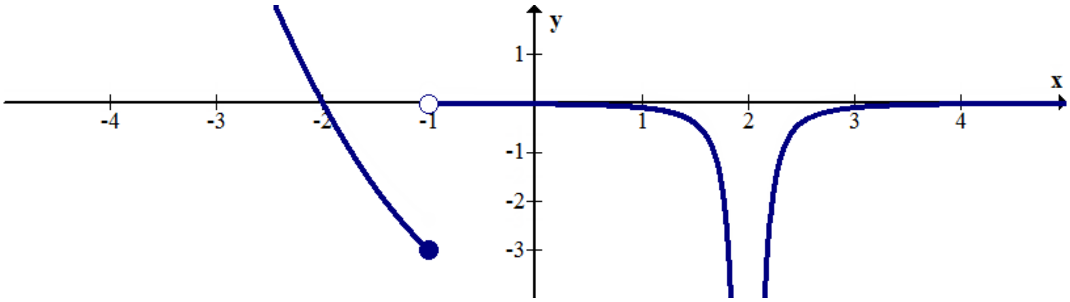


Solution:

The function seems to oscillate infinitely as x approaches 0. One thing that the graph fails to show is that 0 is clearly not in the domain. The graph does not shoot to infinity, nor does it have a simple hole or jump discontinuity. However, it is not continuous at $x = 0$.

Example 5

Describe the discontinuities of the function below:

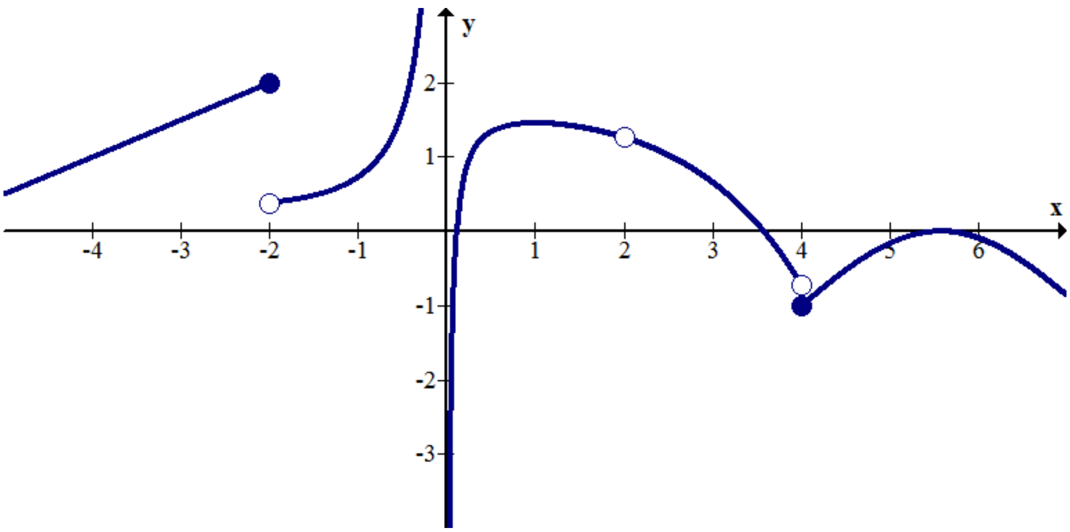


Solution:

There is a jump discontinuity at $x = -1$ and an infinite discontinuity at $x = 2$.

Example 6

Describe the discontinuities of the function below:



Solution:

There are jump discontinuities at $x = -2$ and $x = 4$. There is a removable discontinuity at $x = 2$. There is an infinite discontinuity at $x = 0$.

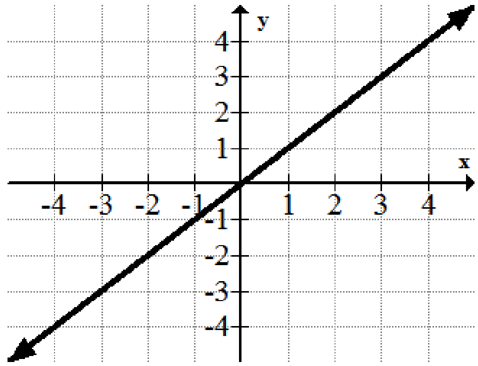
Summary

- There are two types of discontinuities: removable and non-removable. Then there are two types of non-removable discontinuities: jump or infinite discontinuities.
- **Removable discontinuities** are also known as holes. They occur when factors can be algebraically removed or canceled from rational functions.
- **Jump discontinuities** occur when a function has two ends that don't meet, even if the hole is filled in at one of the ends.
- **Infinite discontinuities** occur when a function has a vertical asymptote on one or both sides.

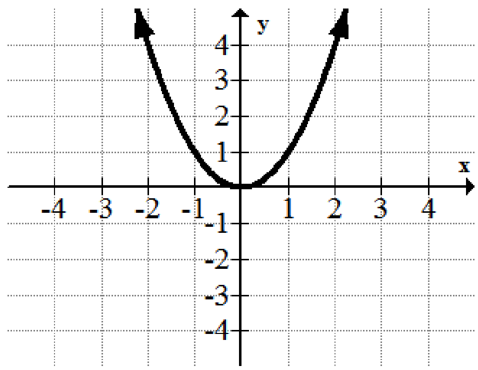
Review

Describe any discontinuities in the functions below:

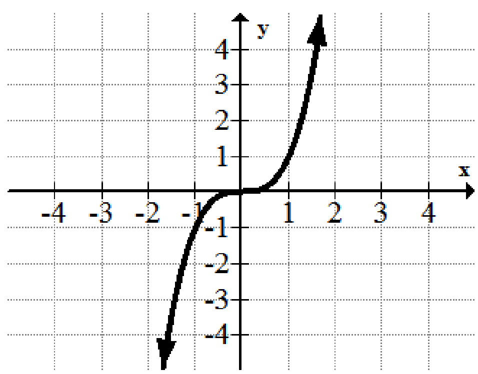
1. $y = x$



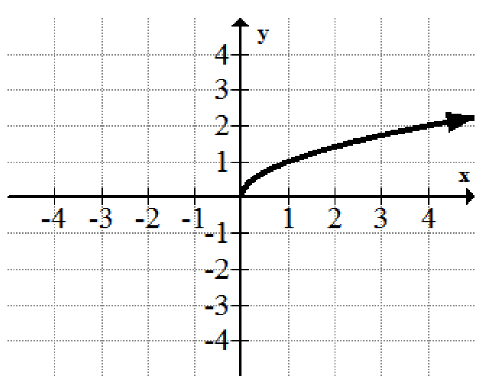
2. $y = x^2$



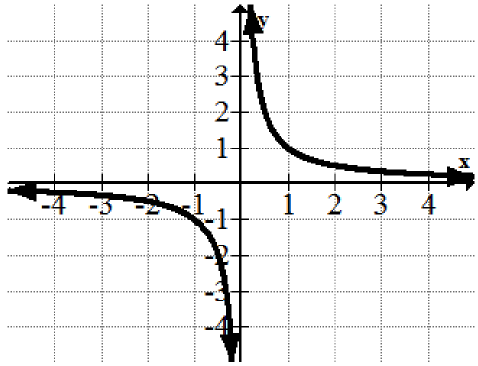
3. $y = x^3$



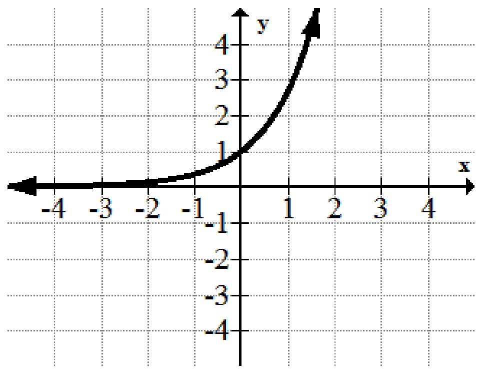
4. $y = \sqrt{x}$



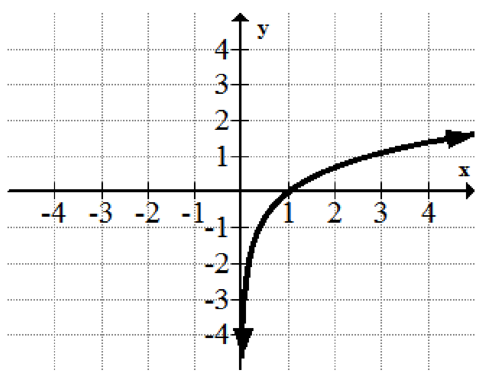
5. $y = \frac{1}{x}$



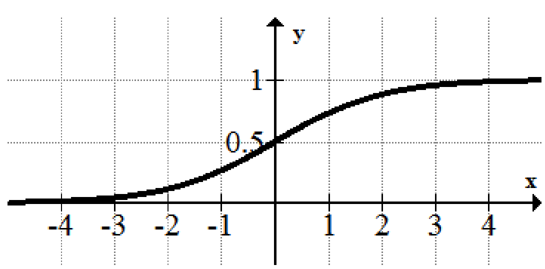
6. $y = e^x$



7. $y = \ln(x)$



8. $y = \frac{1}{1+e^{-x}}$



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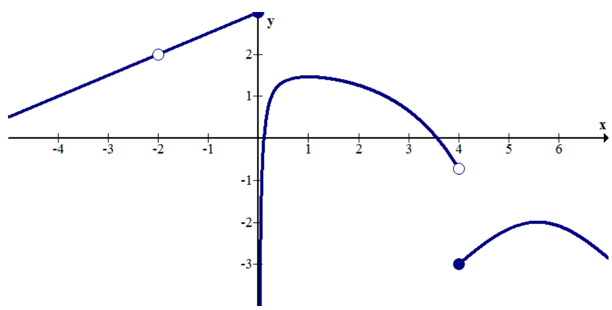

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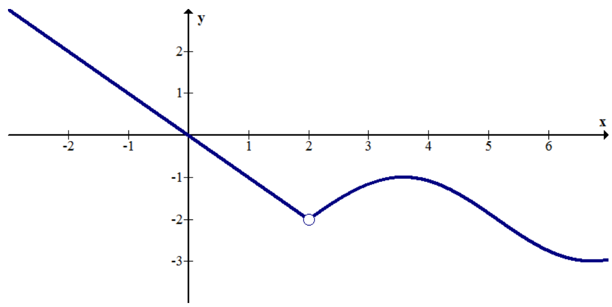

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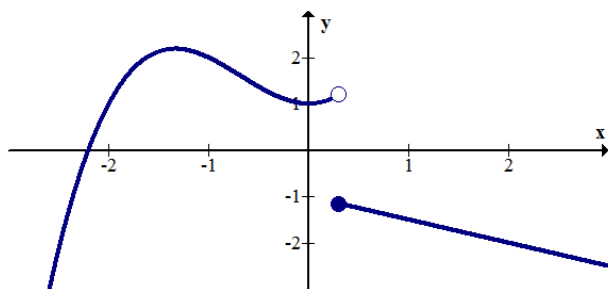


12. $f(x)$ has a jump discontinuity at $x = 3$, a removable discontinuity at $x = 5$, and another jump discontinuity at $x = 6$. Draw a picture of a graph that could be $f(x)$.

13. $g(x)$ has a jump discontinuity at $x = -2$, an infinite discontinuity at $x = 1$, and another jump discontinuity at $x = 3$. Draw a picture of a graph that could be $g(x)$.

14. $h(x)$ has a removable discontinuity at $x = -4$, a jump discontinuity at $x = 1$, and another jump discontinuity at $x = 7$. Draw a picture of a graph that could be $h(x)$.

15. $j(x)$ has an infinite discontinuity at $x = 0$, a removable discontinuity at $x = 1$, and a jump discontinuity at $x = 4$. Draw a picture of a graph that could be $j(x)$.

Review (Answers)

Please see the Appendix.

2.12 Function Combinations and Composition

Learning Objectives

Learn to combine and compose functions.

Introduction

A small art framing company hired a management consultant to analyze its revenue and expenses. The consultant delivered these mathematical functions:

Revenue: $R(x) = 300x - \frac{x^2}{200}$, where x is number of units sold

Cost: $C(x) = 500 + 175x + .7x^2 - .00005x^3$, where x is number of units produced

Net Revenue: $N(r) = 0.91r$, where r is the amount of total revenue in dollars

However, two problems surfaced. The consultant also needed:

1. A profit function (revenue versus cost)

2. Net revenue based on the number of items sold

In order to determine each of these items, the functions can be combined using **function combinations.** A function combination means simply combining two functions using the arithmetic operators $+$, $-$, $\cdot$, and $\div$, as long as the denominator function is not equal to 0.

Thus, profit can be created as an arithmetic combination of the revenue and cost functions:

$$
\begin{aligned}
P(x) &= R(x) - C(x) \\
P(x) &= 300x - \frac{x^2}{200} - \left(500 + 175x + .7x^2 - .00005x^3\right) \\
P(x) &= 125x - .075x^2 + .00005x^3 - 500
\end{aligned}
$$

The 2nd combination of functions is more sophisticated. Now, to calculate net revenue (revenue after taxes) the company had to take two steps:

1. Calculate revenue, based on the number sold.
2. Take that value as input for the net revenue function, and calculate the final value.

Net revenue based on the number sold can be created as a **composition** of net revenue and revenue as follows:

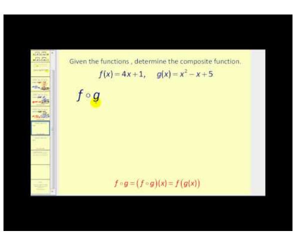
$$
\begin{aligned}
(N \circ R)(x) &= N(R(x)) \\
&= N\left(300x - \frac{x^2}{200}\right) \\
&= 0.91\left(300x - \frac{x^2}{200}\right) \\
&= 273x - .00455x^2
\end{aligned}
$$

Now the two required functions are ready to use. These two functions were created from the original functions using either combinations or composition of functions.

Play, Lean, and Explore Combining Functions: www.ck12.org/a/1824208

Function Composition

A composite function is a combination of two functions. Function composition can be thought of visually as a mapping from the domain to the range of the two functions. Function composition can also be thought of algebraically using function composition notation. The following video describes both ways of thinking about function composition:



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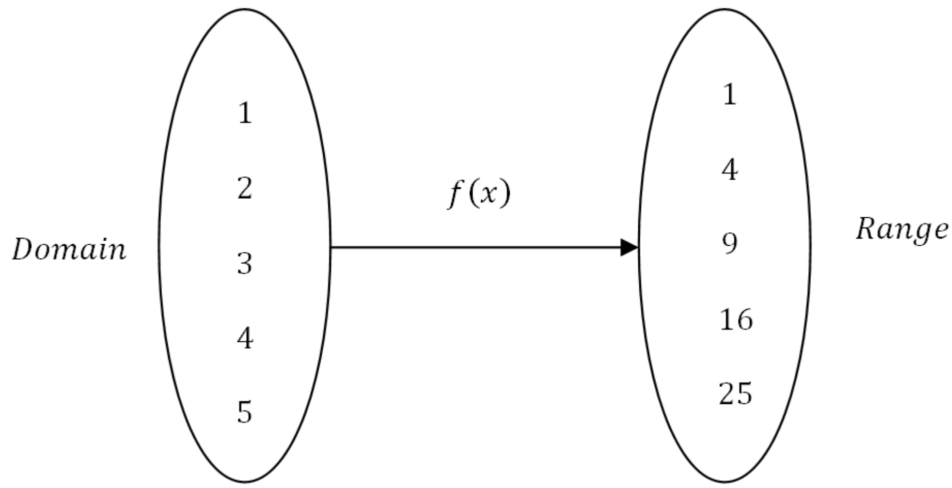
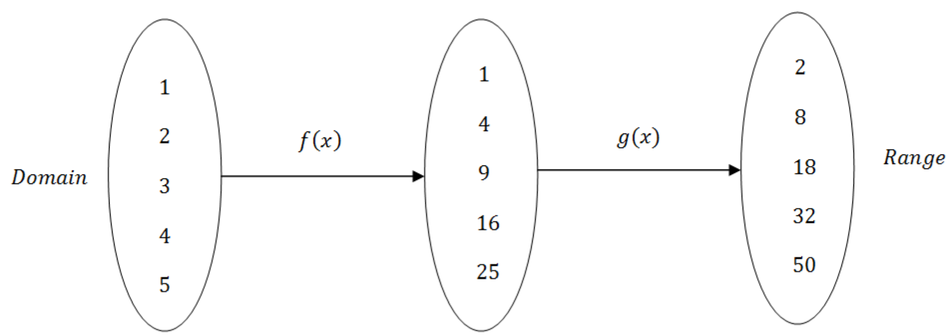
Function Composition Visually

A common way to describe functions is a mapping from the domain to the range:



With function composition, the range of the 1st function becomes the domain of the 2nd function:



Function Composition Notation

Function composition can also be thought of algebraically using function composition notation:

Function Composition Notation

$$(g \circ f)(x) = g(f(x))$$

On both sides of the equation, the f function is linked to the x, and thus operates on the x-values 1st. Then the result is inputted into the g function. Notice that you always work right to left in solving a function composition.

Each of the following functions has a function composition within the function. For instance, in $j(x)$, the outer function could be $\sqrt{x}$ and the inner function could be $x+1$:

$$f(x) = x^2 - 1$$

$$h(x) = \frac{x-1}{x+5}$$

$$g(x) = 3e^x - x$$

$$j(x) = \sqrt{x+1}$$

Examples

Example 1

Using the functions defined above, what is the function composition $g(h(x))$?

Solution:

Substitute $h(x)$ into the g function for x:

$g(h(x)) = g\left(\frac{x-1}{x+5}\right) = 3e^{\left(\frac{x-1}{x+5}\right)} - \left(\frac{x-1}{x+5}\right)$

Example 2

Using the functions defined above, what is the function composition $f\Big(j(h(g(x)))\Big)$?

Solution:

There are many function compositions in this example. In these cases, begin composing the innermost functions 1st, and then continue to compose the functions working inside out. In this problem, 1st substitute $g(x)$ into the h function for x:

$h(g(x)) = \frac{3e^x - x - 1}{3e^x - x + 5}$

Next, consider $f(j(x))$ by substituting $j(x)$ into $f(x)$:

$f(j(x)) = \left(\sqrt{x+1}\right)^2 - 1 = (x+1) - 1 = x$

Notice how this composition simplifies to the variable. Thus, $f\Big(j(h(g(x)))\Big) = h(g(x))$.

Example 3

Show $f(h(x)) \neq h(f(x))$

Solution:

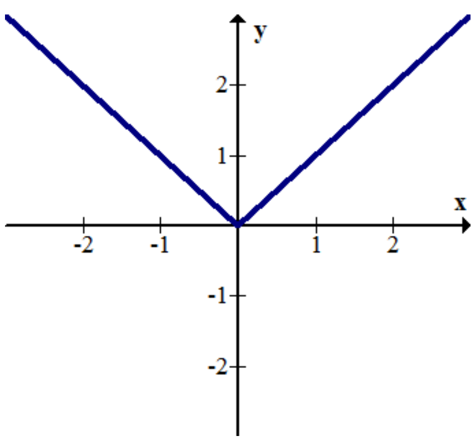
$f(h(x)) = f\left(\frac{x-1}{x+5}\right) = \left(\frac{x-1}{x+5}\right)^2 - 1$

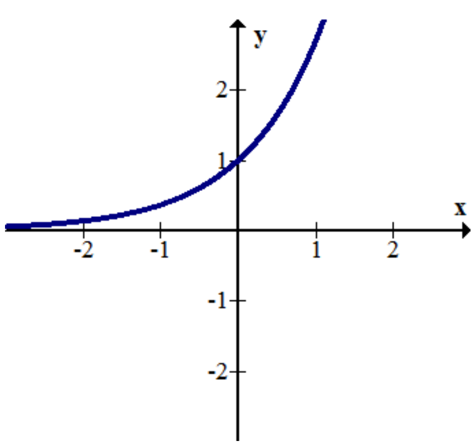
$h(f(x)) = h(x^2-1) = \frac{(x^2-1)-1}{(x^2-1)+5} = \frac{x^2-2}{x^2+4}$

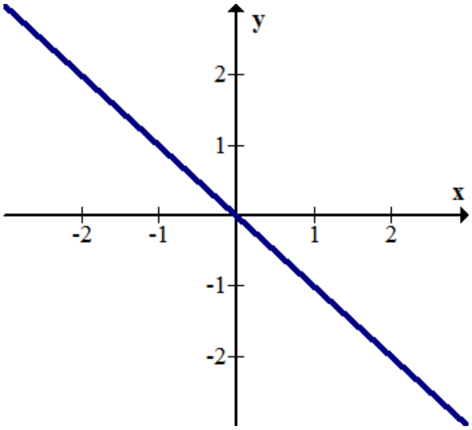
To truly show they are not equal, it is best to find a specific counter example of a number where they differ. Sometimes algebraic expressions may look different, but are actually the same. You should notice that $f(h(x))$is undefined when $x=-5$, because then there would be 0 in the denominator. $h(f(x))$, on the other hand, is defined at $x=-5$. Since the two function compositions differ, you can conclude: $f(h(x)) \neq h(f(x))$.

For the next three example problems use the following functions:

$f(x) = |x|$





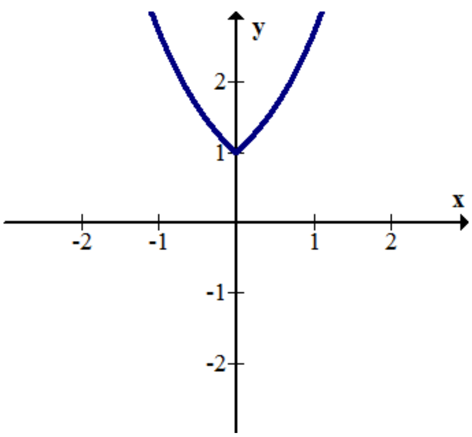


Example 4

Compose $g(f(x))$ and graph the result. Describe the transformation.

Solution:

$g(f(x)) = g(|x|) = e^{|x|}$



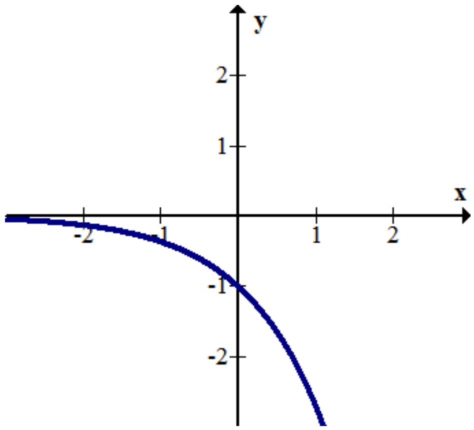
The positive portion of the exponential graph has been reflected over the y-axis, and the negative portion of the exponential graph has been removed.

Example 5

Compose $h(g(x))$ and graph the result. Describe the transformation.

Solution:

$h(g(x)) = h(e^x) = -e^x$



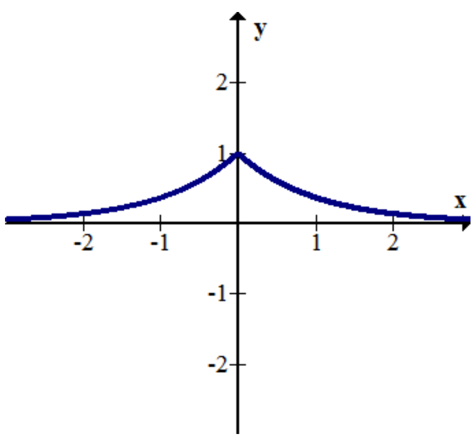
The exponential graph has been reflected over the x-axis.

Example 6

Compose $g\left(h(f(x))\right)$ and graph the result. Describe the transformation.

Solution:

$g\left(h(f(x))\right) = g(h(|x|)) = g(-|x|) = e^{-|x|}$



The negative portion of the exponential graph has been mirrored over the y-axis, and the positive portion of the exponential graph has been truncated.

Summary

- New functions can be made from existing functions through arithmetic combinations or composition. The new functions can combine the effects of the original functions in one step.
- Function composition is when there are two or more functions, and the range of the 1st function becomes the domain of the 2nd function.

Review

For questions 1-9, use the following three functions: $f(x) = |x|, h(x) = -x, g(x) = (x-2)^2 - 3$.

1. Determine $g(x) - h(x)$.

2. Graph $f(x), h(x)$ and $g(x)$.

3. Find $f(g(x))$ algebraically.

4. Graph $f(g(x))$ and describe the transformation.

5. Find $g(f(x))$ algebraically.

6. Graph $g(f(x))$ and describe the transformation.

7. Find $h(g(x))$ algebraically.

8. Graph $h(g(x))$ and describe the transformation.

9. Find $g(h(x))$ algebraically.

10. Graph $g(h(x))$ and describe the transformation.

For 10-16, use the following three functions: $j(x) = x^2, k(x) = |x|, m(x) = \sqrt{x}$.

11. Determine $j(x) + m(x)$.

12. Graph $j(x), k(x)$ and $m(x)$.

13. Find $j(k(x))$ algebraically.

14. Graph $j(k(x))$ and describe the transformation.

15. Find $k(m(x))$ algebraically.

16. Graph $k(m(x))$ and describe the transformation.

17. Find $m(k(x))$ algebraically.

18. Graph $m(k(x))$ and describe the transformation.

19. A toy manufacturer has a new product to sell. The number of units to be sold, n, is a function of the price, p, such that $n(p) = 30 - 25p$. The revenue, r, earned from the sales is a function of the number of units sold, n, such that $r(n) = 1000 - \frac{1}{4}n^2$.Find the function for revenue in terms of price, p.

Review (Answers)

Please see the Appendix.

2.13 Inverses of Functions

Learning Objectives

Learn to identify one-to-one functions.

Learn to determine the inverse of functions.

Introduction

Assume the kidneys can filter out 20% of a drug in the blood every 4 hours. A patient is given one 1,000-milligram dose of a drug. The following function was derived to calculate the amount of the drug in the blood system t days after taking the drug:

$$A(t) = 1000\left(0.75^t\right)$$

A blood test is able to detect the presence of the drug if there is at least 0.01 mg in the blood. How many days will it take before the test will come back negative?

This question requires that we create a new function, an inverse function, whose domain is amount of the drug in the blood system, and whose range is the amount of time since the drug was ingested. For this problem, the inverse function is t, where A is the amount of the drug in the system:

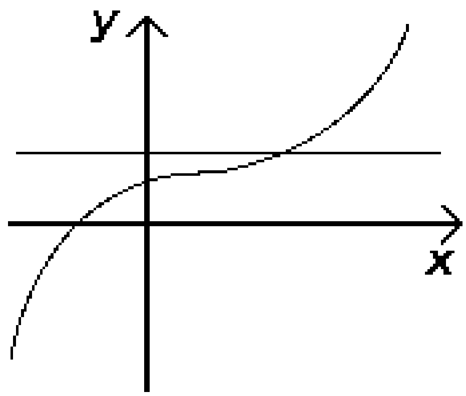
$$t = \frac{\ln A - \ln 1{,}000}{\ln 0.75}$$

We will continue to explore this problem later in this section, in an example. First, we will introduce and discuss the inverse function.

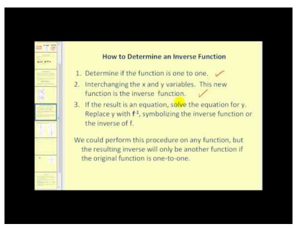
Algebraic Conditions for the Inverse

Prior to determining if a function has an inverse, we must 1st determine if the function is one-to-one. A **one-to-one function** is a function in which every element in the range corresponds to exactly one element in the domain. In other words, there is a one-to-one pairing between the elements in the range and domain.

One method to determine if a function is one-to-one is the horizontal line test. The **horizontal line test** states that a function is one-to-one if any horizontal line drawn through the graph of the function intersects the graph at only one point:



A function has an inverse only if it is a one-to-one function.



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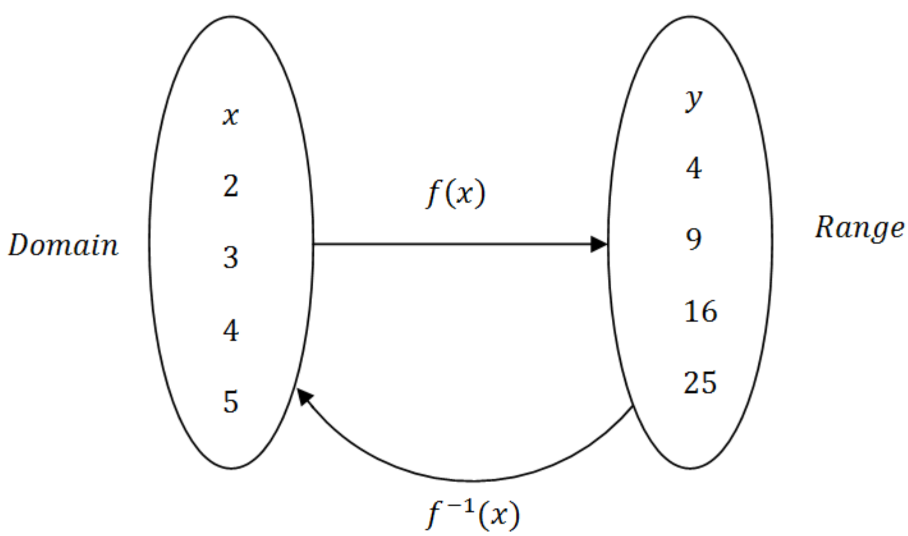
Inverse Notation

A function is written as $f(x)$ and its inverse is written as $f^{-1}(x)$. A common misconception is to see the -1 and interpret it as an exponent and write $\frac{1}{f(x)}$, but this is not correct. Instead, $f^{-1}(x)$ should be viewed as a new function whose domain is the range of $f(x)$ and whose range is the domain of $f(x)$.



In order for two functions to truly be inverses of each other, the following must hold algebraically:

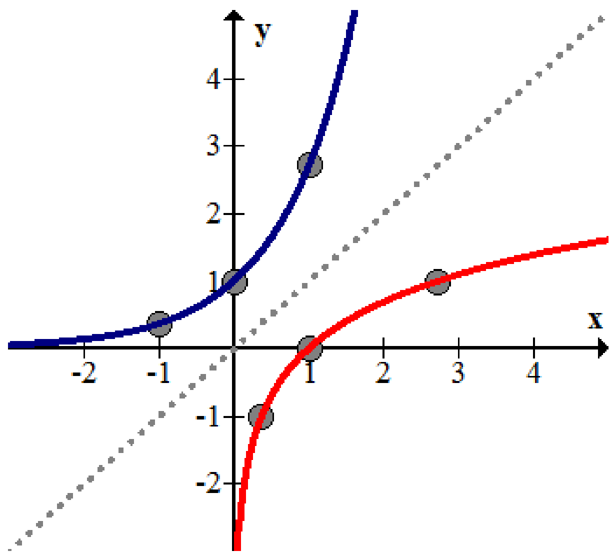
$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

The resulting x is in the domain of f for the 1st equation and in the range of f for the 2nd equation.

Sometimes the inverse of a function can be found algebraically:

1. In the original function, let $y = f(x)$.
2. Then, switch the variables x and y.
3. Next, solve for y in terms of x.
4. Set $y = f^{-1}(x)$. This is the inverse function.
5. Always verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Graphically, inverses are reflections in the line $y = x$. Observe the graphs of $y = e^x$ and $y = \ln x$. Notice how the (x, y) coordinates in one graph become (y, x) coordinates in the other graph:



In order to decide whether an inverse is also actually a function, you can use the vertical line test on the inverse function. You can also use the horizontal line test on the original function.

Learn, Play, and Explore with Inverse Functions: Inverse Functions

Examples

Example 1

Determine the inverse for the function $f(x) = y = (x+1)^2 + 4$, where $x \geq -1$, and then verify the inverse algebraically.

Solution:

To find the inverse, switch x and y then solve for y:

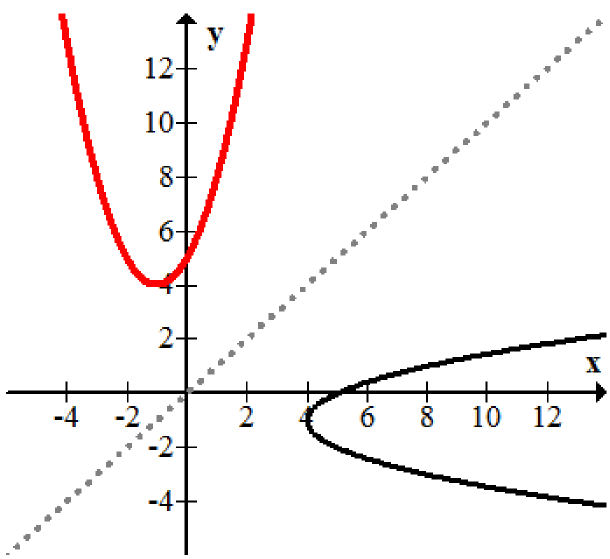
$$x = (y+1)^2 + 4$$
$$x - 4 = (y+1)^2$$
$$\sqrt{x-4} = y+1$$
$$-1 + \sqrt{x-4} = y = f^{-1}(x)$$

To verify algebraically, you must show $x = f(f^{-1}(x)) = f^{-1}(f(x))$:

$$\begin{aligned} f(f^{-1}(x)) &= f\left(-1+\sqrt{x-4}\right) \\ &= ((-1+\sqrt{x-4})+1)^2+4 \\ &= \left(+\sqrt{x-4}\right)^2+4 \\ &= x-4+4 = x \end{aligned}$$

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}((x+1)^2+4) \\ &= -1+\sqrt{((x+1)^2+4)-4} \\ &= -1+\sqrt{(x+1)^2} \\ &= -1+x+1 = x \end{aligned}$$

Note the original function had its domain restricted so it would pass the horizontal line test. Otherwise, the inverse relation would not have passed a vertical line test and would not have been a function.



Example 2

Find the inverse of the function and then verify that $x = f(f^{-1}(x)) = f^{-1}(f(x))$.

$f(x) = y = \frac{x+1}{x-1}$

Solution:

Sometimes it is quite challenging to switch x and y and then solve for y. You must be careful with your algebra.

$$\begin{aligned} x &= \frac{y+1}{y-1} \\ x(y-1) &= y+1 \\ xy - x &= y+1 \\ xy - y &= x+1 \\ y(x-1) &= x+1 \\ y &= \frac{x+1}{x-1} \end{aligned}$$

This function turns out to be its own inverse. Since they are identical, you only need to show that $x = f(f^{-1}(x))$.

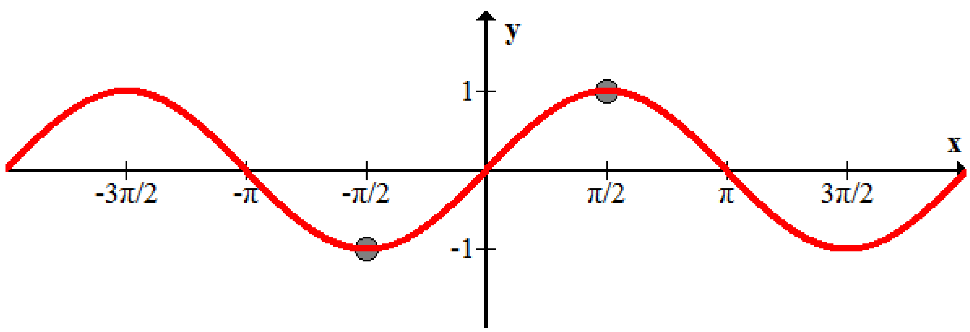
$f\left(\frac{x+1}{x-1}\right) = \frac{\left(\frac{x+1}{x-1}\right)+1}{\left(\frac{x+1}{x-1}\right)-1} = \frac{x+1+x-1}{x+1-(x-1)} = \frac{2x}{2} = x$

Example 3

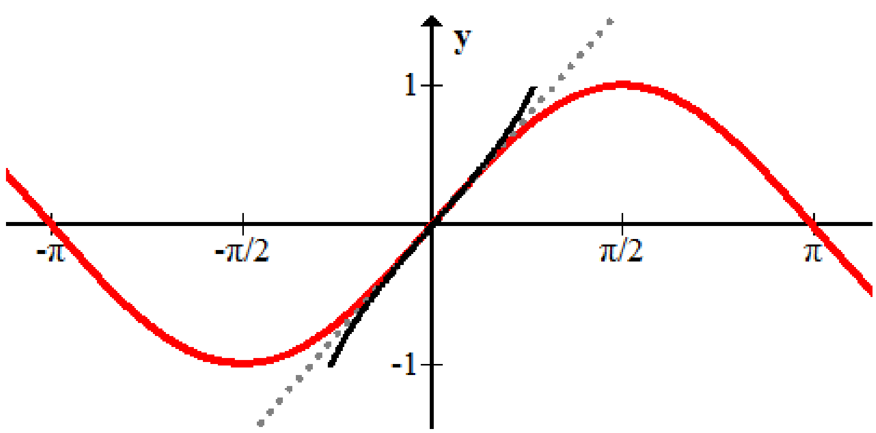
What is the inverse of $f(x) = y = \sin x$?

Solution:

The sine function does not pass the horizontal line test, so its inverse is not a function:



However, if you restrict the domain to just the part of the x-axis between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, then it will pass the horizontal line test and the inverse will be a function:



The inverse of the sine function is called the arcsine function, $f(x) = \sin^{-1}(x)$, and is shown in black. It is truncated so that it only inverts a part of the whole sine wave. You will study periodic functions and their inverses in more detail later.

Example 4

Determine the inverse of $f(x) = 5 + \frac{x}{2}$. Verify that the inverse is actually the inverse.

Solution:

To find the inverse,

$$\begin{aligned}
y &= 5 + \frac{x}{2} \\
x &= 5 + \frac{y}{2} \\
x - 5 &= \frac{y}{2} \\
2(x-5) &= y = f^{-1}(x)
\end{aligned}$$

Verification:

$2\left(5 + \frac{x}{2} - 5\right) = 2\left(\frac{x}{2}\right) = x$

$5 + \frac{2(x-5)}{2} = 5 + x - 5 = x$

They are truly inverses of each other.

Example 5

Determine if $f(x) = \frac{3}{7}x - 21$ and $g(x) = \frac{7}{3}x + 21$ are inverses of one another.

Solution:

Even though $f(x) = \frac{3}{7}x - 21$ and $g(x) = \frac{7}{3}x + 21$ have some inverted pieces, they are not inverses of each other. In order to show this, you must show that the composition does not simplify to x: $\frac{3}{7}\left(\frac{7}{3}x + 21\right) - 21 = x + 9 - 21 = x - 12 \neq x$

Example 6

Determine the inverse of $f(x) = \frac{x}{x+4}$.

Solution:

To find the inverse, switch x and y:

$$\begin{aligned}
f(x) = y &= \frac{x}{x+4} \\
x &= \frac{y}{y+4} \\
x(y+4) &= y \\
xy + 4x &= y \\
xy - y &= -4x \\
y(x-1) &= -4x \\
f^{-1}(x) = y &= -\frac{4x}{x-1}
\end{aligned}$$

Summary

- A function is **one-to-one** if every element in the range corresponds to exactly one element in the domain.
- The **horizontal line test** states that a function is one-to-one if any horizontal line drawn through the graph of the function intersects the graph at only one point.
- A function has an inverse only if it is a one-to-one function.

- Two functions are inverses if $x = f(f^{-1}(x)) = f^{-1}(f(x))$.
- To algebraically solve for the inverse function, switch the variables x and y in the function and then solve for y in terms of x.

Review

Use the function $f(x) = x^3$ for the following problems:

1. Sketch $f(x)$ and $f^{-1}(x)$.

2. Find $f^{-1}(x)$ algebraically. It is actually a function?

3. Verify algebraically that $f(x)$ and $f^{-1}(x)$ are inverses.

Use the function $g(x) = \sqrt{x}, x \geq 0$, for the following problems:

4. Sketch $g(x)$ and $g^{-1}(x)$.

5. Find $g^{-1}(x)$ algebraically. It is actually a function?

6. Verify algebraically that $g(x)$ and $g^{-1}(x)$ are inverses.

Use the function $h(x) = |x|$ for the following problems:

7. Sketch $h(x)$ and $h^{-1}(x)$.

8. Find $h^{-1}(x)$ algebraically. It is actually a function?

9. Verify graphically that $h(x)$ and $h^{-1}(x)$ are inverses.

Use the function $j(x) = 2x - 5$ for the following problems:

10. Sketch $j(x)$ and $j^{-1}(x)$.

11. Find $j^{-1}(x)$ algebraically. It is actually a function?

12. Verify algebraically that $j(x)$ and $j^{-1}(x)$ are inverses.

13. Use the horizontal line test to determine whether or not the inverse of $f(x) = x^3 - 2x^2 + 1$ is also a function.

14. Are $g(x) = \ln(x+1)$ and $h(x) = e^{x-1}$ inverses? Explain.

15. If you were given a table of values for a function, how could you create a table of values for the inverse of the function?

16. In many countries, the temperature is measured in degrees Celsius. In the U.S., we typically use degrees Fahrenheit. For travelers, it is helpful to be able to convert from one unit of measure to another.

1. The temperature at which water freezes will give us one point on a line in which $x$represents the degrees in Celsius and y represents the degrees in Fahrenheit. Water freezes at 0°C and 32°F, so the 1st point is (0, 32). The temperature at which water boils gives us the 2nd point (100, 212), because water boils at 100°C or 212°F. Use this information to show that the equation to convert from Celsius to Fahrenheit is $y = \frac{9}{5}x + 32$ or $F = \frac{9}{5}C + 32$.
2. Find the inverse of the equation above by solving for C, to derive a formula that will allow us to convert from Fahrenheit to Celsius.
3. Show that your inverse is correct by showing that the composition of the two functions simplifies to either F or C (depending on which one you put into the other).

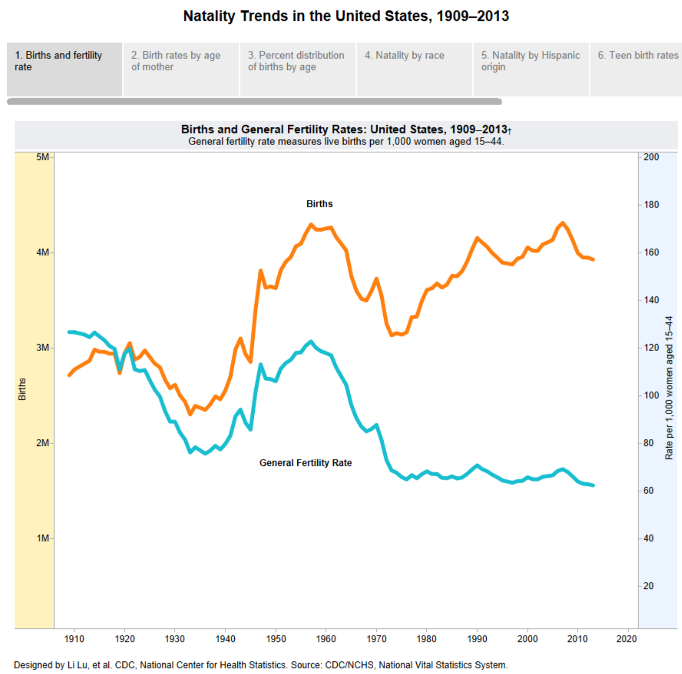
Review (Answers)

Please see the Appendix.

2.14 Project: Functions and Graphs

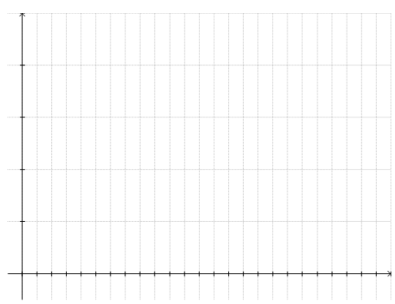
Demography is the statistical study of human populations throughout the world. The changes in a population's size are caused by changes in the birth rate, the death rate, and the net migration rate. The data gathered in such studies are used to create mathematical models that can be used to predict and measure fluctuations in these rates. Then demographic research focuses on why any of these rates change. Understanding a society's demography is an essential tool in determining current and future public health needs.

The Centers for Disease Control and Prevention hosts a website dedicated to providing such data and its visual display to help study these trends. The following dataset will be used to begin this project:

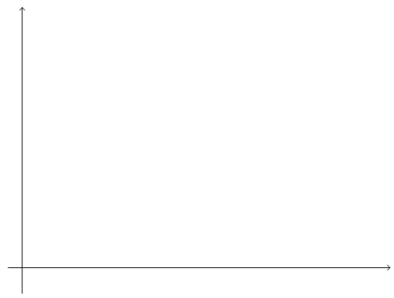


Use the techniques you learned from this chapter to begin an analysis of the Natality Trend Data for the number of births from 1990 to 2013. Use the following guidelines:

1. If the graph of this data defines a function, what is its domain and range?
2. Plot the births from 1990 to 2013.



3. Identify the following characteristics of the natality function:
    a. Intervals where the number of births are decreasing. Write a sentence to interpret this information.
    b. Intervals where the births are increasing. Write a sentence to interpret this information.
    c. Identify the coordinates of the local maximum. Write a sentence to interpret this information.
    d. Identify the coordinates of the local minimum. Write a sentence to interpret this information.
4. Let N(t) be defined by the function in the above graph. Assume several other communities had natality trends with the same pattern. Use function notation to describe the function that models their birth rate from 1990 to 2013.
    a. Country 1: Its birth rate is twice that of the U.S.
    b. Country 2: It has exactly same pattern as the U.S., but each year it has 1 million fewer live births.
    c. Country 3: It had the exact number of births, but its pattern lagged the U.S. by five years.
5. A new function is needed to answer questions about this data. Its domain should be number of births and the range, year. This is the inverse function for N(t). Graph the inverse.



2.15 Summary: Functions and Graphs

Chapter Summary

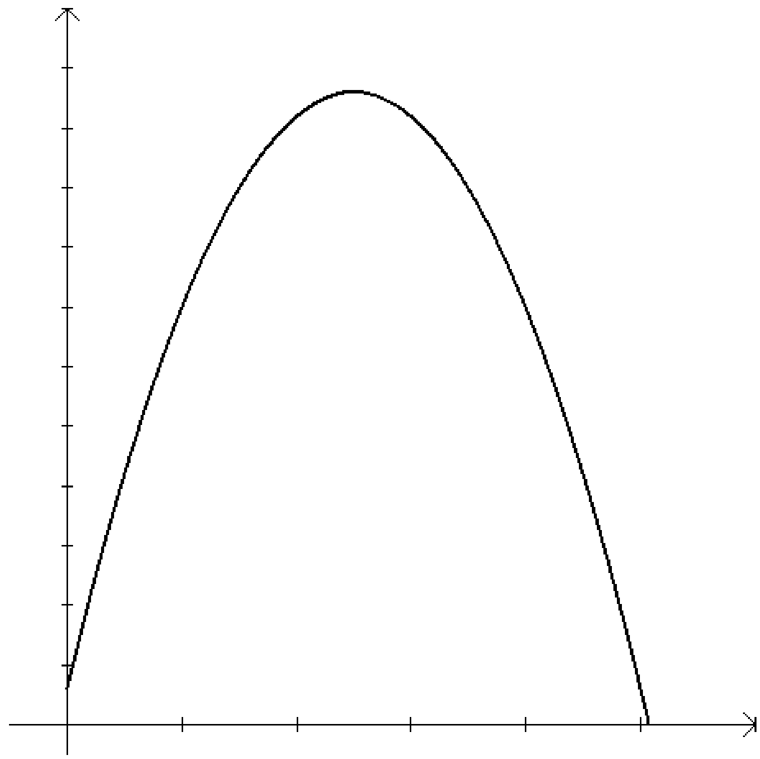
In this chapter, we learned about:

- Parent functions for the following function families: identity function, quadratic function, cubic function, square root function, reciprocal function, exponential function, logarithm function, sine function, absolute value function, and logistic function.
- These parent functions can be transformed using:
  - Vertical shift: $y = f(x) + d$ shift up, $y = f(x) - d$ shift down
  - Horizontal shift: $y = f(x + c)$ shift left, $y = f(x - c)$ shift right
  - Vertical stretch or shrink: $y = a \cdot f(x)$
  - Horizontal stretch or shrink: $y = f(ax)$
  - Reflection: $y = -f(x)$ over the x-axis, $y = f(-x)$ over the y-axis
- Domain is the set of inputs, and range is the set of outputs for a function.
- The zeros or roots of a function are the x-values where a function crosses the x-axis.
- Maximums and minimums are the highest points and the lowest points, respectively, of a function.
- A function is increasing if $f(b) \leq f(c)$ for any b and c in the interval whenever b, and a function is decreasing if $f(b) \geq f(c)$ whenever b.
- A function is even if $f(-x) = f(x)$, and is odd if $f(-x) = -f(x)$.
- A vertical asymptote is a vertical line that visually shows where the function is not defined. A horizontal asymptote is a horizontal line that a function approaches as x approaches positive or negative infinity.
- There are two types of discontinuities: removable and non-removable. There are two types of non-removable discontinuities: jump and infinite.
- A function has an inverse only if it is a one-to-one function. To algebraically solve for the inverse function, switch the variables in the function and solve for y in terms of x.

Chapter Application Problem

Earlier in this chapter, we explored a function used to model the height of a ball. This model is a helpful one to review the scope and depth of techniques discussed in this chapter. Recall that the height of a ball can be given by $h(t) = -16t^2 + 80t + 6$,where t is the time in seconds since the ball left the thrower's hand.

The graph of $h(t)$ can be seen below:



This graph is related to the "squaring" family of functions. The function can be rewritten into vertex form by completing the square:

$$h(t) = -16t^2 + 80t + 6$$
$$h(t) = -16\left(t^2 - 5t + \frac{25}{4}\right) + 6 + 100$$
$$h(t) = -16\left(t - \frac{5}{2}\right)^2 + 106$$

The height function is a transformation of the squaring function, shifted to the right units, reflected in the horizontal axis, stretched vertically by a factor of 16, and shifted vertically up by 106 units. It is also clear that the path of the ball follows this graph during its flight.

Connecting this function to the family of squaring functions provides a wealth of information from the study of that function:

- The ball will reach the maximum height when $t = \frac{5}{2}$.
- The ball's height increases until $t = \frac{5}{2}$, and then its height decreases until its height is 0 feet.
- The domain of the function is about $[0, 5]$, which is the time the ball is in the air.
- Like the squaring function, the path of the ball is symmetrical about the vertical line $t = \frac{5}{2}$,which goes through its maximum. It is clear that the rate at which the ball increases in height until that point is the same as the rate at which it descends after that point. This is understood by any ball player preparing to catch a ball.

- The function has the following intercepts, which further explain the path of the ball: The point (0,6) reflects the fact that the ball was thrown from a height of 6 feet. The point (5.073,0) says that the ball hit the ground at about $t = 5.073$ seconds.
  - x-axis intercept (5.073, 0)
  - y-axis intercept (0,6)
- It is also clear that since this function does not pass the horizontal line test, it does not have an inverse.

Because this type of function has so many applications, we will be study it in detail in the next chapter.

Review

Try the following cumulative review problems to practice the concepts in this chapter:

2.16 References

1. Annca. https://pixabay.com/photo-1787012/ .
2. CK-12 Foundation. .
3. CK-12 Foundation. .
4. CK-12 Foundation. .
5. CK-12 Foundation. .
6. CK-12 Foundation. .
7. CK-12 Foundation. .
8. CK-12 Foundation. .
9. Open Clip Art Library. http://colouringbook.org/art/svg/coloring-book/roller-coaster-coloring-book-colouring-coloring-book-colouring-book-colouringbook-org-art-clip-art-clipart-clipartist-net-openclipart-org-scalable-vector-graphics-svg/ .
10. Rob Young. https://commons.wikimedia.org/wiki/File:Will_%26_Grace_Apartment_Building_(1149735229).jpg .
11. David Stanley. https://www.flickr.com/photos/davidstanleytravel/8227858392/in/photolist-dx4VDs-nKC9j9-My5FPU-9rfFDj-rXVj6c-G9QvSn-FTuBdG-NVoCS3-8qRnga-4ykGxj-jeRZix-Nm65u9-MuS8RM-MtrYpq-nRand5-LSSMHM-GjMCi9-aaGbZM-opao7s-2Cact-FbjHE8-5nGfWn-pniMjM-6C6LQc-xzgo7p-4jAyjp-dQwCx8-Nh11cH-qrnGo4-qq1pzb-j5K9gk-9Pu1uQ-8BxKVf-5J85eG-j5LBAg-cfh4yL-5nLVFj-jeTveb-PSNJ1-K2fTUM-sngP1k-dBmaU6-hMPbWH-pk7mb-6R8PPi-ab6auH-KzX648-4oyMEc-9NHJGZ-Ny57L3 .
12. Paula Evans. .
13. Paula Evans. .
14. daveynin. https://www.flickr.com/photos/daveynin/1299342518 .
15. Goedeker's. https://www.flickr.com/photos/goedekers/12776966415 .
16. Mike Mozart. https://www.flickr.com/photos/jeepersmedia/14896792981 .
17. Paul Keleher. https://commons.wikimedia.org/wiki/File:Boston_Police_Special_Operations_Unit.jpg .
18. Paula Evans. .
19. Jhaymesisviphotography. https://www.flickr.com/photos/jhaymesisvip/6497720753/ .
20. Paula Evans. .
21. Les Chatfield. https://www.flickr.com/photos/elsie/7890051894/ .
22. Iqbal Osman. https://www.flickr.com/photos/82066314@N06/9617266704/in/photolist-fDR1kb-cpUoNL-cpUpej-cpUoSS-cpUoQL-iJdnV6-8nrUTX-3PK6Rd-9PG5sy-8nv3rN-8nv2EJ-8nrUfM-gbv1y2-cpUp4u-fDzJkV-6x15Mh-5zLUeS-fxQWGW-3gtex-fxPKCG-5DsSTE-bQSAS2-8nrUHn-8nrU8p-okzDVK-8nrTZV-8nrUq2-dths1d-4gBBFg-dthiE2-boWcAU-rHqHJZ-dthtvX-g1EDcJ-gwmFi5-boWcbd-gwmN7H-gwm16W-gwmAC2-bsyr7P-bBRunZ-bBR9wB-bBR5QZ-boWzZb-bBRvwZ-bBRuHg-4yWMn-dhzHT6-4sQkZq-9wWVth .
23. . http://blogs.cdc.gov/nchs-data-visualization/us-natality-trends/ , National Center for Health Statistics: Natality Trends in the United States, 1909-2013. .

CHAPTER 3 Power, Polynomial, and Rational Functions

Chapter Outline

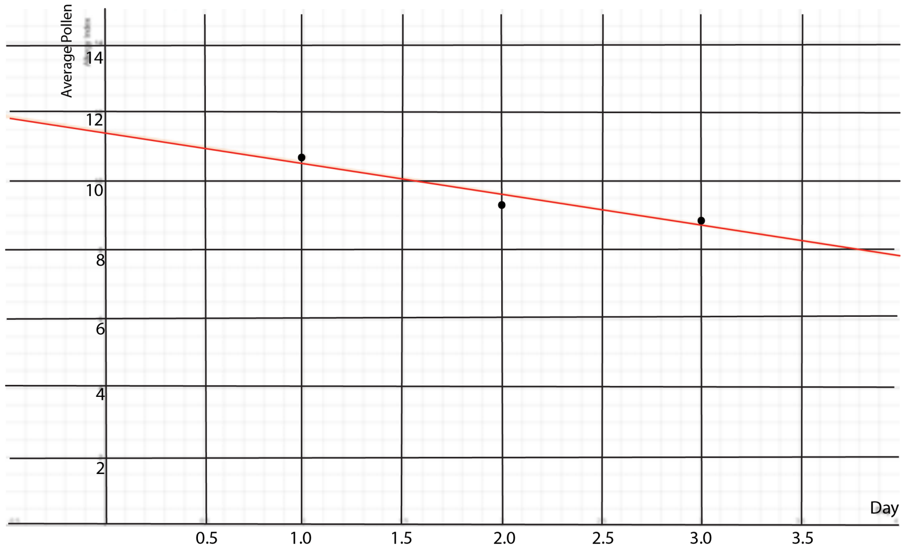
3.1 Introduction: Power, Polynomial, and Rational Functions

Polynomial functions are the most useful functions in the branch of mathematics that supports calculus. These functions are first studied in algebra in the form of the equation of a straight line. For example, consider the sample data for pollen levels in Austin, Texas, during March of 2016:

TABLE 3.1:

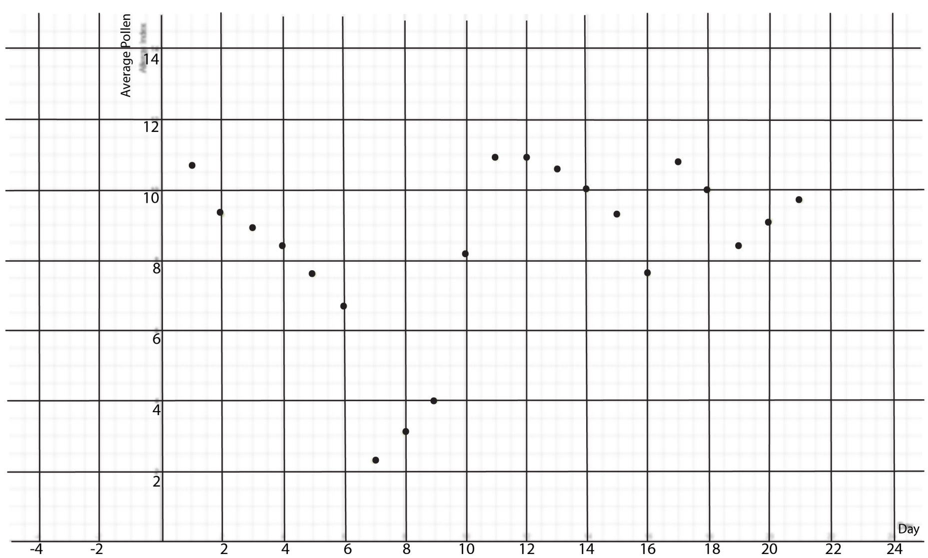
Date	**Pollen Level** (grains per cubic meter of air)
3/3/2016	10.7
3/4/2016	9.3
3/5/2016	8.9

An examination of the data shows that the pollen level decreases at a fairly constant rate. For any point in time within these dates, the linear function $P(t) = -0.9t + 11.43$, where t is the number of days since 3/2/2016, is a good approximation of the pollen level in Austin.



For example, if the pollen level is 10.7 at 8 p.m. on the evening of 3/3/2016, we can then determine that $t = 1.5$ is the number of days that have passed since 8 a.m. on 3/2/2016, and substitute into the linear function. $P(1.5) = (-0.9)(1.5) + 11.43$ calculates to $P(1.5) = 10.083$, which provides a good estimate for the level at 8 p.m. From the graph, it appears that the pollen level would be about 10.083 because the three data points show a steady decrease. Linear functions, then, are models that provide predictive tools for values within an interval or for the short term.

Consider the data for the pollen level over a longer term:



Over a longer period of time, the pollen level does not follow a linear pattern. The pollen level increases, decreases, and reaches maximum and minimum values. Also, for any day, the pollen level is defined. Pollen levels have been in the atmosphere for years, and they will continue for years.

Polynomial functions can be adjusted to measure data. This class of functions can measure behavior while at the same time be predictable. Their domain is the set of all real numbers, so they can be used for any dataset.

This chapter studies the characteristics of polynomial functions. These functions are introduced with the quadratic function. Their domain, range, and behavior are developed along with real-world examples. The chapter concludes with a study of the family of a closely related function: the rational function.

3.2 Quadratic Functions

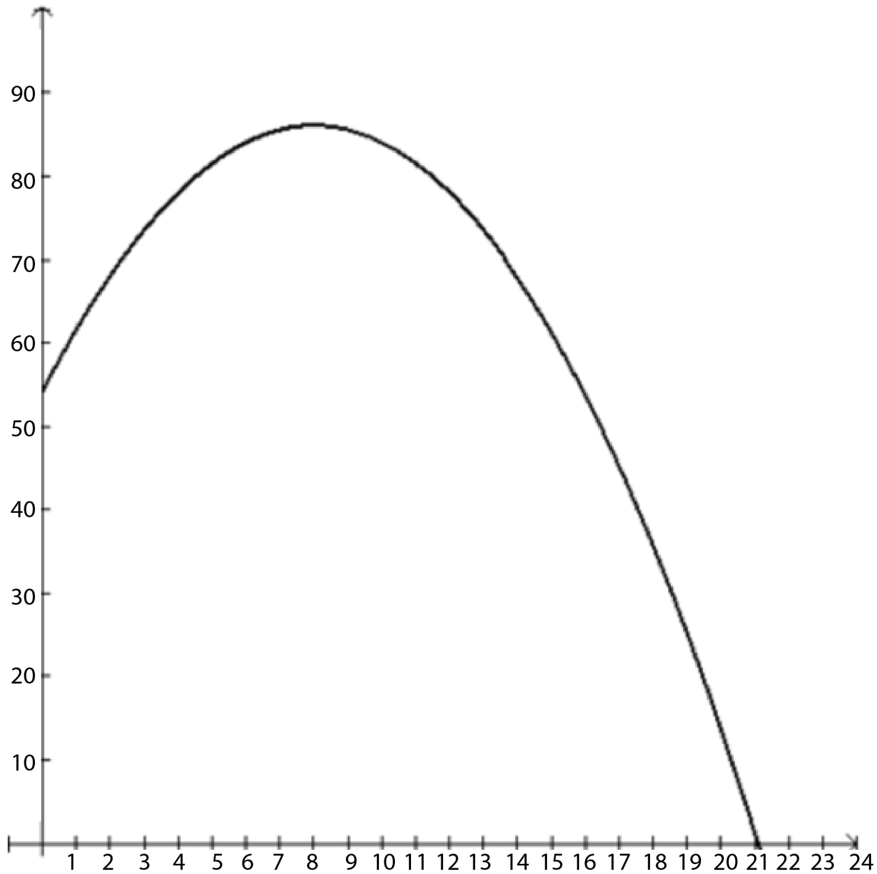
Learning Objectives

Learn about quadratic functions. More specifically, the different forms of writing a quadratic function and the characteristics of its graphs.

Introduction

Quadratic functions are a special group of polynomial functions. They will be used to establish the most important characteristics of polynomials. Consider the function that gives the height of a balloon launched from a roof 54 feet above the ground, $H(t) = -\frac{1}{2}t^2 + 8t + 54$, $0 \le t \le 21$, where t is the number of seconds after launch.

The graph demonstrates that this function has one maximum value of 86 feet, at 8 seconds after launch. The path of the balloon is smooth, continuous, and predictable. In this chapter we will develop techniques so that when predicting behavior based on the quadratic function, we can establish that there is only one extreme value (which can be a minimum or maximum). For this example, the function increases up to $t = 8$ and then decreases, so this function has a maximum at $t = 8$:



Quadratic Functions

Forms of Quadratic Functions

$H(t) = -12t^2 + 8t + 54$ is an example of a quadratic function written in standard form.

- **Standard form:** $f(x) = ax^2 + bx + c$,where a, b, and c are real numbers and $a \neq 0$.
- **Vertex form:** $f(x) = a(x-h)^2 + k$,where $h = \frac{-b}{2a}$ and $k = f(h)$.
- **Factored form:** $f(x) = a(x-r_1)(x-r_2)$, where r_1 and r_2 are the roots of the function.

Characteristics of Quadratic Functions

When the quadratic function is written in vertex form, it is easy to see that it is related to the squaring family of functions. Clearly, $f(x) = a(x-h)^2 + k$ expresses the following transformations applied to the parent function, $f(x) = x^2$:

- Shift horizontally h units.
- If $a > 1$, stretch vertically by a factor of a units.
- If $0 < 1$, shrink vertically by a factor of a units.
- If $a < 0$, reflect across the x-axis and then stretch or shrink by a factor of a units.
- Shift vertically k units.

The characteristics of the squaring family of functions, the quadratic function in standard form, and the parabola can be applied:

- The graph has a vertex at $(h,k) = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.
- The axis of symmetry is the line that passes through the vertex, $x = h$, that creates two symmetric halves of the parabola.
- The function has a y-intercept at $(0,c)$ and zeros at $\left(\frac{-b+\sqrt{b^2-4ac}}{2a}, 0\right)$ and $\left(\frac{-b-\sqrt{b^2-4ac}}{2a}, 0\right)$. If the zeros are real, then they are the $x-$intercepts.
- The graph is symmetrical about the vertical line, $x = h$, called the axis of symmetry.
- The function has an extreme value, k, which occurs when $x = h$.
- If $a < 0$:
  - The function has an absolute maximum.
  - The domain is $(-\infty, \infty)$, and the range is $(-\infty, k]$.
  - The graph increases on $(-\infty, h)$ and decreases on (h, ∞).
  - The end behavior is $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.
- If $a > 0$:
  - The function has an absolute minimum.
  - The domain is $(-\infty, \infty)$, and the range is $[k, \infty)$.
  - The graph decreases on $(-\infty, h)$ and increases on (h, ∞).
  - The end behavior is $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$.

Examples

Example 1

Sketch the graph of the function $y = f(x) = x^2 + 2x - 3$.

Solution:

Step 1: For the $y-$intercept, $(0, f(0)) = (0, -3)$.

For the $x-$intercepts, solve

$$\begin{aligned} x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \\ x &= -3, 1. \end{aligned}$$

Therefore, the $x-$intercepts are (-3, 0) and (1, 0).

Step 2: The vertex (extreme point) is at

$$x = \frac{-b}{2a} = \frac{-2}{2(1)} = -1$$

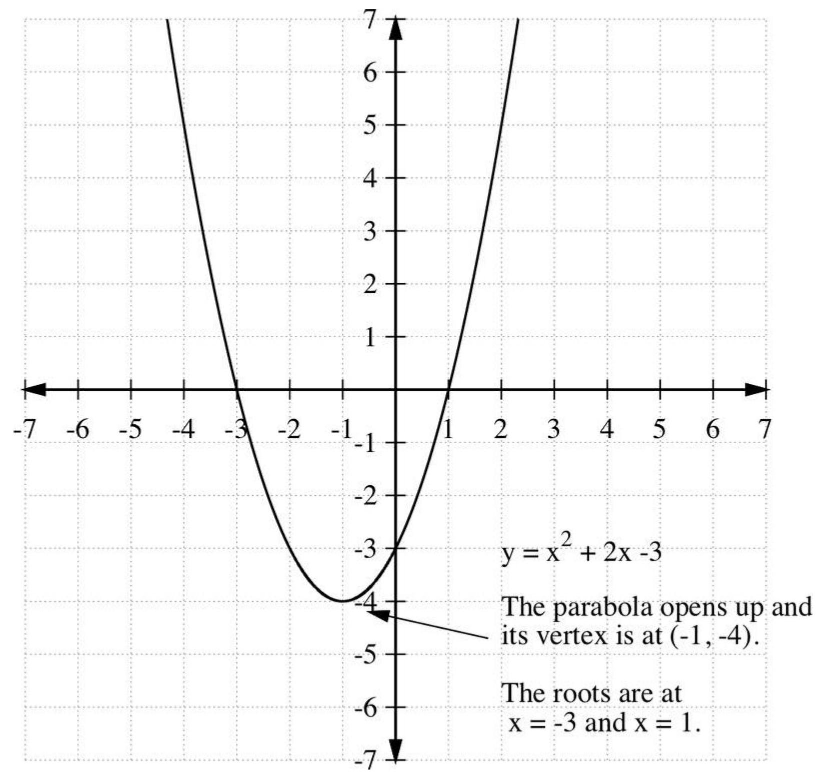
and

$$
\begin{aligned}
f(-1) &= (-1)^2 + 2(-1) - 3 \\
&= -4,
\end{aligned}
$$

so the vertex is (-1, -4).

Step 3: Since the coefficient of x^2 is positive, $a > 0$, the extreme point is a minimum and the parabola opens up. The domain is $(-\infty, \infty)$, and the range is $[-4, \infty)$.

Step 4: Plot these values and sketch a smooth parabola:



Example 2

Rewrite the quadratic $g(x) = x^2 + 6x + 7$ in vertex form and then graph.

Solution:

Step 1: Rewrite the given quadratic in vertex form.

Use the technique of completing the square to write the function in vertex form. Add and subtract $\left(\frac{b}{2}\right)^2$ to the righthand side of the equation:

$$
\begin{aligned}
g(x) &= x^2 + 6x + 7 \\
&= x^2 + 6x + 9 + 7 - 9.
\end{aligned}
$$

Group the 1st three terms on the right side of the equation and factor these terms:

$$\begin{aligned} g(x) &= \left(x^2+6x+9\right)+7-9 \\ &= (x+3)^2-2. \end{aligned}$$

Thus, $a = 1$ and the vertex of this parabola is (-3, -2).

Step 2: Use the vertex to determine the minimum and the range.

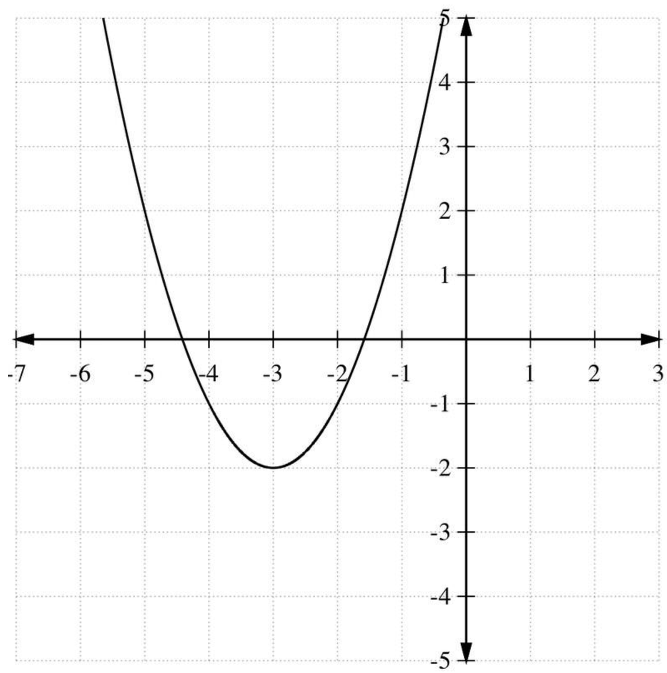
The function has a minimum value of -2 at $x =$ -3. The domain is $(-\infty, \infty)$, and the range is $[-2, \infty)$.

Step 3: Determine the intercepts.

For the y-intercept, $(0, g(0)) = (0, 7)$. For the x-intercept, solve: $0 = x^2+6x+7$ to obtain $\left(-3+\sqrt{2}, 0\right)$ and $\left(-3-\sqrt{2}, 0\right)$.

Step 4: Examine the leading coefficient, $a = 1$, which is positive, so the parabola opens up.

Step 5: Plot these values and sketch a parabola:



Example 3

For $H(t) = -\frac{1}{2}t^2+8t+54$, which models the height of a balloon launched from a height of 54 feet at $t = 0$, use the intercepts, vertex, and symmetry to explain why:

- It is launched from a height of 54 feet.
- Its domain is $0 \le t \le 21$ and the range is $0 \le H(t) \le 86$.
- The maximum height is 86 feet and that level is attained once.

Solution:

Step 1: For the y-intercept, $(0, H(0)) = (0, 54)$.This means that the height of the balloon is 54 feet when the balloon is launched.

For the x-intercepts, solve

$$\begin{aligned}0 &= -\frac{1}{2}t^2+8t+54\\0 &= t^2+16t+54\\t &= 8\pm 2\sqrt{43}.\end{aligned}$$

Since negative time does not make sense, the only intercept that makes sense is $\left(8+2\sqrt{43},0\right)\approx(21.1,0)$.The domain of a quadratic function is $(-\infty,\infty)$ and must be restricted for this problem to $[0,21.1]$. The balloon is launched when $t=0$ seconds and lands when $t\approx 21.1$ seconds.

Step 2: The vertex is at $\left(\frac{-8}{2(-\frac{1}{2})},H\left(\frac{-8}{2(-\frac{1}{2})}\right)\right)=(8,86)$.This is the single extreme point.

Step 3: The leading coefficient is negative, so the parabola opens downward. The extreme point is a local maximum and an absolute maximum. The function increases on the interval $(0,8]$ and decreases on the interval $[8,21.1)$. Since the function increases and then decreases, the maximum height of 86 feet is achieved exactly once.

Example 4

Sketch the graph of the quadratic function $f(x)=-x^2+4x$.

Solution:

Step 1: The $y-$intercept is at $(0,f(0))=(0,0)$. For the $x-$intercept, solve

$$\begin{aligned}-x^2+4x &= 0\\-x(x-4) &= 0\\x &= 0,4.\end{aligned}$$

$(0,0)$ and $(4,0)$ are the $x-$intercepts.

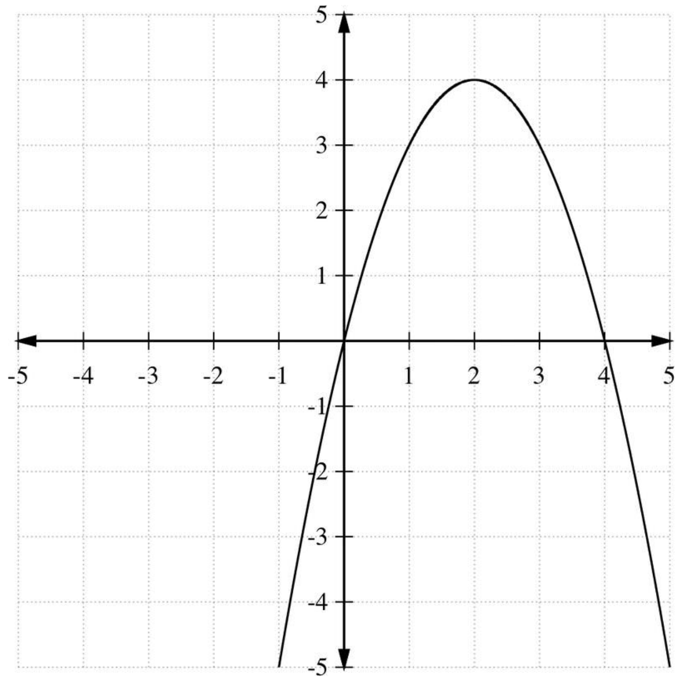
Step 2: The vertex is

$$\begin{aligned}(h,k) &= \left(\frac{-b}{2a},f\left(\frac{-b}{2a}\right)\right)\\(h,k) &= (2,4).\end{aligned}$$

Step 3: $a=-1$, so the parabola opens down and the maximum value is 4 and occurs at $x=2$.

The domain is $(-\infty,\infty)$ and the range is $(-\infty,4]$.

Step 4: Finally, the graph can be obtained by sketching a parabola through the points determined above. Note that the axis of symmetry, $x=2$, would divide this parabola evenly in half.



Example 5

A golf ball is shot from the ground level.

Its height is given by $h(t) = 121t - 4.9t^2$, where t is the time in seconds and $h(t)$ is the height of the golf ball (in meters) above the ground at time t. Find:

1. h when $t = 1$ second,
2. the maximum height reached by the ball, and
3. the graph of the trajectory of the ball.

Solution:

Step 1: At $t = 1$ second, $h(1) = 121(1) - 4.9(1)^2 = 121 - 4.9 = 116.1$ meters.

Step 2: To find the maximum height reached by the ball, the vertex is

$$(t,h(t)) = \left(\frac{-(121)}{2(-4.9)}, f\left(\frac{-(121)}{2(-4.9)}\right)\right)$$
$$(t,h(t)) = (12.3, h(12.3))$$
$$(t,h(t)) = \left(12.3, 121(12.3) - 4.9(12.3)^2\right)$$
$$(t,h(t)) = (12.3, 747).$$

The maximum height is reached at time 12.3 seconds. The maximum height is approximately 747 meters.

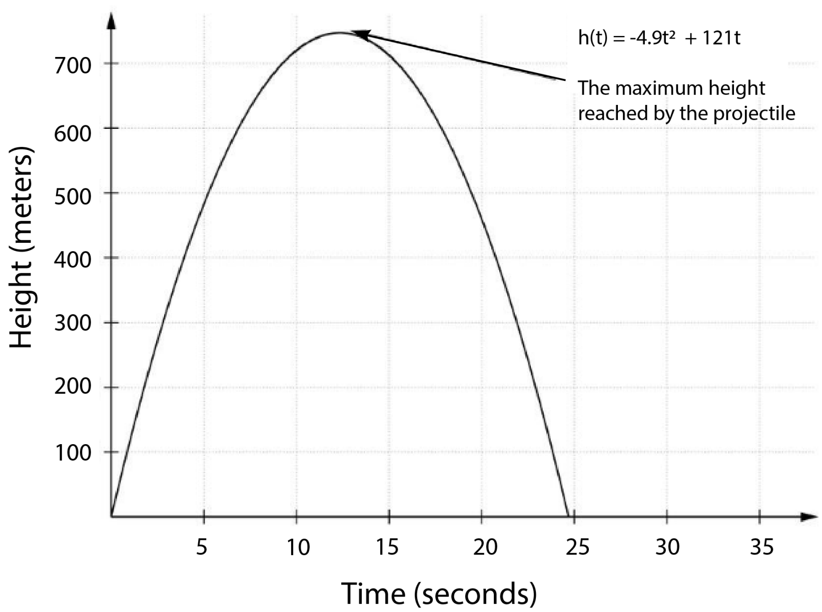
Step 3: To graph the function accurately, the $t-$intercepts are found:

$$121t - 4.9t^2 = 0$$
$$(121 - 4.9t)t = 0.$$

And the $t-$intercepts are at 0 seconds and approximately 24.7 seconds.

The $h-$intercept is $(0, h(0)) = (0,0)$.

Since the leading coefficient is negative, the parabola opens down. From this information, construct the graph. Notice that the graph illustrates that the domain is $[0, 24.7]$ and the range is $[0, 747]$:



Example 6

Find the quadratic function in factored form with the $x-$intercepts $(-7,0)$ and $(3,0)$. The quadratic function also passes through the point $(-1,48)$.

Solution:

Step 1: Use the factored form

$f(x) = a(x-(-7))(x-3) = a(x+7)(x-3)$.

Step 2: Since the point $(-1, 48)$ is on the graph,

$$\begin{aligned} 48 &= a(-1+7)(-1-3) \\ 48 &= a(-24) \\ a &= -2. \end{aligned}$$

Step 3: $f(x) = -2(x+7)(x-3)$.

Summary

- Quadratic functions are a special type of polynomial with forms:
  - Standard form: $f(x) = ax^2 + bx + c$
  - Vertex form: $f(x) = a(x-h)^2 + k$
  - Factored form: $f(x) = a(x-r_1)(x-r_2)$
- Quadratic functions have one extreme value located at the vertex, $(h,k) = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.
- The graph of a quadratic function is a parabola, which opens up if $a > 0$ and down if $a < 0$.

Review

1. What is the U-shaped graph of a quadratic function called?
2. Which direction does a parabola open if the leading coefficient (a) is positive?
3. For $y^2 = x$, if the coefficient of y is positive, which way does the parabola open?
4. What is the name of the lowest point of a parabola that opens up, and the highest point of a parabola that opens down?
5. What is the name of the line passing through the vertex that divides the parabola into two symmetric parts?
6. Sketch the graph of $y = x^2 + 3$.
7. Sketch the graph of $y = -x^2 + 4x - 4$.
8. Sketch the graph of $y = 2x^2 + 8x$.
9. Consider the following quadratic function: $y = -x^2 - 2x + 1$. a) Which direction does it open? b) What is the vertex? c) Is it transformed in any way?
10. Consider the quadratic functions: $y = 2x^2$, $y = 4x^2$, $y = 6x^2$. Which quadratic function would you expect to have the narrowest parabola? Explain your answer.

Sketch the graph of each function:

11. $y = -x^2$
12. $y = 3x^2 + 6x + 1$
13. $y = \frac{1}{2}x^2 + 2x + 4$
14. $y = (x-3)^2 + 4$
15. $y = -x^2 - 8x - 17$

For questions 16-19, refer to the following scenario:

The quadratic function $y = -0.05x^2 + 1.5x$ can be used to represent the path of a football kicked 30 yards down the field. The variable x represents the distance, in yards, the ball has traveled down the field. The height, in yards, of the football in the air is represented by the variable y.

Use the quadratic function to calculate the height of the ball as it travels down the field. Round your answers to the nearest hundredth of a yard.

TABLE 3.2:

Distance Down the Field (yds)	Height in the Air (yds)
0.0	
5.0	
10.0	
15.0	
20.0	
25.0	
30.0	

16. What is the maximum height of the football during the kick?
17. How far down the field has the football traveled when it reaches its maximum height?
18. Use the information in the table to graph the path of the football kick.
19. If you were shown only the graph of this quadratic function, how could you determine the maximum height of the football during the kick, and how far down the field the football has traveled when it reaches its maximum height?

Review (Answers)

Please see the Appendix.

Resources

Technological Tools

You can graph quadratic functions using your computer's graphing program or with a TI-83/84 calculator. Below are basic directions for graphing a quadratic function and finding the vertex and $x-$intercepts, or zeros, using the functions on a TI-83 or TI-84 graphing calculator:

1. Enter the function using the $Y =$ button.
2. Set a standard window with **ZOOM-6** to view the graph. Adjust the window using the **WINDOW** menu. Use the $y-$intercept to help you find the best values for **XMAX, XMIN, YMAX, YMIN,** and the $X-$ and $Y-$ scales for viewing the graph of the parabola.
3. Press **GRAPH**.
4. Adjust and refine the view using **ZOOM** or by changing the **WINDOW** settings.

Finding the Vertex

You can use functions built into the calculator to find the vertex of any parabola you graph:

1. Follow the directions above to graph a quadratic function.
2. From the Graph screen, press **2ND TRACE** (this is the **CALC** Menu).
3. Scroll down and choose **MINIMUM** or **MAXIMUM**.
4. The calculator prompts **LEFT BOUND**? Use the arrow keys (<or >) to place the cursor to the left of the vertex, and press **ENTER.**
5. The calculator then prompts **RIGHT BOUND**? Use the arrows to place the cursor to the right of the vertex and press **ENTER**.
6. Finally, the calculator will prompt for a guess with **GUESS**? Press **ENTER** again.
7. The calculator will display the $x-$ and $y-$coordinates of the vertex.

Finding the Zeros

1. From the Graph screen, press **2ND TRACE** (This is the **CALC** Menu).
2. Choose **ZERO**.
3. The calculator prompts **LEFT BOUND**? Use the arrow keys (<or >) to place the cursor to the left of the zero and press **ENTER**.
4. The calculator then prompts **RIGHT BOUND**? Use the arrows to place the cursor to the right of the zero and press **ENTER**.
5. Finally, the calculator will prompt for a guess with **GUESS**? Press **ENTER** again.
6. The calculator will display the $x-$coordinate of the zero (x-intercept).
7. Repeat steps 1-6 to find the other zero of the function.

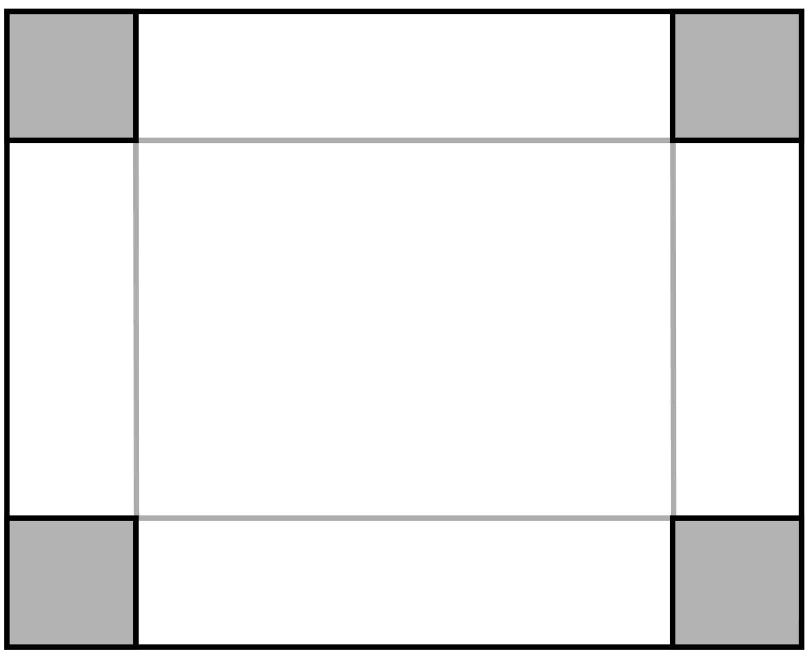
3.3 Polynomial Functions

Learning Objectives

Learn about polynomial functions and characteristics of their graphs.

Introduction

A box manufacturer constructs open boxes by cutting squares from each corner of an 8-inch by 10-inch rectangular piece of cardboard, and folding up the sides.



The volume of the box is given by $V(x) = x(8-2x)(10-2x)$,where x is the length of each square cut from the paper. The manufacturer wants to know the size of the square to cut from each corner that will yield the box with greatest volume. To answer this question, it is helpful to look at the equation in standard form. Polynomials in standard form begin with the term of the highest degree (in the case of polynomials, the one with the greatest exponent) and continue until the last term is the smallest degree. First, we must distribute in order to determine the degree of each term:

$$V(x) = x(8-2x)(10-2x) = 4x^3 - 36x^2 + 80x.$$

This reveals that $V(x)$ is a polynomial function. Polynomials represent a large group of models with similar characteristics. Analyzing these functions provides tools to answer questions like the one posed by the box manufacturer.

Standard Form of Polynomial Functions

If $P(x)$ is a polynomial function, then

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0,$$

where the coefficients $a_n,\ a_{n-1},\ \ldots,\ a_1,\ a_0$ are real numbers and the exponents are positive integers.

The 1st nonzero coefficient, a_n, is the **leading coefficient**. The term $a_n x^n$ is called the leading or dominant term. The **degree** of the polynomial is n. For example, the quadratic equation $f(x) = -2x^2 + 3x - 5$ has a leading coefficient of -2 and a leading term of $-2x^2$, and is of degree $n = 2$. The polynomial $f(x) = 1$ is a polynomial with a leading coefficient of 1, and the leading term is $1x^0 = 1$, so the degree is $n = 0$.

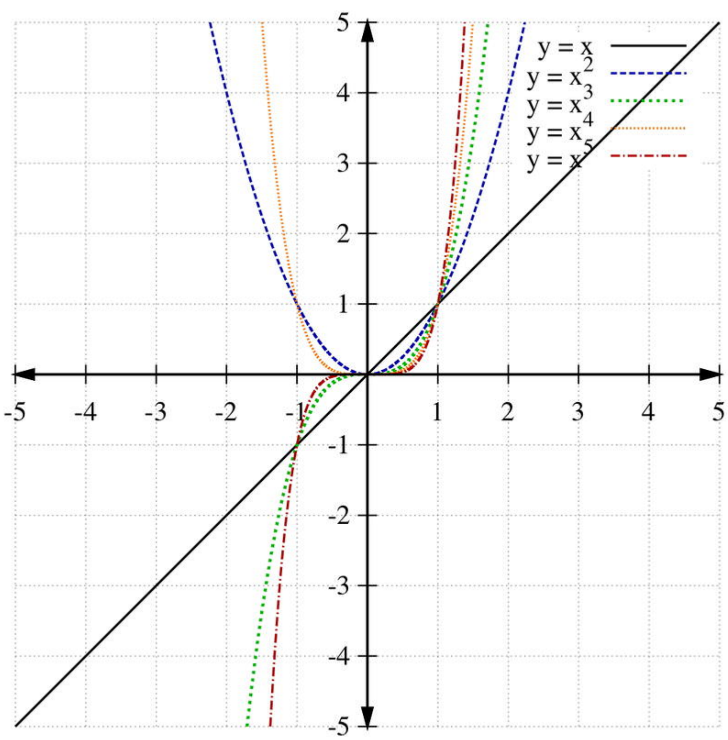
A very interesting property of polynomial functions is that they are all continuous—that is, they have no holes or breaks in the graph. In addition to their continuity, the domain of all polynomial functions is the set of all real numbers expressed in interval notation as $(-\infty, \infty)$.

To understand what a polynomial is, it is helpful to consider examples of functions that are not polynomials. For example, $f(x) = \frac{2}{\sqrt{x}}$ is not a polynomial. Its form $f(x) = \frac{2}{\sqrt{x}} = 2x^{-\frac{1}{2}}$ shows that the exponent of the single term is not a positive integer.

Power Function (even, odd)

A **power function** is a function of the form $f(x) = ax^n$, where $a \neq 0$ and n is a real number.

For power functions that are polynomial functions, if n is even, then the power function is also called "even," and if n is odd, then the power function is "odd." The graphs of the 1st five power functions are shown below:



Notice that each power function has only one $x-$ and $y-$intercept at the origin (0, 0).

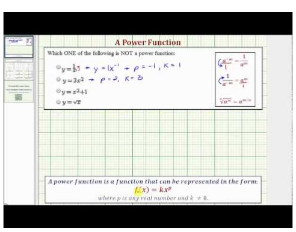
The end behavior of power functions can be classified as follows:

- For even powers of n and $a > 0$, the power function $f(x) = ax^n$ is U-shaped. The function approaches positive infinity as x approaches positive infinity and as x approaches negative infinity.
- For odd powers of n and $a > 0$, the power function resembles the cubing family of functions. The function approaches positive infinity as x approaches positive infinity. Likewise, the function approaches negative infinity as x approaches negative infinity.

As with quadratics and polynomials, the leading coefficient a "stretches" the graph of the function vertically when $|a| > 1$, and "contracts" the graph vertically when $0 < |a| < 1$. This leading coefficient also impacts the end behavior of the graph depending upon the degree of the polynomial. When a is negative, the graph is still stretched or contracted as noted, but it is also reflected about the x-axis.

One of the most interesting features of the polynomial function is its graph. The zeros, the coefficients, and the degrees of the terms provide key information about how the function will be graphed. There are a number of methods that can be used to graph polynomial functions, including:

- applying transformations and
- graphing the zeros.



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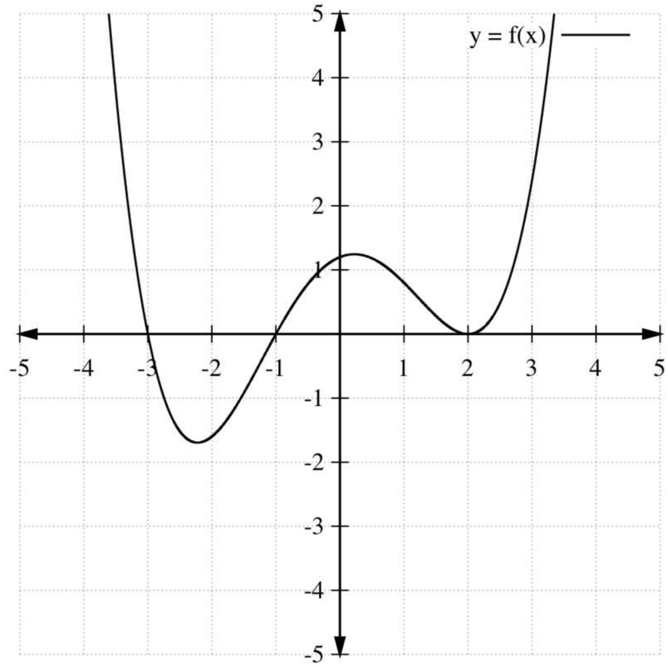
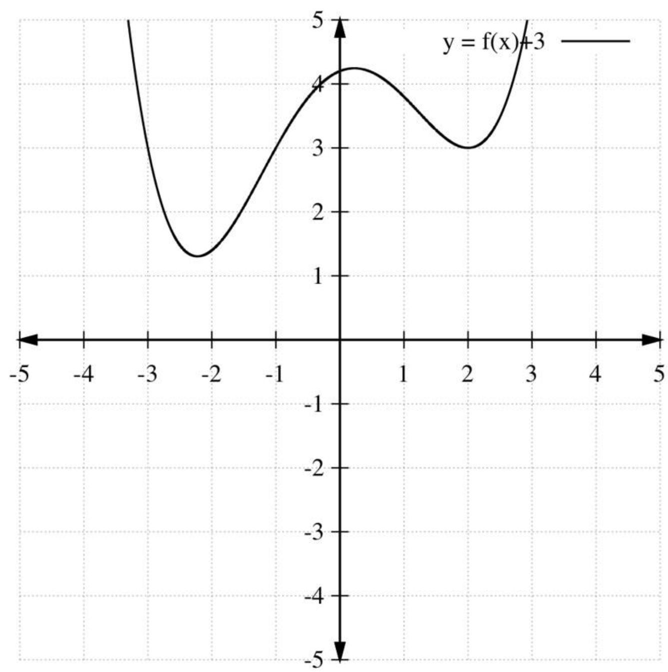
The Graphs of Polynomial Functions Using Transformations

The sum of several power functions with positive integer powers is called a polynomial. Polynomial functions can be graphed using transformations of a known graph. There are a number of ways that transformations can be applied to the graphs of power functions.

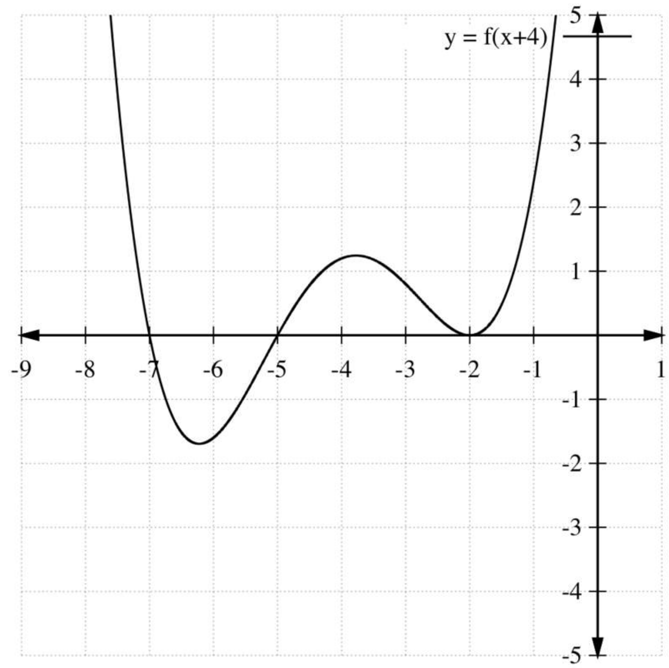
The graph of $f(x)$ is shown below. Use the graph of $f(x)$ to graph each of the following: a) $f(x)+3$, b) $f(x+4)$, and c) $f(-x)+3$.



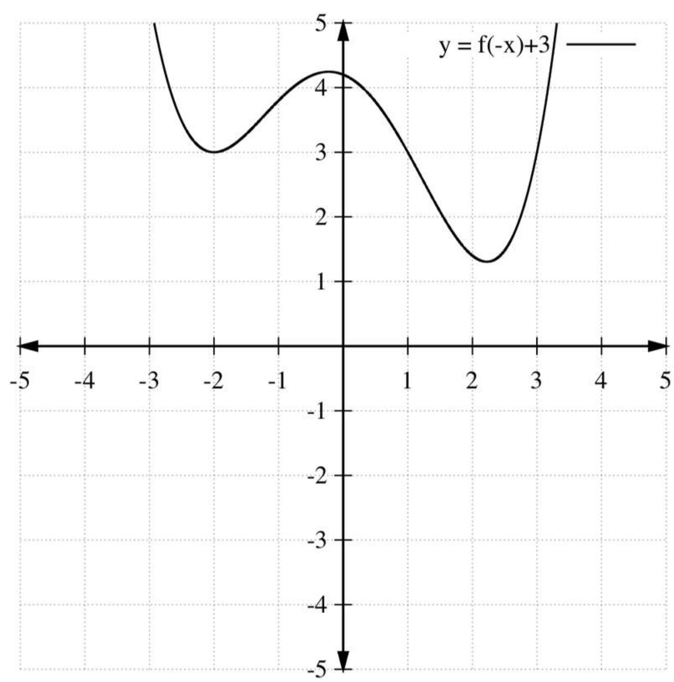
a) $f(x)+3$ is a vertical shift of $f(x)$ up by 3 units:



b) $f(x+4)$ is a horizontal shift of $f(x)$ to the left by 4 units:



c) $f(\text{-}x)+3$ is a reflection of $f(x)$ about the y-axis and a vertical shift up by 3 units:



Graph Polynomial Functions Using Zeros

Recall that the real zeros of a function or the roots of a polynomial equation coincide with the $x-$intercepts. For example, the polynomial function

$$h(x) = x^3 + 2x^2 - 5x - 6$$

in factored form is

$$h(x) = x^3 + 2x^2 - 5x - 6 = (x+1)(x-2)(x+3).$$

To find the zeros, we set

$$h(x) = 0$$

and solve for x:

$$(x+1)(x-2)(x+3) = 0.$$

We set each factor equal to zero:

$$(x+1) = 0,$$

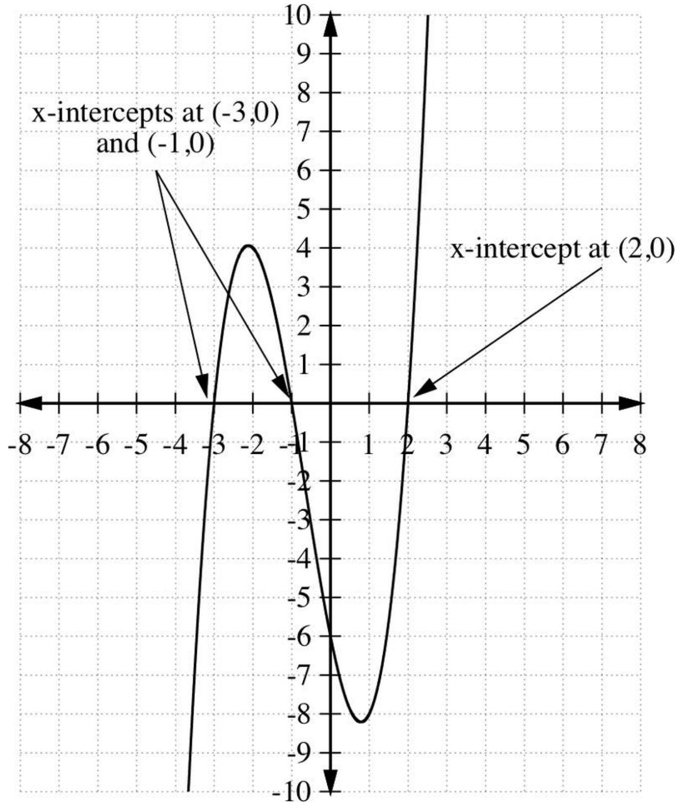
$$(x-2) = 0,$$

$$(x+3) = 0.$$

Solving all three provides us with our three solutions:

$$x = -3, -1, 2.$$

So we say that the roots of the equation are $\{-3, -1, 2\}$. They are the zeros of the function $h(x)$. The real zeros of $h(x)$ are also the $x-$intercepts of the graph $y = h(x)$.



This example illustrates a strategy for graphing a polynomial if the zeros of the function are known. To create the general shape of the graph, it is not always necessary to know the actual y-values, but simply whether they are above or below the x-axis, or positive or negative. To determine this, "test values" between the zeros to evaluate the function and determine if its graph is above or below the x-axis between the zeros. While this cannot tell you the height ($y-$value) of the function between the zeros, you can use the zeros to sketch enough information about the graph to begin a thorough analysis.

Characteristics of Polynomial Functions

It is helpful to consider the following facts when graphing any polynomial function:

Number of Zeros

If $f(x)$ is a polynomial function with degree $n \geq 1$, then:

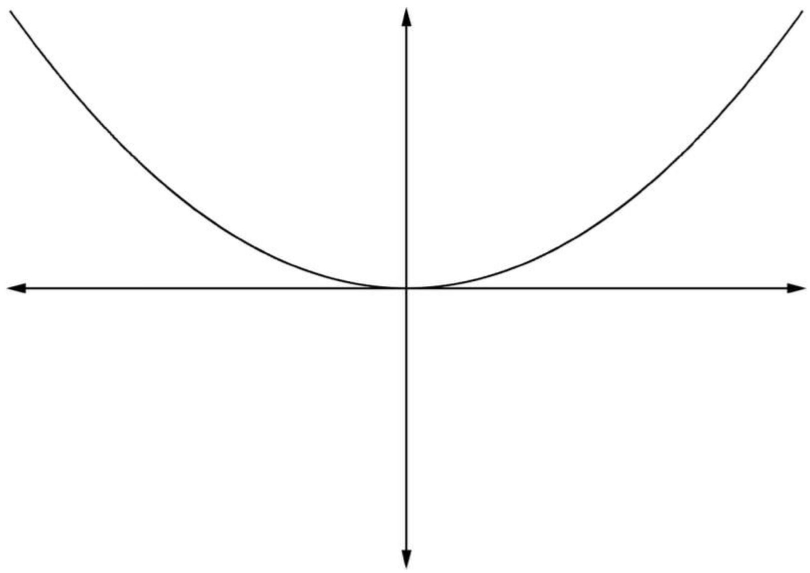
- The maximum number of real zeros ($x-$intercepts) is n.
- The maximum number of turning points of the graph, which is where the graph has either a local minimum or local maximum, is $n-1$.

Every polynomial function with degree $n \geq 1$ has at least one zero and at most n zeros (counting imaginary or complex zeros).

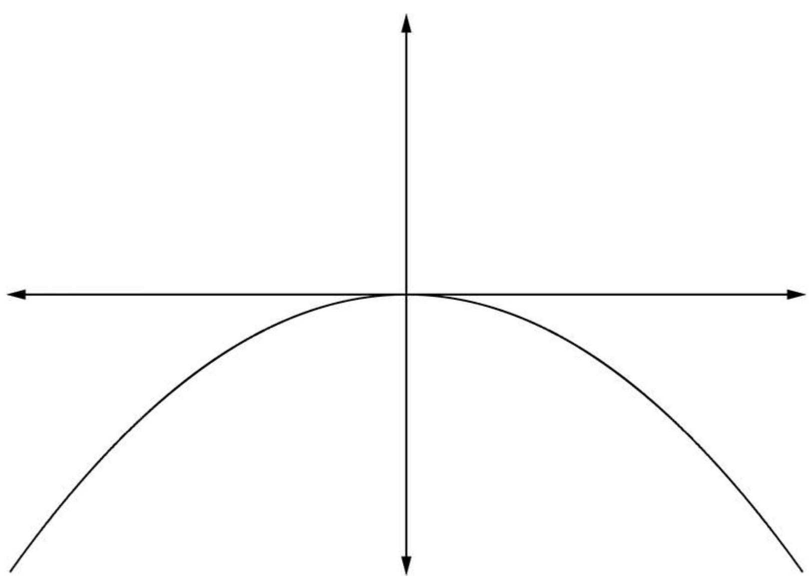
End Behavior by Examining the Leading Coefficient

If $a_n x^n$ is the leading term of a polynomial function, then the behavior of the graph as $x \to \infty$ or $x \to -\infty$ is the same as the end behavior of one of the following:

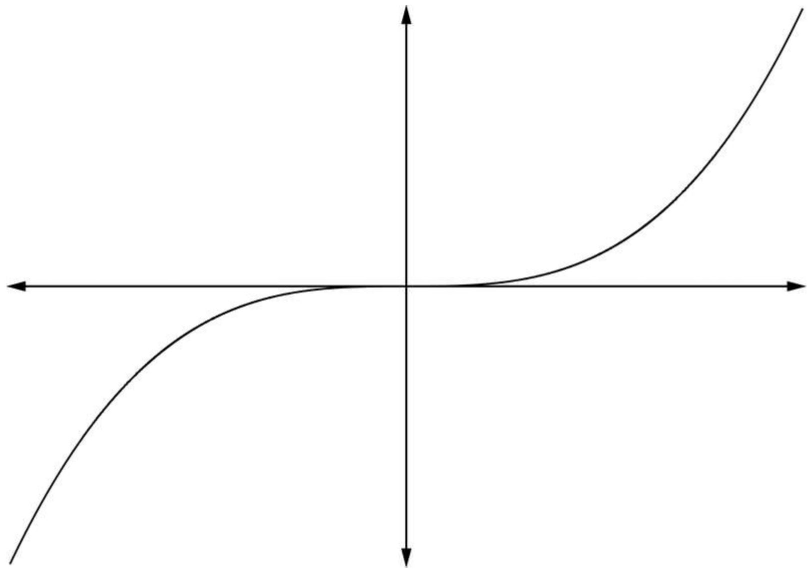
1. If $a_n > 0$ and n even, then $f(x) \to \infty$ as $x \to \infty$ and $x \to -\infty$.



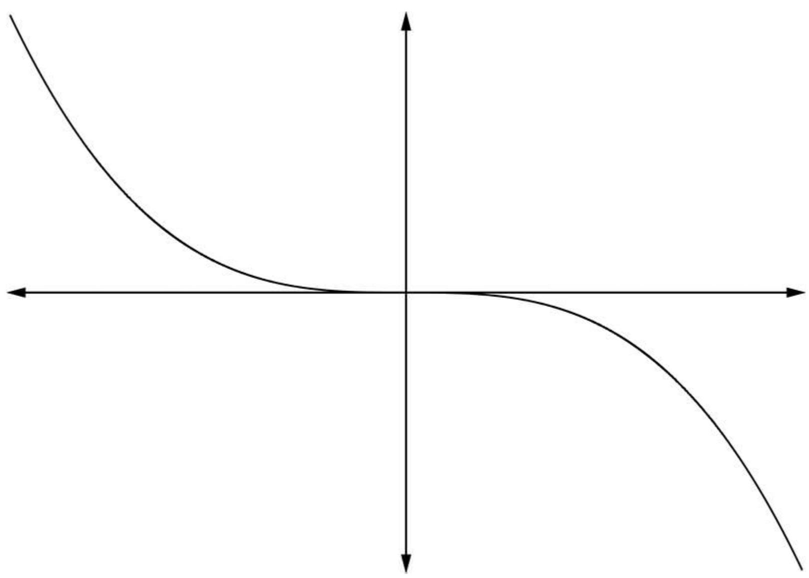
2. If $a_n < 0$ and n even, then $f(x) \to -\infty$ as $x \to \infty$ and $x \to -\infty$.



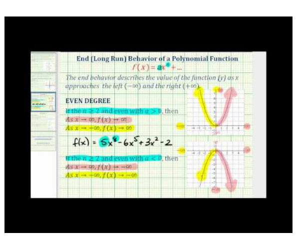
3. If $a_n > 0$ and n odd, then $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.



4. If $a_n < 0$ and n odd, then $f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$.



Using the leading coefficient in this way implies that as $x \to \infty$ or $x \to -\infty$, then all polynomial functions look like the power function related to the leading term of the polynomial function, which is a powerful and useful generalization for analysis.



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Graphing a Polynomial Function

To summarize, the following procedure can be followed when graphing a polynomial function:

1. Examine the leading coefficient to determine the end behavior of the graph.
2. Find the $x-$intercept(s) of $f(x)$ by setting $f(x) = 0$ and then solving for x.
3. Find the $y-$intercept of $f(x)$ by setting $y = f(0)$ and finding y.
4. Use the $x-$intercept(s) to divide the $x-$axis into intervals, and then choose test points to determine the sign of $f(x)$ on each interval.
5. Plot the test points.
6. If necessary, find additional points to determine the general shape of the graph.

Examples

Example 1

Find the zeros and sketch a graph of the polynomial function

$$g(x) = -(x-2)(x-2)(x+1)(x+5)(x+5)(x+5).$$

Solution:

Step 1: The polynomial function can be written as

$$g(x) = -(x-2)^2(x+1)(x+5)^3.$$

Step 2: To solve the equation, set $g(x)$ equal to zero:

$$-(x-2)^2(x+1)(x+5)^3 = 0.$$

Step 3: Set each term equal to 0:

$$\begin{aligned} x-2 &= 0, \\ x+1 &= 0, \\ x+5 &= 0. \end{aligned}$$

Step 4: Solve each:

$$\begin{aligned} x &= 2, \\ x &= -1, \\ x &= -5. \end{aligned}$$

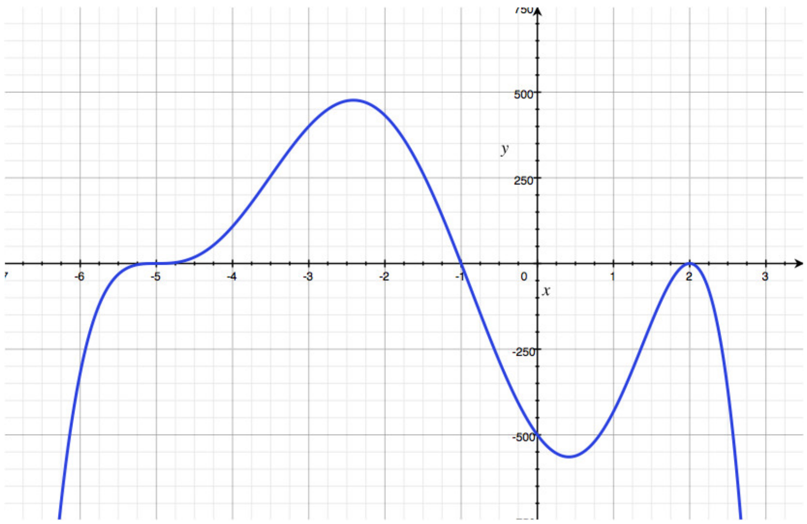
Notice the occurrence of the zeros in the function. The factor $(x-2)$ occurred twice (because it was squared), the factor $(x+1)$ occurred once, and the factor $(x+5)$ occurred three times. We say that the zero we obtain from the factor $(x-2)$ has a multiplicity $k = 2$, and the factor $(x+5)$ has a multiplicity $k = 3$.

Step 5: To graph $g(x)$, use the zeros to create a table of intervals and see whether the function is above or below the $x-$axis in each interval:

TABLE 3.3:

Interval	Test value x	$g(x)$	Sign of $g(x)$	Location of graph relative to $x-$axis
$(-\infty, -5)$	-6	-320	-	Below
$x=-5$	-5	0	NA	
(-5, -1)	-2	432	+	Above
$x=-1$	-1	0	NA	
(-1, 2)	0	-500	-	Below
$x=2$	2	0	NA	
$(2, \infty)$	3	-2048	-	Below

Step 6: Finally, use this information and the test points to sketch a graph of $g(x)$:



Example 2

Find the zeros and sketch a graph of the polynomial function

$$f(x) = x^4 - x^2 - 56.$$

.

Solution:

Step 1: Factor:

$$\begin{aligned} f(x) &= x^4 - x^2 - 56 \\ &= (x^2 - 8)(x^2 + 7). \end{aligned}$$

Step 2: Set $f(x) = 0$:

$$(x^2 - 8)(x^2 + 7) = 0.$$

Step 3: Solve each factor for 0:

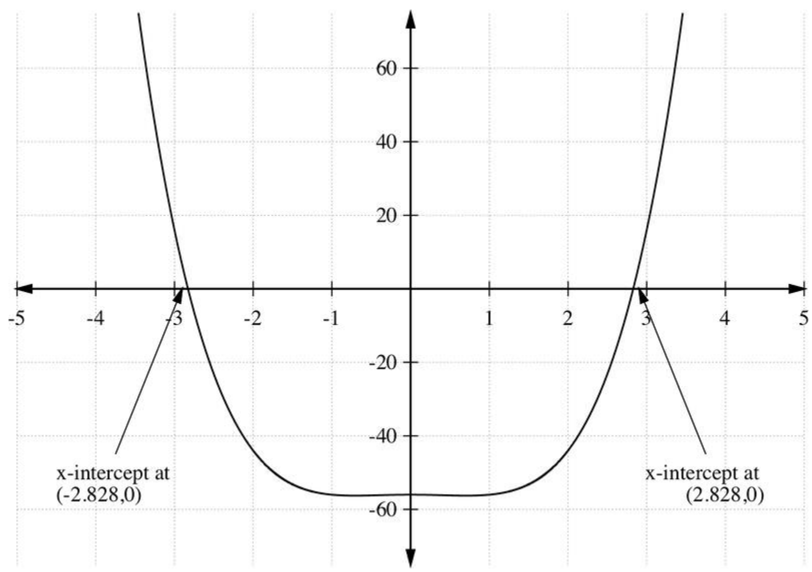
$$\begin{aligned} x^2 - 8 &= 0 \\ x^2 &= 8 \\ x &= \pm\sqrt{8} \\ &= \pm 2\sqrt{2}, \end{aligned}$$

and the 2nd factor gives

$$\begin{aligned} x^2 + 7 &= 0 \\ x^2 &= -7 \\ x &= \pm\sqrt{-7} \\ &= \pm i\sqrt{7}. \end{aligned}$$

Step 4: So the solutions are $\pm 2\sqrt{2}$ and $\pm i\sqrt{7}$, a total of four zeros of $f(x)$.

Only the real zeros of a function correspond to the $x-$intercepts of its graph. For this example, only the two zeros $\pm 2\sqrt{2}$ correspond to actual $x-$intercepts, but $+i\sqrt{7}$ and $-i\sqrt{7}$ do not, since they are complex.



Example 3

Graph the polynomial function $f(x) = -3x^4 + 2x^3$.

Solution:

Step 1: Since the leading term is $-3x^4$, then $a_n = -3 < 0$, and $n = 4$ even. Thus, the end behavior of the graph as $x \to \infty$ and $x \to -\infty$ is that of the 2nd case where $f(x) \to -\infty$.

Step 2: For the zeros of the function, solve $f(x) = 0$ for x:

$$\begin{aligned} -3x^4 + 2x^3 &= 0 \\ -x^3(3x - 2) &= 0. \end{aligned}$$

Step 3: Solve each term set to 0:

$$-x^3 = 0 \quad \text{or} \quad 3x - 2 = 0$$
$$x = 0 \quad \text{or} \quad x = \frac{2}{3}.$$

So there are two $x-$intercepts, at $x = 0$ and at $x = \frac{2}{3}$, with multiplicity $k = 3$ for $x = 0$ and multiplicity $k = 1$ for $x = \frac{2}{3}$.

The $y-$intercept is

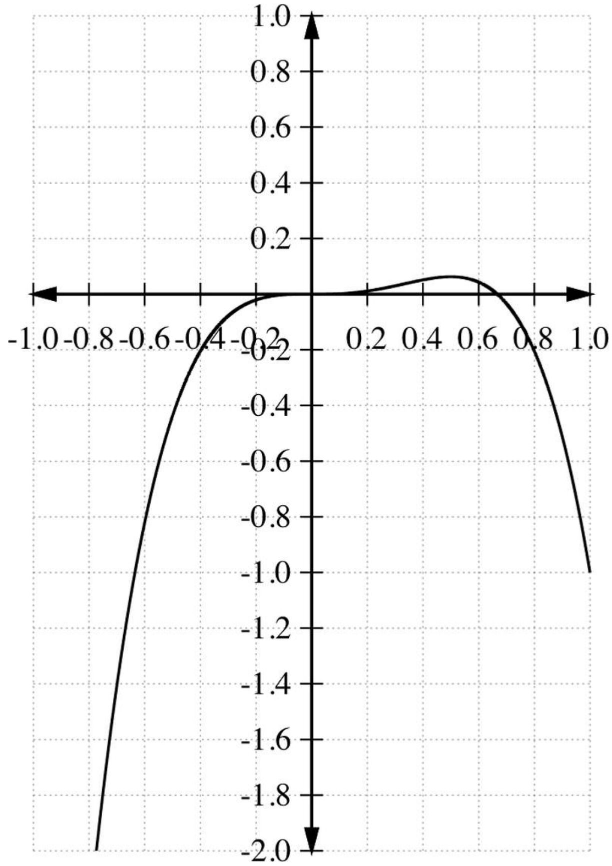
$$(0, f(0)) = (0, 0).$$

Step 4: Since the $x-$intercepts are 0 and $\frac{2}{3}$, they divide the $x-$axis into three intervals: $(-\infty, 0)$, $\left(0, \frac{2}{3}\right)$, and $\left(\frac{2}{3}, \infty\right)$. Note the sign of $f(x)$ in these intervals:

TABLE 3.4:

Interval	**Test Value** x	$f(x)$	**Sign of** $f(x)$	**Location of points on the graph**
$(-\infty, 0)$	-1	-5	-	below the $x-$axis
$\left(0, \frac{2}{3}\right)$	$\frac{1}{2}$	$\frac{1}{16}$	+	above the $x-$axis
$\left(\frac{2}{3}, \infty\right)$	1	-1	-	below the $x-$axis

Step 5: The test points give three additional points to plot: $(-1, -5)$, $\left(\frac{1}{2}, \frac{1}{16}\right)$, and (1, -1). The graph is a synthesis of all of this information:



Example 4

Graph the polynomial function

$$g(x) = 2x^3 + 3x^2 - 50x - 75.$$

.

Solution:

Step 1: Notice that the leading term is $2x^3$, where $n = 3$ is odd and $a_n = 2 > 0$. This tells us that the end behavior will take the shape of the 3rd case, where $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, for end behavior. For the $x-$ and $y-$intercepts:

$$\begin{aligned} g(x) &= 0 \\ 2x^3 + 3x^2 - 50x - 75 &= 0. \end{aligned}$$

Step 2: Factor the polynomial by grouping:

$$\begin{aligned} (2x+3)(x^2) - (2x+3)(25) &= 0 \\ (2x+3)(x^2 - 25) &= 0 \\ (x+5)(x-5)(2x+3) &= 0. \end{aligned}$$

Step 3: Setting each of the factors equal to zero, we can calculate that the zeros are -5, 5, and $\frac{-3}{2}$. Divide the $x-$axis into four intervals:

$$(-\infty,-5) \qquad \left(-5,\frac{-3}{2}\right) \qquad \left(\frac{-3}{2},5\right) \qquad (5,\infty)$$

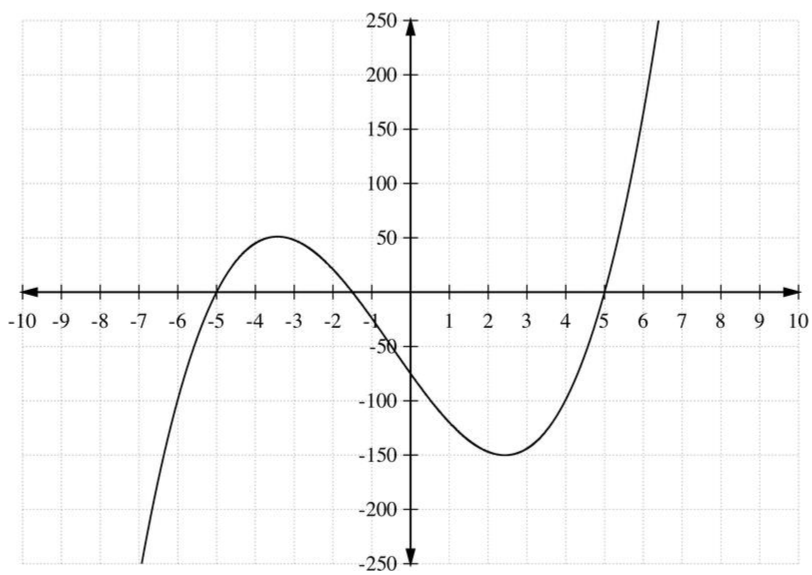
The $y-$intercept is $(0, f(0)) = (0, -75)$.

Step 4: Now choose test points from each interval and find $g(x)$:

TABLE 3.5:

Interval	Test value x	Value of $g(x)$ and its sign	Location of points on graph
$(-\infty,-5)$	-6	-99	below the $x-$axis
$(-5,\frac{-3}{2})$	-2	21	above the $x-$axis
$(\frac{-3}{2},5)$	0	-75	below the $x-$axis
$(5,\infty)$	6	165	above the $x-$axis

Step 5: From the information obtained, we can roughly sketch the graph (below):



Example 5

Now, the manufacturer's question can be answered. The volume function is:

$$V(x) = x(8-2x)(10-2x) = 4x^3 - 36x^2 + 80x,$$

which yields the following intercepts and end behavior:

$(0,0)$, $(4,0)$, $y = (x-3)(x+3)(x-2)$, $f(x) \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$.

From the following graph, the maximum volume is obtained when squares that are 1.5 inches by 1.5 inches are cut from the corners of the cardboard:

Review

Find the real roots for the following cubic polynomials, and graph:

1. $y = (x-3)(x+3)(x-2)$
2. $y = (x+5)(x+2)(x-2)$
3. $y = (3x+3)\left(x^2+2x+6\right)$
4. $y = (x-4)\left(2x^2+3\right)$
5. $y = (x-1)\left(-2x^2-5x-10\right)$

Graph the functions below using a graphing calculator or program and determine the number of real roots. Give at least one factor of each polynomial from the graphed solution:

6. $y = x^3-3x^2-2x+6$
7. $y = x^3+x^2-3x-3$
8. $y = x^3+2x^2-16x-32$
9. $y = 2x^3+13x^2+9x+6$
10. $y = 2x^3+15x^2+4x-21$

Describe the following graphs (include zeros, end behavior, and test points in between):

11. $y = x^4-5x^2+2x+2$
12. $y = x^4+3x^3-x-3$
13. $y = -x^4+x^3+4x^2-x+6$
14. $y = -x^4-5x^3-5x^2+5x+6$
15. $y = -2x^4-4x^3-5x^2-4x-4$

Review (Answers)

Please see the Appendix.

3.4 Synthetic Division of Polynomials

Learning Objectives

Learn how to use both long division and a 'shortcut' version of long division called 'synthetic division' to help find the x-intercepts of higher-degree polynomials.

Introduction

Consider the task of needing to find the zeros of the function $f(x) = x^4 + x^3 - 25x^2 - 37x + 60$. Factoring seems to be a daunting task. In fact, computer algebra systems are now free and available to factor this polynomial easily.

While we have the option of using software to do the work, however, several classic techniques remain useful. One technique is synthetic division of polynomials, which is based on the process of long division.

Polynomial Long Division

We use long division of polynomials to determine the factors of a polynomial in a similar way that we use long division to divide integers.

Whenever you want to divide a polynomial by a polynomial, you can use a process called polynomial long division. This process is similar to long division for integers. If we would like to divide

$$\frac{x^2 + 3x + 2}{x + 1},$$

we can transfer the fraction form to the long division format:

$$x+1\overline{)x^2+3x+2}$$

.

Step 1: Divide the 1st term in the numerator (x^2) by the 1st term in the denominator (x). Write this result above the division bar in your answer. In this case, $\frac{x^2}{x}=x$:

$$\begin{array}{r l}
 & x \\
x+1 & \overline{)x^2+3x+2}
\end{array}$$

.

Step 2: Multiply the denominator ($x+1$) by the result from Step 1 (x), and put the new result below your numerator. Then, ***subtract*** to get your new polynomial. *This is similar to the process of regular number long division!*

$$\begin{array}{r l}
 & x \\
x+1 & \overline{)x^2+3x+2} \\
 & \underline{x^2+x\quad\downarrow} \\
 & \qquad 2x+2
\end{array}$$

Step 3: Divide the 1st term in the new polynomial ($2x$) by the 1st term in the denominator (x). Put this result above the division bar in your answer. Multiply, subtract, and repeat this process until you cannot repeat it anymore:

$$\begin{array}{r l}
 & x\ +2 \\
x+1 & \overline{)x^2+3x+2} \\
 & \underline{x^2+x\quad\downarrow} \\
 & \qquad 2x+2 \\
 & \qquad \underline{2x+2} \\
 & \qquad\qquad 0
\end{array}$$

.

Since there is no remainder, $\frac{x^2+3x+2}{x+1}=x+2$.

Here is another example of using long division to divide:

$$\frac{x^2+6x-7}{x-1}$$

Step 1: Divide the 1st term in the numerator by the 1st term in the denominator, and put this in your answer. Therefore, $\frac{x^2}{x} = x$.

$$\begin{array}{r|l} & x \\ \hline (x-1) & x^2+6x-7 \end{array}$$

Step 2: Multiply the denominator by this number (variable) and put it below your numerator, subtract and get your new polynomial:

$$\begin{array}{rl} & x \\ (x-1)| & \overline{x^2+6x-7} \\ & \underline{x^2-x} \\ & \quad 7x. \end{array}$$

Step 3: Repeat the process until you cannot repeat it anymore:

$$\begin{array}{rl} & x \quad +7 \\ (x-1)| & \overline{x^2+6x-7} \\ & \underline{x^2-x \quad \downarrow} \\ & \quad 7x-7 \\ & \underline{\quad 7x-7} \\ & \qquad 0. \end{array}$$

Therefore, $\frac{x^2+6x-7}{x-1} = x+7$.

Two more examples of long division:

$$\frac{2x^2+7x+5}{2x+5}$$

Step 1: Divide the 1st term in the numerator by the 1st term in the denominator; put this in your answer. Therefore, $\frac{2x^2}{2x} = x$.

$$\begin{array}{rl} & x \\ (2x+5)| & \overline{2x^2+7x+5} \end{array}$$

Step 2: Multiply the denominator by this number (variable) and put it below your numerator, subtract and get your new polynomial:

$$\begin{array}{rl} & x \\ (2x+5)| & \overline{2x^2+7x+5} \\ & \underline{2x^2+5x} \\ & \quad 2x. \end{array}$$

Step 3: Repeat the process until you cannot repeat it anymore:

$$\begin{array}{r} x \quad +1 \\ (2x+5)\overline{)2x^2+7x+5} \\ \underline{2x^2+5x \ \downarrow} \\ 2x+5 \\ \underline{2x+5} \\ 0. \end{array}$$

Therefore, $\frac{2x^2+7x+5}{2x+5} = x+1$.

Last example of long division:

$$\frac{x^2-5x+6}{x-2}$$

Step 1: Divide the 1st term in the numerator by the 1st term in the denominator, put this in your answer. Therefore, $\frac{x^2}{x} = x$.

$$\begin{array}{r} x \\ (x-2)\overline{)x^2-5x+6} \end{array}$$

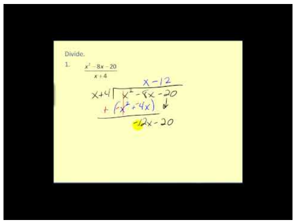
Step 2: Multiply the denominator by this number (variable) and put it below your numerator, subtract and get your new polynomial:

$$\begin{array}{r} x \\ (x-2)\overline{)x^2-5x+6} \\ \underline{x^2-2x} \\ -3x. \end{array}$$

Step 3: Repeat the process until you cannot repeat it anymore:

$$\begin{array}{r} x \quad -3 \\ (x-2)\overline{)x^2-5x+6} \\ \underline{x^2-2x \ \downarrow} \\ -3x+6 \\ \underline{-3x+6} \\ 0. \end{array}$$

Therefore, $\frac{x^2-5x+6}{x-2} = x-3$.



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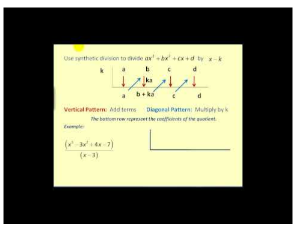
Synthetic Division

Synthetic division is an alternative to long division. It is a shorthand version of long division when the divisor is a 1st-degree (linear) polynomial. It can also be used to divide a polynomial by a possible factor, $x-k$. However, synthetic division cannot be used to divide polynomials of a higher degree into another polynomial.



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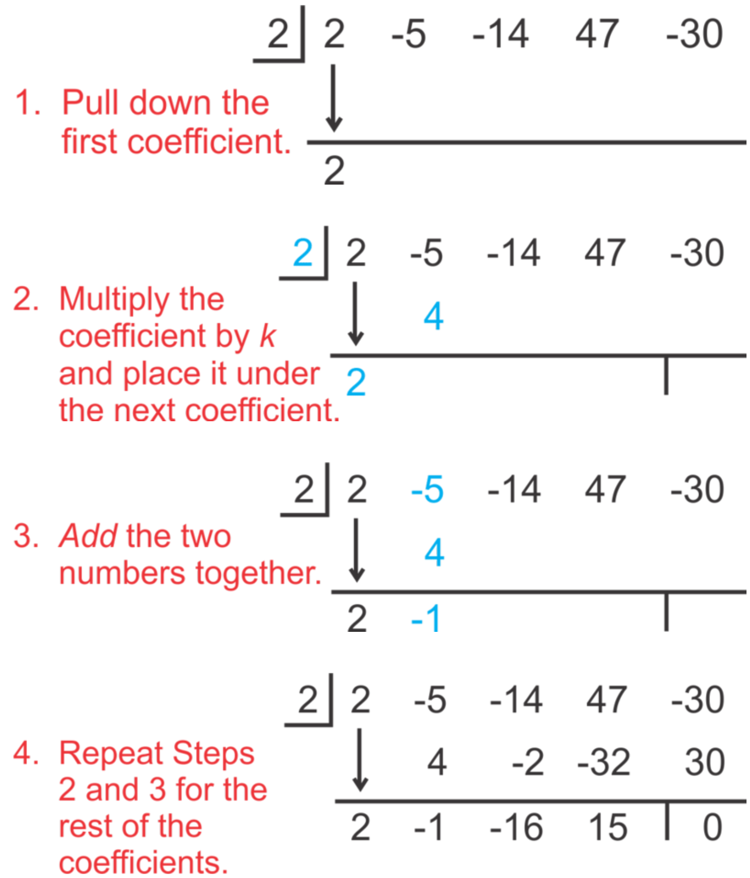
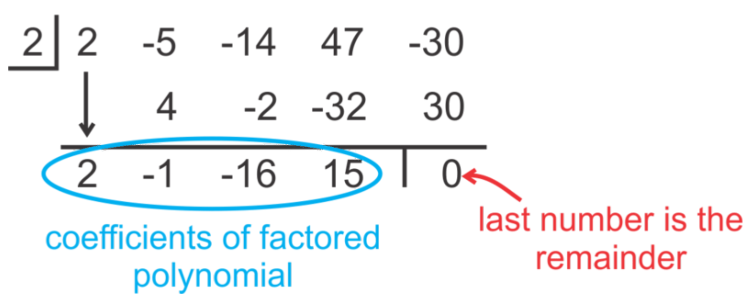
If you are asked to divide $2x^4-5x^3-14x^2+47x-30$ by $x-2$, you could divide by long division.

$$\begin{array}{l}
\quad\quad 2x^3 \quad -x^2 \quad -16x \quad +15 \\
(x-2)\ |\overline{2x^4-5x^3-14x^2+47x-30} \\
\underline{2x^4-4x^3 \quad \downarrow} \\
-1x^3-14x^2 \\
\underline{\ -x^3+2x^2} \\
-16x^2+47x \\
\underline{\ -16x^2+32x} \\
15x-30 \\
\underline{\ 15x-30} \\
0
\end{array}$$

Alternatively, you could use synthetic division. Start by extracting all of your coefficients while also noting the k from the divisor, $x-k$. Then, the setup is as follows:



To "read" the answer, use the numbers as follows:



This process shows that $2x^4-5x^3-14x^2+47x-30=(x-2)\left(2x^3-x^2-16x+15\right)$, because the remainder is zero. The polynomial $2x^3-x^2-16x+15$ is read from the bottom row of the synthetic division process. Its degree is one less than the original. We could also write this in division form as:

$$\frac{2x^4-5x^3-14x^2+47x-30}{x-2}$$
$$=2x^3-x^2-16x+15+\frac{0}{x-2}.$$

Thus, $2x^4-5x^3-14x^2+47x-30=(x-2)(2x^3-x^2-16x+15)$.

Examples

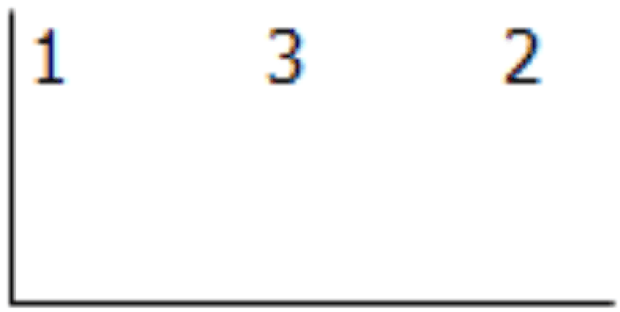
Example 1

Divide

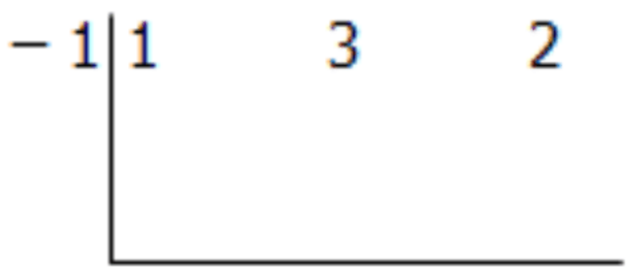
$$x+1\overline{)x^2+3x+2}$$

Solution:

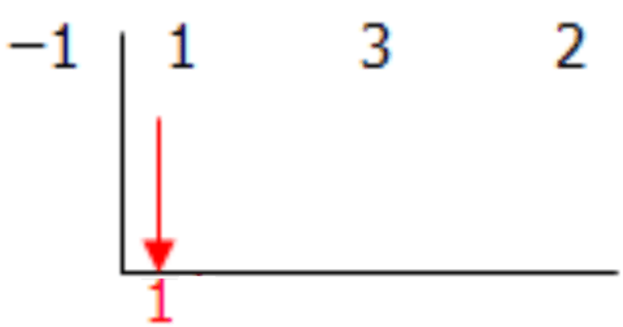
Step 1: Write the coefficients in an upside-down division sign:



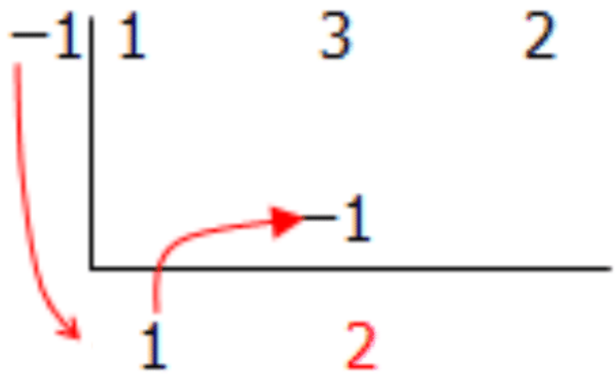
Step 2: Put the opposite of the number from the divisor to the left of the division symbol. In this case, the divisor is $x+1$, so you will use -1:



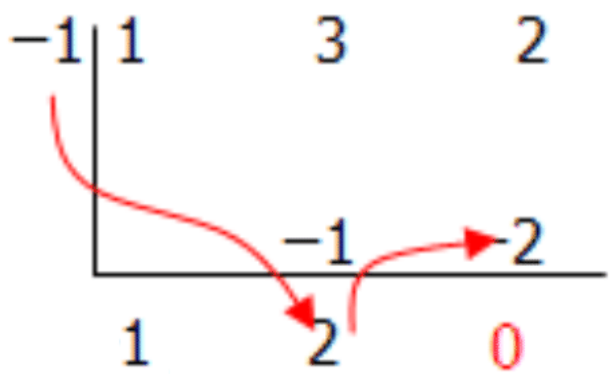
Step 3: Take your leading coefficient and bring it down below the division symbol:



Step 4: Multiply this number by the number to the left of the division symbol, and place it in the next column. Add the two numbers together, and place this new number below the division sign:



Step 5: Multiply this 2nd number by the number to the left of the division symbol, and place it into the 3rd column. Add the two numbers together, and place this new number below the division sign:



The numbers below the division sign represent your coefficients. Therefore, $\frac{x^2+3x+2}{x+1} = x+2$.

Example 2

Determine if 4 is a zero of $5x^3 + 6x^2 - 24x - 16$.

Solution:

To determine if 4 is a zero is the same as trying to determine if $(x-4)$ is a factor. Using synthetic division:

4	5	6	-24	-16
	↓	20	104	320
	5	26	80	304

The remainder is 304, so $(x-4)$ is not a factor. Notice that

$$f(4) = 5(4)^3 + 6(4)^2 - 24(4) - 16 = 304.$$

This observation leads to the Remainder Theorem.

Remainder Theorem:

If $f(k) = r$, then r is the remainder when dividing $f(x)$ by $(x-k)$. This implies that the point (k,r) would be on the graph of $f(x)$.

Example 3

Determine if $2x-5$ is a factor of $4x^4 - 9x^2 - 100$.

Solution:

To use synthetic division, the factor must be in the form $(x-k)$. The technique to handle this situation is to create a factor with the same zero as $(2x-5)$, which is $(x-\frac{5}{2})$. Also, notice that $4x^4 - 9x^2 - 100$ can be written as

$$4x^4 - 9x^2 - 100 = 4x^4 + 0x^3 - 9x^2 + 0x - 100.$$

Since synthetic division is shorthand process, all of these terms must be considered and their coefficients included:

5/2	4	0	-9	0	-100
	↓	10	25	40	100
	4	10	16	40	0

Therefore, $\frac{5}{2}$ is a zero and its corresponding binomial, $2x-5$, is a factor. Note that this method confirms that $2x-5$ is a factor of the polynomial, since the remainder is zero, but it does not tell you the quotient if you were to divide the original polynomial by $2x-5$.

Example 4

Perform the division: $\frac{3x^3-2x^2+11x-9}{x^2-1} = 3x-2+\frac{14x-11}{x^2-1}$.

Solution:

To use synthetic division, notice that the factor is not in the form $(x-k)$. To find the linear factors, solve $x^2-1=0$, to get $x=1$ and -1. Therefore, synthetic division can be performed twice because there are two real roots:

Next, divide the resulting polynomial, $3x^2-5x+16$, with the remainder, -25, by the other real root, 1:

Since the degree of the original dividend polynomial, $3x^3-2x^2+11x-9$, is 3 and the degree of the divisor polynomial, x^2-1, is 2, the quotient polynomial is degree 1 because $3-2=1$.Therefore, the 3 and -2 from the synthetic division are part of the resulting quotient polynomial, and 14 and -11 from the synthetic division are part of the remainder.

Thus, the final answer is $\frac{3x^3-2x^2+11x-9}{x^2-1} = 3x-2+\frac{14x-11}{x^2-1}$.

Example 5

Divide $2x^4-11x^3+12x^2+9x-2$ by $2x+1$. Write the resulting polynomial with the remainder (if there is one).

Solution:

To use synthetic division, the factor must be in the form $(x-k)$. To handle this situation, create a factor with the same zero. In this case, it would be $\left(x+\frac{1}{2}\right)$ with the zero $-\frac{1}{2}$. Do this by factoring out the coefficient of x in the denominator and dividing the numerator by that number.

$$\frac{(2x^4 [U+0088][U+0092] 11x^3+12x^2+9x [U+0088][U+0092] 2)}{(2x+1)} = \frac{(2x^4 [U+0088][U+0092] 11x^3+12x^2+9x [U+0088][U+009}{\left(2\left(x+\frac{1}{2}\right)\right)}$$
$$= \left(\frac{1}{2}\right)\frac{(2x^4 [U+0088][U+0092] 11x^3+12x^2+9x [U+0088][U}{\left(x+\frac{1}{2}\right)}$$
$$= \frac{(x^4 [U+0088][U+0092] \left(\frac{11}{2}\right)x^3+6x^2+\left(\frac{9}{2}\right)x [U+0088][U+0}{\left(x+\frac{1}{2}\right)}$$

Now you can use synthetic division.

$-\frac{1}{2}$	1	$-\frac{11}{2}$	6	$\frac{9}{2}$	-1
	↓	$-\frac{1}{2}$	3	$-\frac{9}{2}$	0
	1	-6	9	0	-1

Thus, the answer is $x^3-6x^2+9x-\frac{1}{x+\frac{1}{2}}$.

Example 6

Is $x-6$ a factor, or is 6 a root, of the function $f(x)=x^3-8x^2+72$? If so, find the real roots of the resulting polynomial.

Solution:

Put a zero placeholder for the $x-$term. Divide by 6.

6	1	-8	0	72
	↓	6	-12	-72
	1	-2	-12	0

The resulting polynomial is $x^2-2x-12$. While this quadratic does not factor, the quadratic formula is used to find the other roots.

$$x=\frac{2\pm\sqrt{(-2)^2-4(1)(-12)}}{2(1)}=\frac{2\pm\sqrt{52}}{2}=\frac{2\pm2\sqrt{13}}{2}=1\pm\sqrt{13}$$

The solutions to this polynomial are 6, $1+\sqrt{13}\approx4.61$, and $1-\sqrt{13}\approx-2.61$.

Example 7

Divide to prove the following equation true: $\frac{4x^3+3x^2+10x+4}{x^2+2}=4x+3+\frac{2x-2}{x^2+2}$.

Solution:

The 1st step is to create factors equivalent to x^2+2.

$x^2+2=0$
$x=\pm i\sqrt{2}$

Using synthetic division, divide by $i\sqrt{2}$.

$i\sqrt{2}$	4	3	10	4
	↓	$4i\sqrt{2}$	$-8+3i\sqrt{2}$	$-6+2i\sqrt{2}$
	4	$3+4i\sqrt{2}$	$2+3i\sqrt{2}$	$-2+2i\sqrt{2}$

Divide the results from the last step by $-i\sqrt{2}$.

$-i\sqrt{2}$	4	$3+4i\sqrt{2}$	$2+3i\sqrt{2}$	$-2+2i\sqrt{2}$
	↓	$-4i\sqrt{2}$	$-3i\sqrt{2}$	$-2i\sqrt{2}$
	4	3	2	-2

So, $\frac{4x^3+3x^2+10x+4}{x^2+2}=4x+3+\frac{2x-2}{x^2+2}$.

Summary

- Synthetic division is a summary of the long division process. It records the coefficients of the state of a polynomial during the long division process.
- The Remainder Theorem states if $f(k)=r$, then r is the remainder when dividing $f(x)$ by $(x-r)$. This implies that the point (k,r) would be on the graph of $f(x)$.

Review

Use long division to divide each of the following:

1. $(x^2+7x+12)\div(x+3)$
2. $(x^2+4x+3)\div(x+3)$
3. $(a^2-4a-45)\div(a-9)$
4. $(3x^2+5x-2)\div(3x-1)$

Use synthetic division to divide the polynomials below. Write out the remaining polynomial:

5. $(x^3+6x^2+7x+10)\div(x+2)$
6. $(4x^3-15x^2-120x-128)\div(x-8)$
7. $(4x^2-5)\div(2x+1)$
8. $(2x^4-15x^3-30x^2-20x+42)\div(x+9)$
9. $(x^3-3x^2-11x+5)\div(x-5)$
10. $(3x^5+4x^3-x-2)\div(x-1)$
11. Which of the division problems above generate no remainder? What does that mean?
12. What is the difference between a zero and a factor?
13. a) Find $f(-2)$ if $f(x)=2x^4-5x^3-10x^2+21x-4$.

b) Now, divide $2x^4 - 5x^3 - 10x^2 + 21x - 4$ by $(x+2)$ synthetically. What do you notice?

Find all real zeros of the following polynomials, given one zero:

14. $12x^3 + 76x^2 + 107x - 20;$ zero: -4

15. $x^3 - 5x^2 - 2x + 10;$ zero: 5

16. $6x^3 - 17x^2 + 11x - 2;$ zero: 2

Find all real zeros of the following polynomials, given two zeros:

17. $x^4 + 7x^3 + 6x^2 - 32x - 32;$ zeros: -4, -1

18. $6x^4 + 19x^3 + 11x^2 - 6x;$ zeros: 0, -2

19. The volume of a rectangular prism is $3x^3 - 11x^2 - 56x - 48$, and the height is $3x + 4$. What is the area of the base?

20. A group of geologists have taken samples of a substance from a proposed mining site and must identify the substance. Each sample is roughly cylindrical. The volume of each sample as a function of cylinder height (in centimeters) is $V(h) = \frac{1}{4}\pi h^3$. The mass (in grams) of each sample in terms of height can be modeled by $M(h) = \frac{1}{4}h^3 - h^2 + 5h$. Write an expression that represents the density of the samples. (Hint: $D = \frac{M}{V}$)

Review (Answers)

Please see the Appendix.

3.5 Real Zeros of Polynomials

Learning Objectives

Learn about using long and synthetic division, along with the Remainder, Factor, and Rational Zero Theorems, to find roots.

Introduction

Unlike those often found in textbooks, problems in real life do not always fit easily into quadratic or even cubic equations. Financial models, population models, fluid activity, etc., often require polynomials of higher degree in order to approximate the overall behavior. Although it is challenging to model some of these more complex interactions, the effort can be well worth it. Mathematical models of stocks are used constantly to "look into the future" and make the kinds of educated guesses that create safe and stable investment portfolios.

To graph more complicated polynomial functions, we need powerful methods to find the zeros. Synthetic division can verify the zeros, but knowing where they are located is often a mystery. Several theorems summarize these methods:

- Remainder Theorem
- Factor Theorem
- Rational Zero Theorem
- Descartes's Rule of Sign

These theorems and rules help make the discovery of the roots of polynomial functions easier.

The last theorem listed, credited to Rene Descartes (pictured) in his work *La Géométrie*, helps us calculate the upper bound on the number of roots and their signs.

The Remainder Theorem

If a polynomial $f(x)$ of degree $n > 0$ is divided by $x - c$, then the remainder R is a constant and it is equal to the value of the polynomial when c is substituted for x. That is,

$$f(c) = R.$$

The Factor Theorem

An extension of the Remainder Theorem is the Factor Theorem. It states that if $f(x)$ is a polynomial of degree $n > 0$ and $f(c) = 0$, then $x - c$ is a *factor* of the polynomial $f(x)$. Conversely, if $x - c$ is a factor, then c is a *zero* of f.

The Rational Zero Theorem

Given the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

$a_n \neq 0$ and n is a positive integer. If the coefficients are integers and $\frac{p}{q}$ is a rational zero in lowest terms, then p is a divisor of a_0 and q is a divisor of a_n.

Descartes's Rule of Signs

Given any polynomial $p(x)$, the maximum number of positive roots can be determined by this method:

1. Write the terms of $p(x)$ in descending order (i.e., from the highest degree term to the lowest degree term).
2. Count the number of sign changes in the sequence of terms in $p(x)$. Call the number of sign changes r.
3. Note that the maximum number of positive roots of $p(x)$ is less than or equal to r. Further, the possible number of positive roots is $r, r-2, r-4, \ldots$

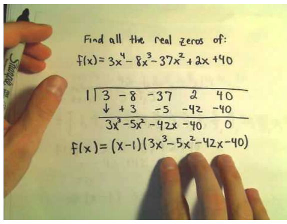
Similarly, given any polynomial $p(x)$, the maximum number of negative roots can be determined by this method:

1. Substitute $-x$ for x in $p(x)$ so that you have $p(-x)$ (i.e., change the sign of all terms in $p(x)$ with odd powers).
2. Write the terms of $p(-x)$ in descending order.
3. Count the number of sign changes in the sequence of terms as above.
4. Note that the maximum number of negative roots is less than or equal to the number of sign changes.

Finding Zeros

When asked to find all of the zeros of a polynomial function, follow the steps as follows:

1. Apply the Rational Zero Theorem by listing all of the possible rational zeros by dividing each of the factors of the constant term by the factors of the leading coefficient.
2. Apply Descartes's Rule of Signs to determine the number of possible positive and negative real zeros.
3. Use the Factor Theorem or synthetic division to determine if any of the possible rational zeros are actual zeros.
4. Find all of the zeros by dividing the polynomial by the binomial factor found using the Factor Theorem or synthetic division.



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Click image to the left or use the URL below.

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Examples

Example 1

Use synthetic division and the Remainder and Factor theorems to find the quotient $Q(x)$ and the remainder R if $f(x) = 2x^3 - 3x^2 + 6$ is divided by $x - 5$.

Solution:

Step 1: Perform synthetic division:

$$\begin{array}{r|rrrr} 5\ ) & 2 & -3 & 0 & 6 \\ & \downarrow & 10 & 35 & 175 \\ \hline & 2 & 7 & 35 & 181. \end{array}$$

Step 2: Apply the obtained coefficients to the quotient $Q(x)$:

$$2x^3 - 3x^2 + 6 = (2x^2 + 7x + 35)(x - 5) + 181.$$

Note: The remainder is 181, and it can also be obtained if we substitute $x = 5$ into $f(x)$:

$$\begin{aligned} f(5) &= 2(5)^3 - 3(5)^2 + 6 \\ &= 250 - 75 + 6 \\ &= 181. \end{aligned}$$

Example 2

Use the Rational Zero Theorem and synthetic division to find all the possible rational zeros of the polynomial

$$f(x) = x^3 - 2x^2 - x + 2.$$

Solution:

Step 1: The Rational Zero Theorem states if $\frac{p}{q}$ is a rational root, then p is a divisor of the constant term of the polynomial $f(x)$ and q is a divisor of its leading coefficient. So p is a divisor of 2 and q is a divisor of 1. Hence, p can take the following values: -1, 1, -2, 2, and q can be either -1 or 1. Therefore, the possible values of $\frac{p}{q}$ are

$$\frac{p}{q}: -1, 1, -2, 2.$$

Step 2: There are four possible zeros. Of these four, not more than three can be zeros of f because f is a polynomial with degree 3. To test which of the four possible candidates are zeros of f, we use synthetic division.

Step 3: We use synthetic division to check to see if 2 is indeed a factor:

$$\begin{array}{r|rrrr} 2 & 1 & -2 & -1 & 2 \\ & \downarrow & 2 & 0 & -2 \\ \hline & 1 & 0 & -1 & 0. \end{array}$$

Hence, 2 is a zero of f. Further, by division,

$$\begin{aligned} f(x) &= (x-c)Q(x) + R(x) \\ &= (x-2)(x^2-1) + 0. \end{aligned}$$

Step 4: The remaining zeros of f are the zeros of $Q(x) = x^2 - 1$, which can be factored to

$$\begin{aligned} Q(x) &= x^2 - 1 \\ &= (x-1)(x+1). \end{aligned}$$

Thus, the remaining zeros are -1 and 1. The rational zeros of f are -1, 1, and 2.

Example 3

Graph the polynomial function $h(x) = 2x^3 - 9x^2 + 12x - 5$.

Solution:

Step 1: Notice that the leading term is $2x^3$, where $n = 3$ is odd and $a_n = 2 > 0$. This tells us that the end behavior will take the shape of a power function with an odd exponent. Apply the Factor Theorem and synthetic division, and we can find the rational roots of $h(x)$.

Step 2: Use the Rational Zero Theorem and find that the possible rational zeros are

$$\frac{p}{q}: -1, 1, -2, 2, -5, 5, -\frac{5}{2}, \frac{5}{2}.$$

Step 3: Test each number by the Factor Theorem:

$$h(-1) = 2(-1)^3 - 9(-1)^2 + 12(-1) - 5 = -28$$

Since the remainder is -28, -1 is not a root.

Now let's test $x = 1$:

$$h(1) = 2(1)^3 - 9(1)^2 + 12(1) - 5 = 0$$

Since the remainder is 0, 1 is a root, so use synthetic division to divide.

$$\begin{array}{r|rrrr} 1 & 2 & -9 & 12 & -5 \\ & \downarrow & 2 & -7 & 5 \\ \hline & 2 & -7 & 5 & 0. \end{array}$$

Step 4: Rewrite $h(x)$ since 1 is a zero:

$$h(x) = (2x^2 - 7x + 5)(x - 1).$$

Step 5: Factor the quadratic and solve for the roots:

$$2x^2 - 7x + 5 = (2x - 5)(x - 1)$$

$$h(x) = (2x - 5)(x - 1)^2$$
$$2x - 5 = 0 \text{ or } x - 1 = 0.$$

Thus, 1 and $\frac{5}{2}$ are the $x-$intercepts of $h(x)$.

Step 6: The $y-$intercept is

$$h(0) = -5.$$

Further, synthetic division or the Remainder Theorem can be also used to form a table of values for the graph of $h(x)$:

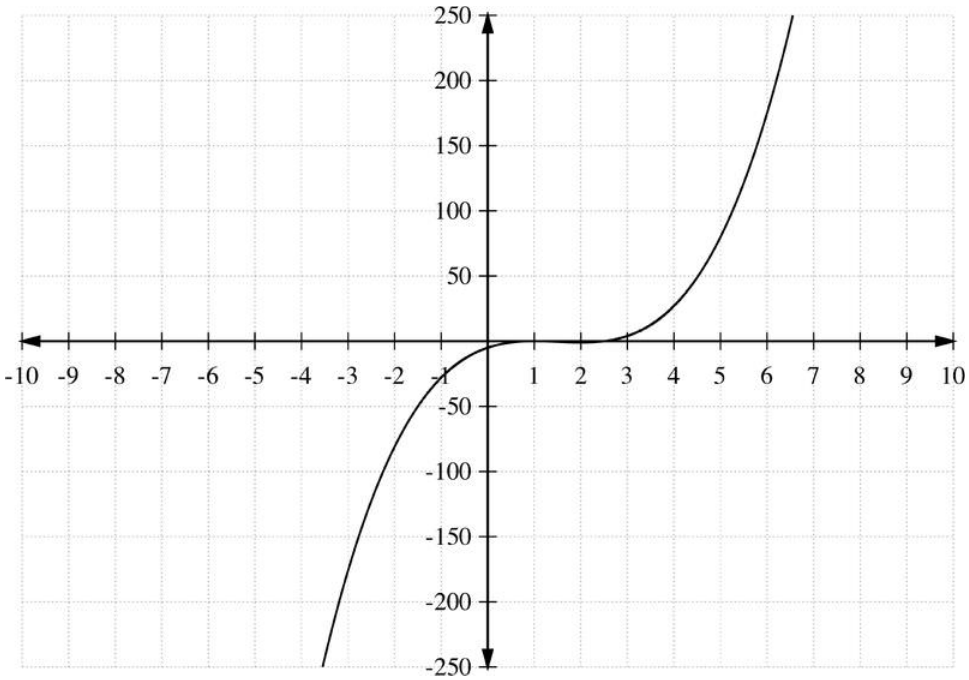
x	-1	0	1	2	$\frac{5}{2}$	3
$h(x)$	-28	-5	0	-1	0	4

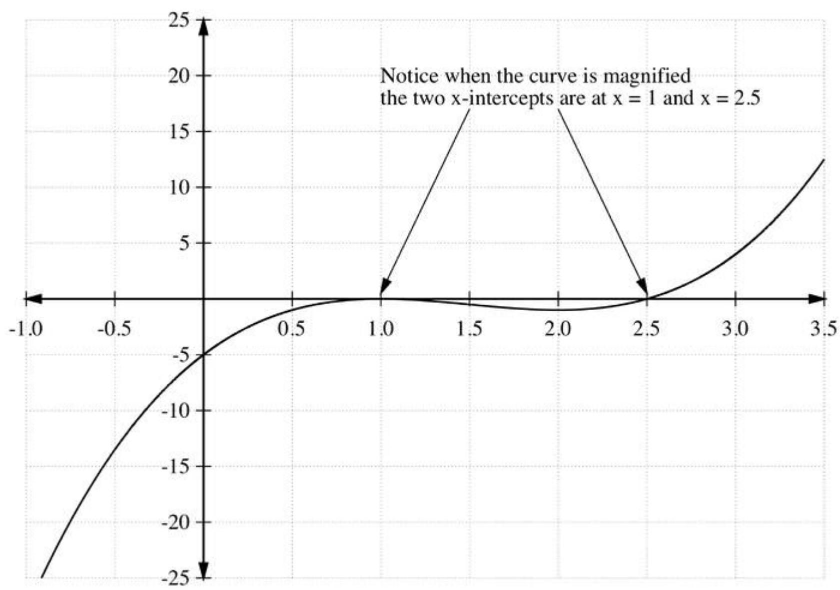
Step 7: Choose test points from each interval and find $h(x)$:

TABLE 3.6:

Interval	**Test Value** x	$h(x)$	**Sign of** $h(x)$	**Location of points on the graph**
$(-\infty, 1)$	-1	-28	-	below the $x-$axis
$\left(1, \frac{5}{2}\right)$	$\frac{3}{2}$	$\frac{-1}{2}$	-	below the $x-$axis
$\left(\frac{5}{2}, \infty\right)$	3	4	+	above the $x-$axis

Using this information, the graph of $h(x)$ is shown in the two graphs below. Notice that the 2nd graph is a magnification of $h(x)$ in the vicinity of the $x-$axis:





Example 4

a) Show that $x+3$ is a factor of $g(x) = x^4 + 2x^3 - 3x^2 + 4x + 12$.

Solution:

By the Factor Theorem, if $f(c) = 0$, then $x - c$ is a factor of the polynomial. In other words, if synthetic division produces a remainder equal to zero, then c is a factor of the polynomial.

$$g(-3) = (-3)^4 + 2(-3)^3 - 3(-3)^2 + 4(-3) + 12 = 0$$

So, -3 is a root and $x+3$ is a factor of $g(x)$.

b) Find the quotient $Q(x)$ and express $f(x)$ in factored form.

Solution:

Use synthetic division with $c = -3$:

$$\begin{array}{r|rrrrr} -3 & 1 & 2 & -3 & 4 & 12 \\ & \downarrow & -3 & 3 & 0 & -12 \\ \hline & 1 & -1 & 0 & 4 & 0. \end{array}$$

Since $g(-3) = 0$ and the quotient is $Q(x) = x^3 - x^2 + 4$, $g(x)$ can be written as

$$g(x) = (x - (-3))(x^3 - x^2 + 4)$$
$$g(x) = (x+3)(x^3 - x^2 + 4).$$

Example 5

Use the Rational Zero Theorem to find all the rational zeros of the polynomial

$$f(x) = x^3 - 2x^2 - 5x + 6.$$

Solution:

Step 1: Assume $\frac{p}{q}$ is a rational zero of f. By the Rational Zero Theorem, p is a divisor of 6 and q is a divisor of 1. Thus, p and q can assume the following respective values:

$$p: 1, -1, 2, -2, 3, -3, 6, -6$$

and

$$q: -1, 1.$$

Therefore, the possible rational zeros will be

$$\frac{p}{q}: -1, 1, -2, 2, -3, 3, -6, 6.$$

Notice that with these choices for p and q, there could be $8 \cdot 2 = 16$ rational zeros. But 8 of them are duplicates. For example, $\frac{1}{-1} = \frac{-1}{1} = -1$.

Step 2: Test all these values by synthetic division (do this on your own for practice), and we finally find that $1, -2,$ and 3 are zeros of f. That is,

$$f(x) = x^3 - 2x^2 - 5x + 6$$
$$= (x-1)(x+2)(x-3).$$

Example 6

Use Descartes's Rule of Signs to identify the maximum number of possible positive and negative roots for the function

$$f(x) = -2x^3 + x^2 - 3x^5 + 5x - 1.$$

Solution:

Step 1: Rewrite $f(x)$ in descending order:

$$f(x) = -3x^5 - 2x^3 + x^2 + 5x - 1.$$

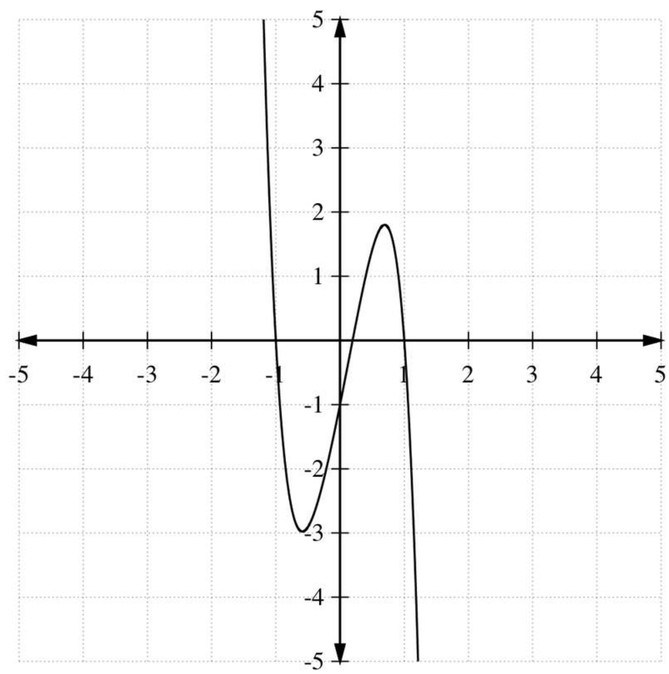
The number of sign changes of $f(x)$ is 2, so the number of positive roots is either 2 or 0.

Step 2: For the negative roots, write

$$f(-x) = -3(-x)^5 - 2(-x)^3 + (-x)^2 + 5(-x) - 1$$
$$f(-x) = 3x^5 + 2x^3 + x^2 - 5x - 1.$$

Step 3: The number of sign changes of $f(-x)$ is 1, so the maximum number of negative roots is 1.

Step 4: The graph of $f(x)$ below shows that there is one negative root and two positive roots:



Example 7

Find the root(s) of $f(x) = 4x^2 - 3x - 7$.

Solution:

Factor and apply the zero product rule:

$4x^2 - 3x - 7 = 0$
$(4x - 7)(x + 1) = 0$
$4x - 7 = 0$ or $x + 1 = 0$

$x = \frac{7}{4}$ or $x = -1$
(each zero has a multiplicity of one).

Example 8

Find the real root(s) of $f(x) = x^4 + 1$.

Solution:

Since there are no even roots of negative numbers, this function has no real zeros:

$$x^4 = -1$$
$$x = \sqrt[4]{-1}.$$

Summary

- Three theorems work together to identify and verify zeros of a polynomial:
  - Remainder Theorem, which is used to quickly evaluate a polynomial;
  - Factor Theorem, which relates a zero remainder to a factor of a polynomial;
  - Rational Zero Theorem, which creates a list of possible rational zeros.
- Descartes's Rule of Sign lists possible numbers of positive and negative zeros, which helps to narrow possibilities.

Review

Find all the possible rational solutions for the following polynomials. Use the Rational Zero Theorem:

1. $f(x) = x^3 + 6x^2 - 18x + 20$
2. $f(x) = 4x^4 + x^2 - 15$
3. $f(x) = -2x^3 + 7x^2 - x + 8$
4. $f(x) = x^4 - 3x^3 - 4x^2 + 15x + 9$
5. $f(x) = 8x^4 - 5x^3 + 16x^2 + 37x - 24$

Find all the real-number solutions for each function below. Use any method you like:

6. $f(x) = 6x^3 - 17x^2 + 11x - 2$
7. $f(x) = x^4 + 7x^3 + 6x^2 - 32x - 32$
8. $f(x) = 16x^3 + 40x^2 - 25x - 3$
9. $f(x) = 2x^3 - 9x^2 + 21x - 18$
10. $f(x) = 4x^3 - 16x^2 + 39x - 295$
11. $f(x) = 6x^4 + x^3 - 29x^2 + 17x + 5$
12. $f(x) = x^5 + 7x^4 - 3x^3 - 65x^2 - 8x + 156$
13. $f(x) = 4x^4 + 20x^3 - 23x^2 - 120x + 144$
14. $f(x) = 9x^4 - 226x^2 + 25$
15. Solve $f(x) = 3x^4 - x^2 - 14$ by factoring. How many real solutions does this function have? What type of solution(s) could the others be?

Review (Answers)

Please see the Appendix.

3.6 Fundamental Theorem of Algebra

Learning Objectives

Learn about the Fundamental Theorem of Algebra, and how to apply it to various functions. You will also review using the quadratic equation to find complex roots, and explore the use of the conjugate root theorem.

Introduction

Suppose the manufacturer of the boxes above wanted to know the size of the sides to ensure the volume of each box was 48 cubic inches. The mathematical approach to solving this problem is well established, but the first question to consider is: how is it possible to know that a solution exists?

The Fundamental Theorem of Algebra is the key to answering this question and determining whether finding the solution is worthwhile. The theorem states every polynomial equation has at least one solution.

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree $n \geq 1$, then $f(x)$ has at least one zero in the complex number domain. In other words, there is at least one complex number c such that $f(c) = 0$.

Several proofs exist for the Fundamental Theorem of Algebra, but all require much more advanced mathematics. This theorem is considered to be one of the most important theorems in mathematics. A corollary of this important theorem is the Factorization Theorem.

The Factorization Theorem:

If

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$, and n is a positive integer, then

$$f(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n),$$

where the numbers c_i are complex numbers. This leads to the n-Roots Theorem.

The n-Roots Theorem

If $f(x)$ is a polynomial of degree n, where $n \neq 0$, then $f(x)$ has, at most, n zeros.

Notice that this theorem does not restrict that the zeros must be distinct. In other words, multiplicity of the zeros is allowed. For example, the quadratic equation $f(x) = x^2 + 6x + 9$ has one zero, -3, with multiplicity $k = 2$. In general, if

$$f(x) = (x - c)^k q(x) \quad \text{and} \quad q(c) \neq 0,$$

then c is a zero of the polynomial f and of multiplicity k. For example,

$$f(x) = (x - 2)^3 (x + 5)$$

has 2 as one zero with $k = 3$ and -5 as a zero with $k = 1$.

Finding a Polynomial Function with Given Roots

On the other hand, if you are given the roots of the polynomial, you can create the equation. By substituting each of the zeros into factors and then distributing, zeros can allow us to determine the polynomial function. This process may remind you of using the zeros to determine a quadratic equation. For example, if we are asked to find a 4th-degree polynomial with the roots 1, 6, -3, and -8, we could write $f(x) = (x - 1)(x - 6)(x + 3)(x + 8)$.

Conjugate Pairs Theorem

If $f(z)$ is a polynomial of degree n, with $n \neq 0$ and with real coefficients, and if $f(z_0) = 0$, where $z_0 = a + bi$, then $f(\overline{z_0}) = 0$ such that $\overline{z_0}$ is the complex conjugate of z_0.

Recall that the complex conjugate for a complex number $a + bi$ is $a - bi$.

This theorem says that if a complex number is a zero of a polynomial with real coefficients, then its complex conjugate must also be a zero of the same polynomial. In other words, complex roots (or zeros) exist in conjugate pairs for the same polynomial. For example, the polynomial function

$$f(x) = x^2 - 2x + 2$$

has two zeros: one is the complex number $1 + i$. By the Conjugate Pairs Theorem (also called the Conjugate Root Theorem), $1 - i$ is also a zero of $f(x) = x^2 - 2x + 2$. We can easily prove that by multiplication:

$$\begin{aligned}[x-(1+i)][x-(1-i)] &= (x-1-i)(x-1+i)\\ &= x^2-x+xi-x+1-i-xi+i+1\\ &= x^2-2x+2\end{aligned}$$

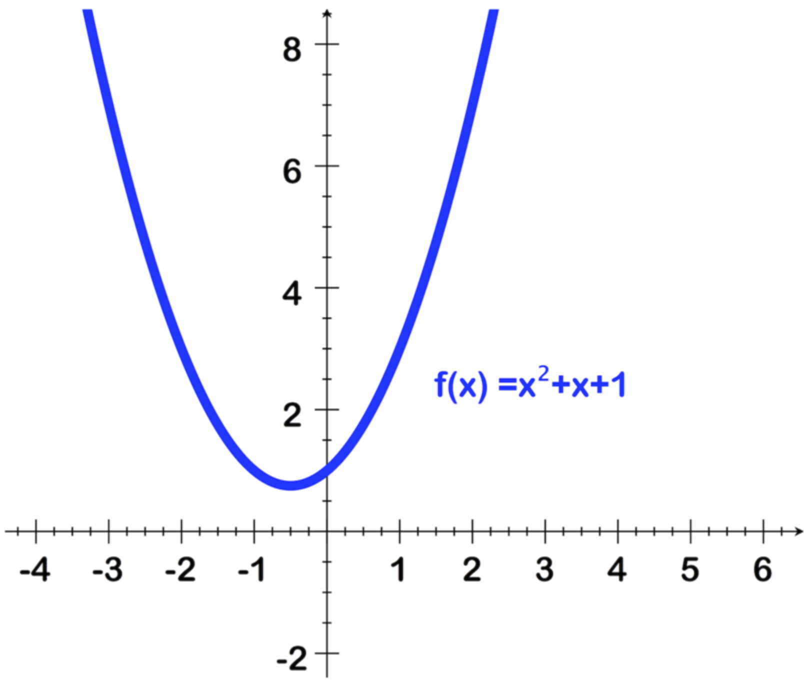
Examples

Example 1

Write $g(x)=x^2+x+1$ in factored form over the complex numbers.

Solution:

Step 1: Notice that $g(x)$ has no real roots, since its graph has no x-intercepts:



Step 2: Using the quadratic formula, the roots of $g(x)$ are

$$\begin{aligned}x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\ &= \frac{-1\pm\sqrt{(1)^2-4(1)(1)}}{2(1)}\\ &= \frac{-1\pm\sqrt{-3}}{2}\\ &= -\frac{1}{2}+\frac{\sqrt{3}}{2}i \text{ or } -\frac{1}{2}-\frac{\sqrt{3}}{2}i.\end{aligned}$$

Step 3: Write $g(x)$ in factored form over the complex numbers:

$$g(x)=\left[x-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right]\left[x-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right].$$

Example 2

Find a polynomial with real coefficients of minimum degree with the zeros: $\frac{-1}{3}, 1-i$ and $2i$.

Solution:

Since the numbers $2i$ and $1-i$ are zeros, then their conjugate pairs must also be zeros by the Conjugate Pairs Theorem. Thus, $-2i$ and $1+i$ must also be roots of $f(x)$. Therefore,

$$f(x) = \left(x+\frac{1}{3}\right)[x-(1-i)][x-(1+i)][x-(2i)][x-(-2i)].$$

Simplify:

$$f(x) = \left(x+\frac{1}{3}\right)(x-1+i)(x-1-i)(x-2i)(x+2i).$$

After multiplying,

$$f(x) = \frac{1}{3}(3x^5-5x^4+16x^3-18x^2+16x+8),$$

which is a 5th-degree polynomial. Notice that the total number of zeros is also 5.

Example 3

What is the multiplicity of each zero of the polynomial below?

$$g(x) = x^4-6x^3+18x^2-54x+81$$

Solution:

Step 1: Using a graphing calculator or program or the Rational Zero Theorem, it can be determined that $x=3$ is a zero of $g(x)$. By synthetic division,

$$\begin{array}{r|rrrrr} 3 & 1 & -6 & 18 & -54 & 81 \\ & \downarrow & 3 & -9 & 27 & -81 \\ \hline & 1 & -3 & 9 & -27 & 0 \end{array}$$

$$g(x) = x^4-6x^3+18x^2-54x+81 = (x-3)(x^3-3x^2+9x-27).$$

Step 2: Using synthetic division again on the resulting quotient, $x=3$ is seen to be a zero of multiplicity 2:

$$\begin{array}{r|rrrr} 3 & 1 & -3 & 9 & -27 \\ & \downarrow & 3 & 0 & 27 \\ \hline & 1 & 0 & 9 & 0 \end{array}$$

$$g(x) = (x-3)(x-3)(x^2+9)$$

Step 3: Factor x^2+9 over the complex numbers:

$$g(x) = (x-3)^2(x-3i)(x+3i).$$

Then, 3 has multiplicity 2, or 3 is said to be a double zero, and $3i$ and $-3i$ are each zeros with multiplicity 1.

Example 4

Identify or estimate the values of the zeros and state their multiplicities for

$y = (x+2)^2(x-1)$.

Solution:

To identify the roots and their multiplicities:

Step 1: Set the function equal to 0:

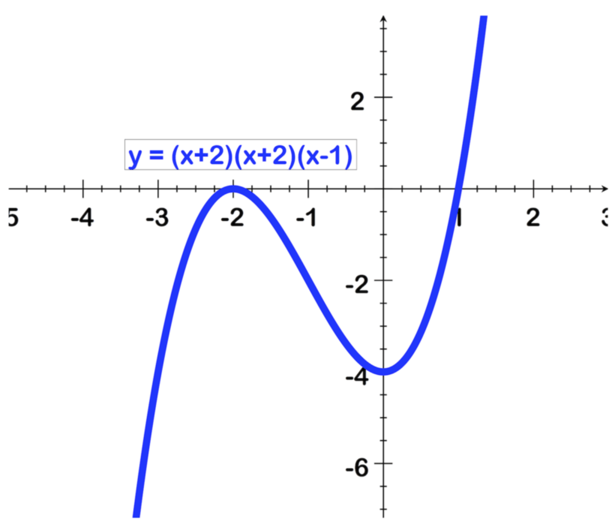
$$(x+2)(x+2)(x-1) = 0.$$

Step 2: Solve. The roots then are $x = 1$ and $x = -2$.

Step 3: Since the $x = -2$ root appears twice, it has a multiplicity of 2, whereas the $x = 1$ root appears only once, so its multiplicity is 1.

Example 5

Identify the zeros for the function graphed below and state their multiplicities:



Solution:

Step 1: Identify points where the function crosses the $x-axis$. On this graph, the function passes through at the root $x = 1$.

Step 2: Identify points where the function touches the $x-axis$ but does not cross. This occurs at the root $x = -2$. If a root has an even multiplicity, it will not intersect but will turn away from the axis, and if it has an odd multiplicity, it will pass through.

Step 3: The zeros of this function are 1 with multiplicity 1, and -2 with multiplicity 2. Note that this makes sense as it is the same function as the previous example.

Example 6

Find a polynomial of degree 5 with real coefficients and the zeros: 1 (multiplicity 2), -2, and $-2i$.

Solution:

Step 1: Create the factors for the given zeros and add the conjugate of -2i:

$$(x-1)(x+2)(x-1)(x+2i)$$

$(x-1)(x-1)(x+2)(x+2i)(x-2i)$.

Step 2: Multiply:

$$f(x)=x^5+x^3+2x^2-12x+8.$$

Summary

b@x

- The Fundamental Theorem of Algebra guarantees that a polynomial of degree $n \geq 1$ has at least one complex zero.
- Its corollaries provide tools to find the zero:
  - The Factorization Theorem states that a polynomial of degree n can be factored over the complex numbers: $f(x)=a_n(x-c_1)(x-c_2)\cdots(x-c_n)$.
  - The Conjugate Pairs Theorem states that if a polynomial has real coefficients, and $z_0=a+bi$ is a zero, then $z_0=a-bi$ is also a zero.
- The n-Roots Theorem states that if $f(x)$ is a polynomial of degree n, where $n \neq 0$, then $f(x)$ has at most n zeros.

Review

For 1-4, find the polynomial with the given roots:

1. 2 (with multiplicity 2), 4 (with multiplicity 3), 1, $\sqrt{2}i, -\sqrt{2}i$.

2. 1, -3 (with multiplicity 3), -1, $\sqrt{3}i, -\sqrt{3}i$.

3. 5 (with multiplicity 2), -1 (with multiplicity 2), $2i, -2i$.

4. $i, -i$, $\sqrt{2}i, -\sqrt{2}i$

Factor each polynomial completely. Use the various theorems from this section, synthetic division, and/or a calculator or graphing program to solve.

5. $f(x)=x^5+4x^4-2x^3-14x^2-3x-18$

6. $g(x) = x^4 - 1$

7. $h(x) = x^6 - 12x^5 + 61x^4 - 204x^3 + 532x^2 - 864x + 576$

8. $j(x) = x^7 - 11x^6 + 49x^5 - 123x^4 + 219x^3 - 297x^2 + 243x - 81$

9. $k(x) = x^5 + 3x^4 - 11x^3 - 15x^2 + 46x - 24$

10. $m(x) = x^6 - 12x^4 + 23x^2 + 36$

11. $n(x) = x^6 - 3x^5 - 10x^4 - 32x^3 - 81x^2 - 85x - 30$

12. $p(x) = x^6 + 4x^5 + 7x^4 + 12x^3 - 16x^2 - 112x - 112$

13. How can you tell the number of roots that a polynomial has from its equation?

14. Explain the meaning of the term "multiplicity."

15. A polynomial with real coefficients has one root that is $\sqrt{3}i$. What other root(s) must the polynomial have?

Review (Answers)

Please see the Appendix.

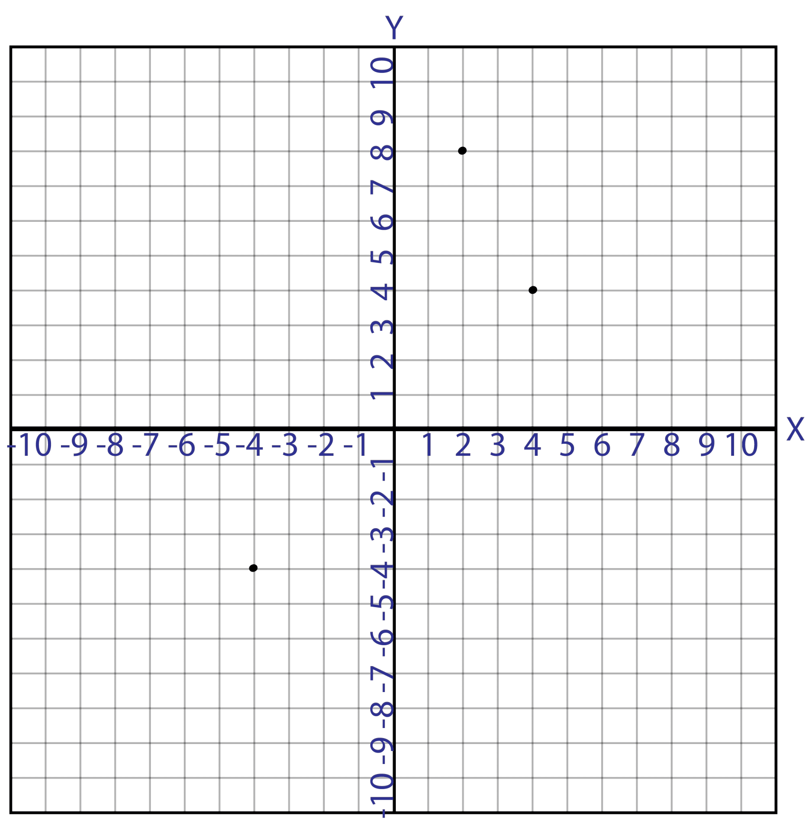
3.7 Approximating Real Zeros of Polynomial Functions

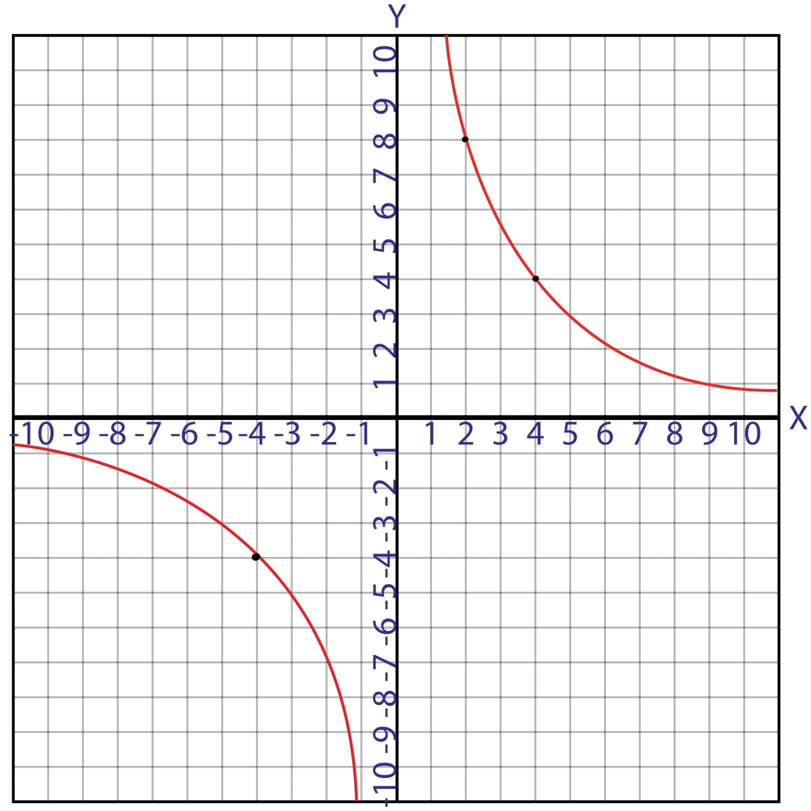
Learning Objectives

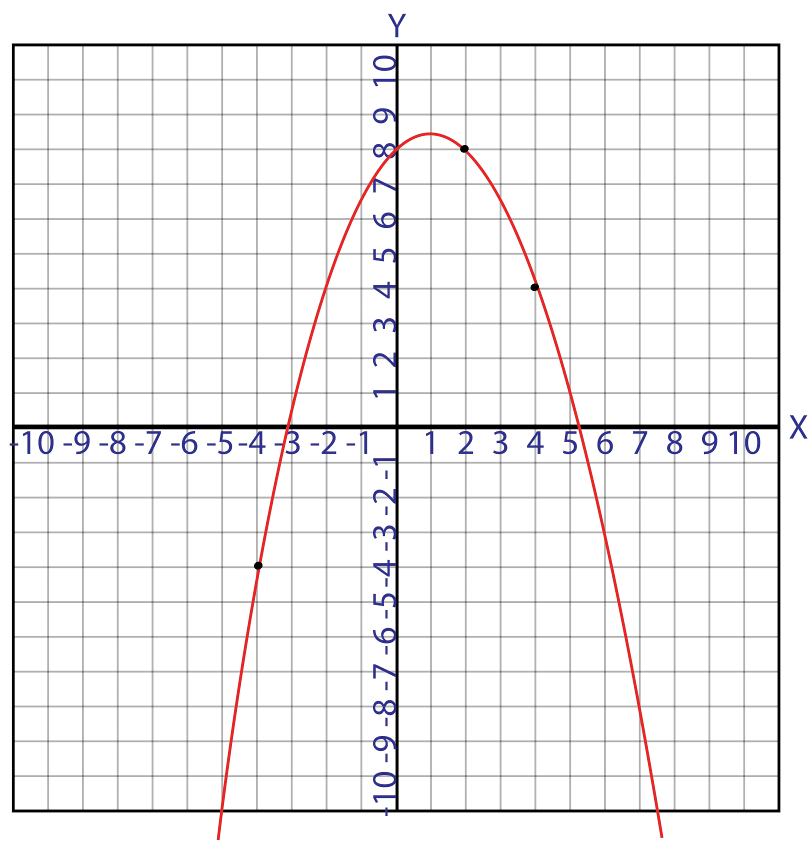
Learn about the Intermediate Value Theorem and the Bounded Roots Theorem in order to approximate the real zeros of a polynomial function.

Introduction

In elementary algebra courses, graphs are often constructed by plotting points and then connecting the points. In precalculus, the points are used to fine-tune a graph. To do this, the 1st step is to classify a function into a family, and choose the appropriate characteristics of that family to build a graph. Interestingly, for many graphs, connecting the points would not be the best choice. To see why, assume that a graph needs to be constructed for a function, $f(x)$. Three points are plotted. The following graphs are possibilities:







This situation demonstrates the need to know more about the function's characteristics to be able to make this decision. It also requires some theorems to know under which conditions the mathematician can "connect the dots" to complete the sketch of a graph.

Intermediate Value Theorem

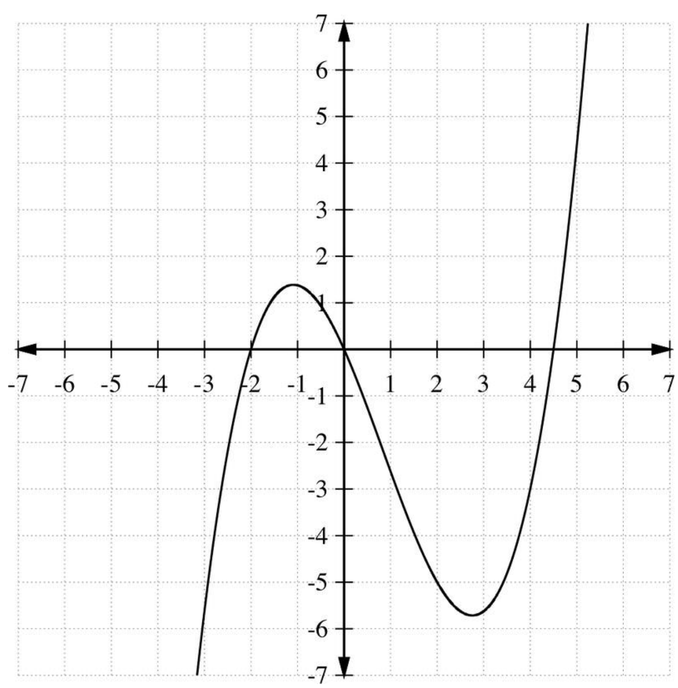
The Intermediate Value Theorem provides the conditions to sketch a continuous graph between the plotted points. It requires that a function be continuous. Recall that an informal definition of continuous is that a function is continuous over a certain interval if it has no breaks, jumps, asymptotes, or holes in that interval. Polynomial functions are continuous for all real x. Rational functions are often not continuous over the set of real numbers because they do have the discontinuities, asymptotes, or holes in the graph.

Knowing that a function is continuous over some interval $[a,b]$ allows a mathematician to use the Intermediate Value Theorem.

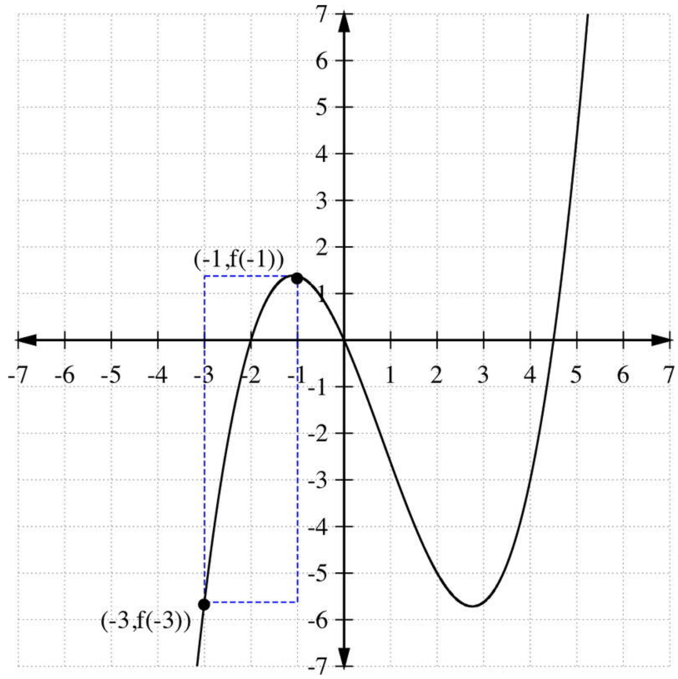
Intermediate Value Theorem

If $f(x)$ is continuous on some interval $[a,b]$, and n is between $f(a)$ and $f(b)$, then there is some $c \in [a,b]$ such that $f(c) = n$.

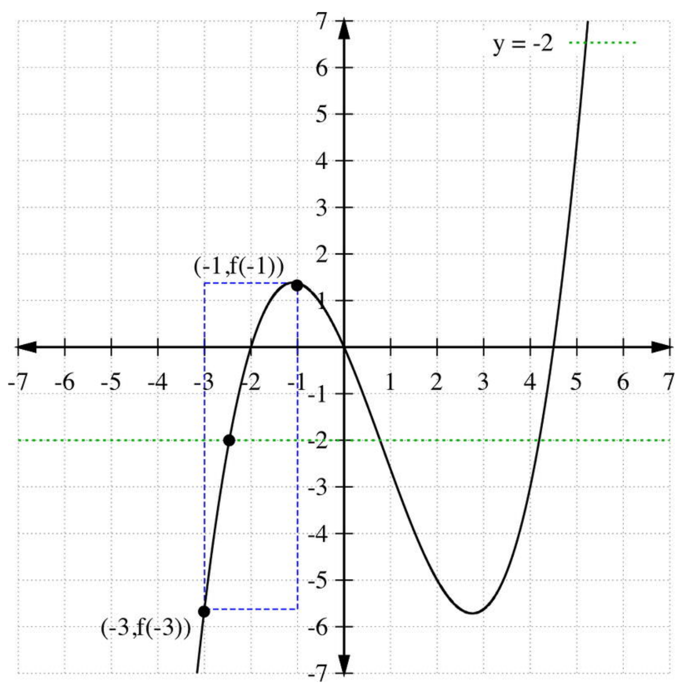
The following graphs highlight how the Intermediate Value Theorem works. Consider the graph of the function $f(x) = \frac{1}{4}\left(x^3 - \frac{5x^2}{2} - 9x\right)$ on the interval [-3, -1]:



$f(-3) = -5.625$ and $f(-1) = 1.375$. Since $f(x)$ is a polynomial, thus continuous for all real values of x, then a continuous curve can be drawn from $(-3, -5.625)$ to $(-1, 1.375)$.



This theorem can be used to estimate the coordinates of points on the graph. For example, any y-value between -5.625 and 1.375, say $y = -2$, corresponds to some $x-$value on the interval [-3, -1].



The Bounded Roots Theorem

The Bounded Roots Theorem is a corollary to the Intermediate Value Theorem:

Bounded Roots Theorem

If f is continuous on $[a,b]$ and there is a sign change between $f(a)$ and $f(b)$ (that is, $f(a)$ is positive and $f(b)$ is negative, or vice versa), then there is a $c \in (a,b)$, such that $f(c) = 0$.

The Bounded Roots Theorem is a corollary to the Intermediate Value Theorem because it is simply a special case where $n = 0$.

In the previous example, $f(x) = \frac{1}{4}\left(x^3 - \frac{5x^2}{2} - 9x\right)$, since $f(-3) < 0$ and $f(-1) > 0$, and $f(x)$ is continuous on $[-3,-1]$, then f has a root on the given interval. The graph estimates the root to be at $x = -2$.

Approximate Zeros of Polynomial Functions

The Intermediate Value Theorem is a simple but extremely powerful theorem. One of the most critical applications of the theorem is illustrated here with bounded roots:

Given a continuous function $g(x)$,

1. Find two points such that $g(a) < 0$ and $g(b) > 0$. Once these two points are found, iterate, using the steps below to home in on the root of $g(x)$ on the interval $[a,b]$. (Note, we will assume $g(b) > g(a)$; the same algorithm works with minor adjustments if the opposite is true.)
2. Evaluate $g\left(\frac{a+b}{2}\right)$.
   a. If $g\left(\frac{a+b}{2}\right) = 0$, then the root is $x = \frac{a+b}{2}$.
   b. If $g\left(\frac{a+b}{2}\right) < 0$, replace a with $\frac{a+b}{2}$, and repeat steps 1-2 using $\left[\frac{a+b}{2}, b\right]$.
   c. If $g\left(\frac{a+b}{2}\right) > 0$, replace b with $\frac{a+b}{2}$, and repeat steps 1-2 using $\left[a, \frac{a+b}{2}\right]$.

This algorithm will not usually find the exact root of $g(x)$, but the root found will be within a tolerable error range. For example, if this process is repeated enough times so that $|a-b| < 0.01$, the root of $g(x)$ is correct within 0.005.

Examples

Example 1

Show that $f(x) = -3x^3 + 5x$ has at least one root in the interval [1, 2].

Solution:

Since $f(x)$ is a polynomial, it is continuous on $[1,2]$. $f(1) = 2$ and $f(2) = -14$. Since $n = 0 \in [-14,2]$, there must exist some point $c \in (1,2)$ such that $f(c) = 0$ by the Intermediate Value Theorem or the Bounded Roots Theorem. Therefore, $f(x)$ has a root in (1, 2).

Example 2

The table below shows several sample values of a polynomial $p(x)$:

x	−4	−2	0	1	4	6	8	10	15	18
$p(x)$	44.15	6.62	−4.12	−4.09	1.16	0	−8.74	−24.07	−49.89	3.41

Based on the information in the table,

(a) What is the minimum number of roots of $p(x)$?

(b) What are bounds on the roots of $p(x)$ identified in (a)?

Solution:

Since $p(x)$ is a polynomial, it is continuous everywhere. Use the Bounded Roots Theorem to identify roots by looking at intervals where $p(x)$ changes sign.

(a) There are four sign changes of $p(x)$ in the table, so $p(x)$ has a minimum of four roots.

(b) The roots are in the following intervals $x \in [-2,0], x \in [1,4], x \in [15,18]$, and the table shows one root is at $x = 6$.

Example 3

Show the 1st five iterations of finding the root of $h(x) = x^2 - x - 1$ using the starting values $a = 0$ and $b = 2$.

Solution:

Step 1: Since $h(x)$ is continuous, verify that there is a root between $x = 0$ and $x = 2$. $h(0) = -1$ and $h(2) = 1$, so by the Bounded Roots Theorem there is a root in the interval [0, 2]. Check $h\left(\frac{0+2}{2}\right) = h(1) = -1$. Since $-1 < 0$, the root is between $x = 1$ and $x = 2$. Continue with the new interval, [1, 2].

Step 2: Now the interval is [1, 2]. $h\left(\frac{1+2}{2}\right) = h(1.5) = -0.25$. Since $-0.25 < 0$, choose the interval [1.5, 2].

Step 3: $h\left(\frac{1.5+2}{2}\right) = h(1.75) \approx 0.31$. Since $0.31 > 0$, choose the interval [1.5, 1.75].

Step 4: $h\left(\frac{1.5+1.75}{2}\right) = h(1.625) \approx 0.02$. Since $0.02 > 0$, the root is between 1.5 and 1.625.

Step 5: $h\left(\frac{1.5+1.625}{2}\right) = h(1.5620) \approx -0.12$. Since $-0.12 < 0$, we know the root is between 1.5620 and 1.625.

This example shows that after five iterations, the possible location of the root, the midpoint of the interval, is accurate within approximately 0.03 units.

Summary

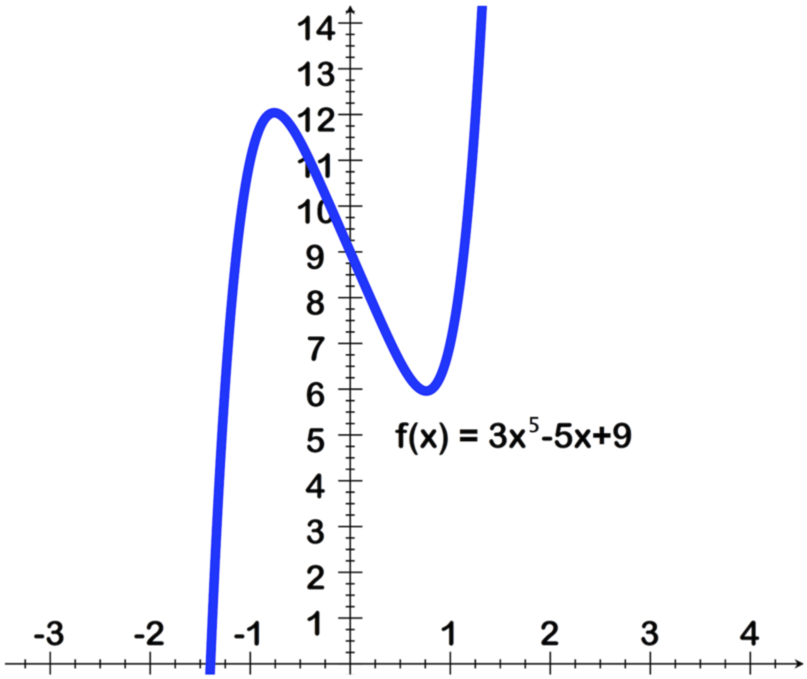
- **Intermediate Value Theorem**
  If $f(x)$ is continuous on some interval $[a,b]$, and n is between $f(a)$ and $f(b)$, then there is some $c \in [a,b]$ such that $f(c) = n$.
- **Bounded Roots Theorem**
  If f is continuous on $[a,b]$ and there is a sign change between $f(a)$ and $f(b)$ (that is, $f(a)$ is positive and $f(b)$ is negative, or vice versa), then there is a $c \in (a,b)$ such that $f(c) = 0$.

Review

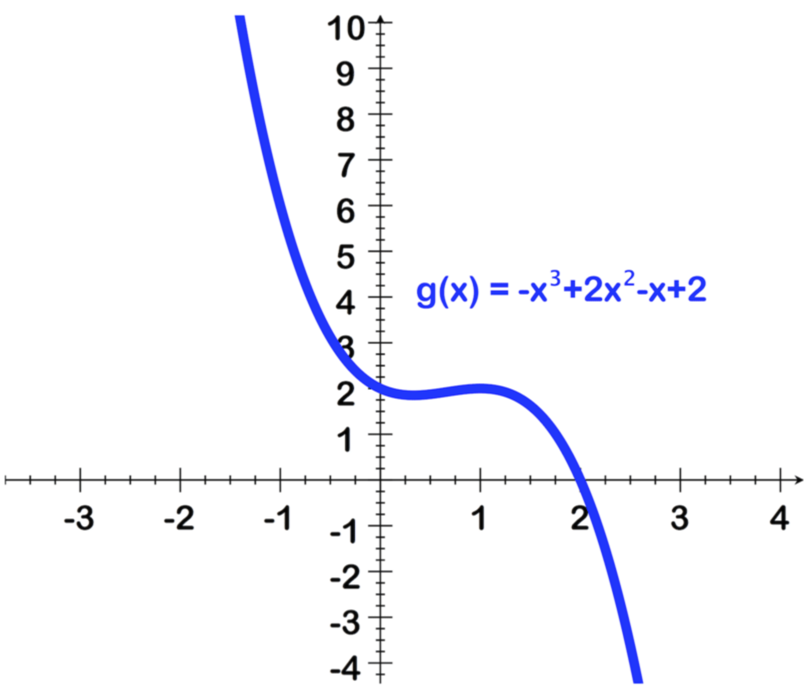
For questions 1-5, use the image of the graph and accompanying equation to find the following:

a. Leading coefficient and degree of the polynomial,
b. Number of real zeros and their approximate values using the graph,
c. Number of imaginary zeros.

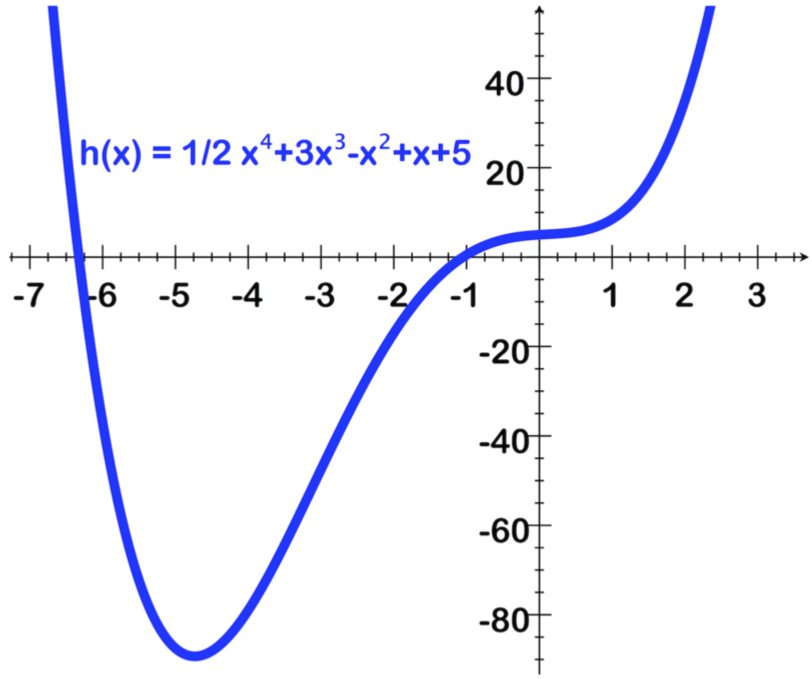
1.



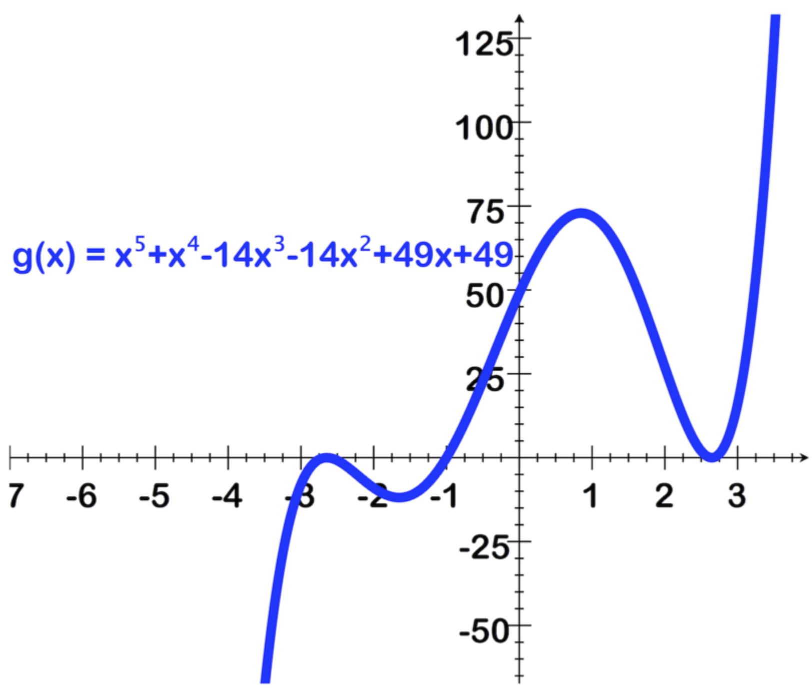
2.



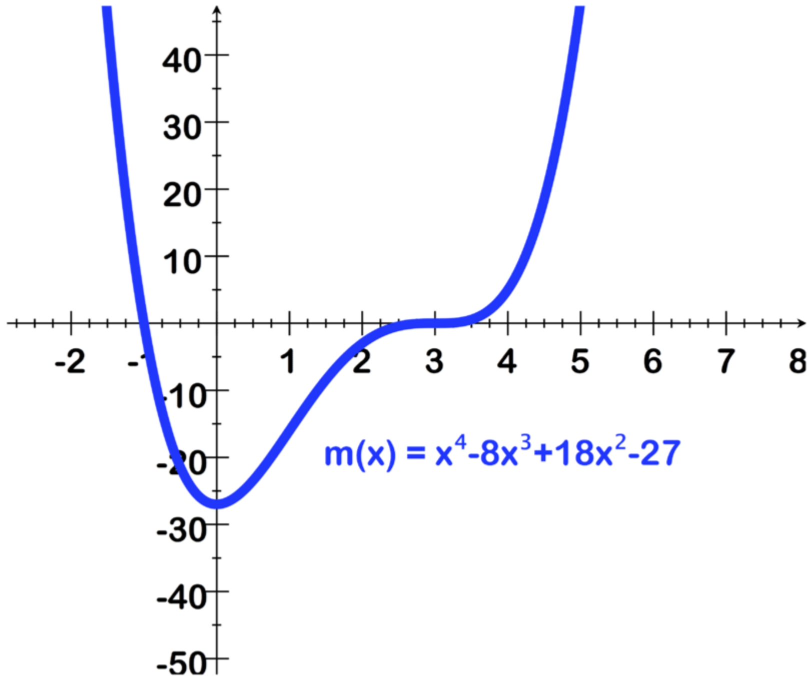
3.


h(x) = 1/2 x^4+3x^3-x^2+x+5
40
20
-20
-40
-60
-80
-7 -6 -5 -4 -3 -2 -1 1 2 3

4.


g(x) = x^5+x^4-14x^3-14x^2+49x+49
125
100
75
50
25
-25
-50
7 -6 -5 -4 -3 -2 -1 1 2 3

5.



For questions 6-10, use the Intermediate Value Theorem to show the bounds on the roots of each function. The bounds should be integer values.

6. $f(x) = 2x^3 - 3x + 4$
7. $g(x) = -5x^2 + 8x + 12$
8. $h(x) = \frac{1}{2}x^4 - x^3 - 3x^2 + 1$
9. $k(x)$ is a polynomial and selected values of $k(x)$ are given in the following table:

x	-3	-2	-1	0	1	2	3
$k(x)$	-23.5	-1	0.5	-1	.5	-1	-23.5

10. Stephen argues the function $r(x) = \frac{4x+1}{x+3.5}$ has two zeros based on the following table and an application of the Bounded Roots Theorem. What is faulty about Stephen's reasoning?

x	-5	-4	-3	-2	-1	0	1	2	3	4
$r(x)$	12.67	30	-22	-4.67	-1.20	0.29	1.11	1.64	2.0	2.27

For problems 11-12, apply the numerical algorithm five times to find a bound on the zeros of the following functions, given the indicated starting values. Give a final estimate for the zero.

11. $k(x) = x^4 - 3x + 1$ on $[0, 1]$
12. $b(x) = -0.1x^5 + 3x^3 - 5x^2$ on $[1, 3]$

Review (Answers)

Please see the Appendix.

3.8 Rational Functions

Learning Objectives

Learn about rational functions and how to graph these functions.

Introduction

Students at a college are currently required to demonstrate their understanding of their classwork at a level of 75% or higher to move on to more advanced material.

Suppose a study of the link between grade and study time suggests that the amount of homework time T in hours required for each student based on an understanding level of $p\%$ is given by

$$T(p) = \frac{(18p)}{(100-p)}.$$

The domain of $T(x)$ is $[0, 100)$. If the college administration decides to raise the minimum level of understanding to 82%, how will this affect the students' homework time?

This mathematical model is an example of a rational function. These functions have characteristics that are very different from those of polynomial functions, but like that class of functions, specific techniques can be established for analysis.

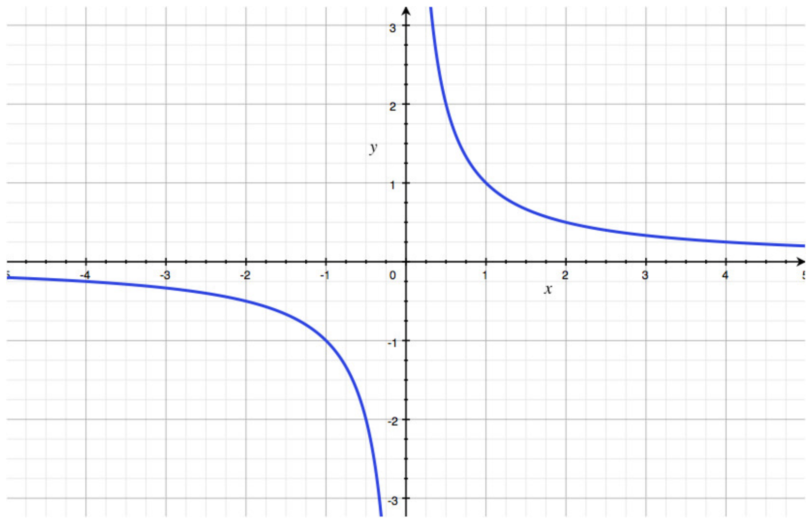
Standard Form of Rational Functions

A function can be written in the form

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials, and $Q(x) \neq 0$ is called a rational function. The domain of a rational function includes all real numbers x so that the denominator is not zero.

An example of a rational function is $f(x) = \frac{1}{x}$.



Asymptotes

An asymptote is a line or curve to which a function's graph approaches continuously but does not reach (with certain exceptions). There are three types of asymptotes: vertical, horizontal, and oblique (also known as slant). Functions cannot intersect a vertical asymptote. The horizontal asymptote denotes the end behavior of the function as $x \to \pm\infty$. There are certain instances in which the graph of a function intersects the horizontal asymptote.

Vertical Asymptotes

Find vertical asymptotes by setting the denominator to zero and then solving for x. For

$$r(x) = \frac{P(x)}{Q(x)},$$

the solution to $Q(x) = 0$ will give the vertical asymptote(s).

So if

$$f(x) = \frac{x+2}{(x-1)(x+3)},$$

setting

$$(x-1)(x+3) = 0$$

will give the vertical asymptotes at $x = 1$ and $x = -3$.

Horizontal Asymptote

The horizontal asymptote is a line parallel to the $x-$axis, which the function approaches but does not intersect as $x \to \infty$ and $x \to -\infty$. (Note that a rational function can cross a horizontal asymptote at other places on the graph.) To find the horizontal asymptote, follow the procedure below:

How to Find the Horizontal Asymptote

- Expand the numerator and denominator if they are written in a factored form.
- There are three possibilities:
  - If the degree of the numerator is smaller than the degree of the denominator, then the horizontal asymptote equals $y = 0$, the $x-$axis itself.
  - If the degree of the denominator and the numerator are the same, then the horizontal asymptote equals the ratio of the leading coefficients.
  - If the degree of the numerator is larger than the degree of the denominator, then there is no horizontal asymptote.

Oblique Asymptotes

When the degree of the numerator is 1 greater than the degree of the denominator, there is no horizontal asymptote. However, there is another type of asymptote, the oblique or slant asymptote. Note that there is no linear oblique asymptote if the degree of the numerator is 2 or more greater than the degree of the denominator.

To determine the oblique asymptote, divide the polynomial in the numerator by the polynomial in the denominator. The resulting quotient polynomial is the oblique asymptote.

Graphing Rational Functions Using Transformation

Just like polynomials, rational functions can be graphed using transformations. Some transformations change the asymptotes while others do not.

- $r(x)+c$ is a vertical shift that moves each horizontal asymptote up c units (or down if $c < 0$).
- $r(x-c)$ is a horizontal shift that moves each vertical asymptote right c units (or left if $c < 0$).
- $a \cdot r(x)$ is a vertical stretch that moves horizontal asymptotes by a multiple of a (so this moves the horizontal asymptote closer to the $x-$axis if $a < 1$).
- $r(a \cdot x)$ is a horizontal compression that moves the vertical asymptotes closer to $y-$axis by a factor of $\frac{1}{a}$.
- $r(-x)$ is a reflection about the $y-$axis. All vertical asymptotes are also reflected.
- $-r(x)$ is a reflection about the $x-$axis. All horizontal asymptotes are also reflected.

Examples

Example 1

What is the domain of $f(x) = \frac{1}{x}$?

Solution:

Determine any values that would make the denominator zero. In this case, when $x = 0$, the function is undefined.

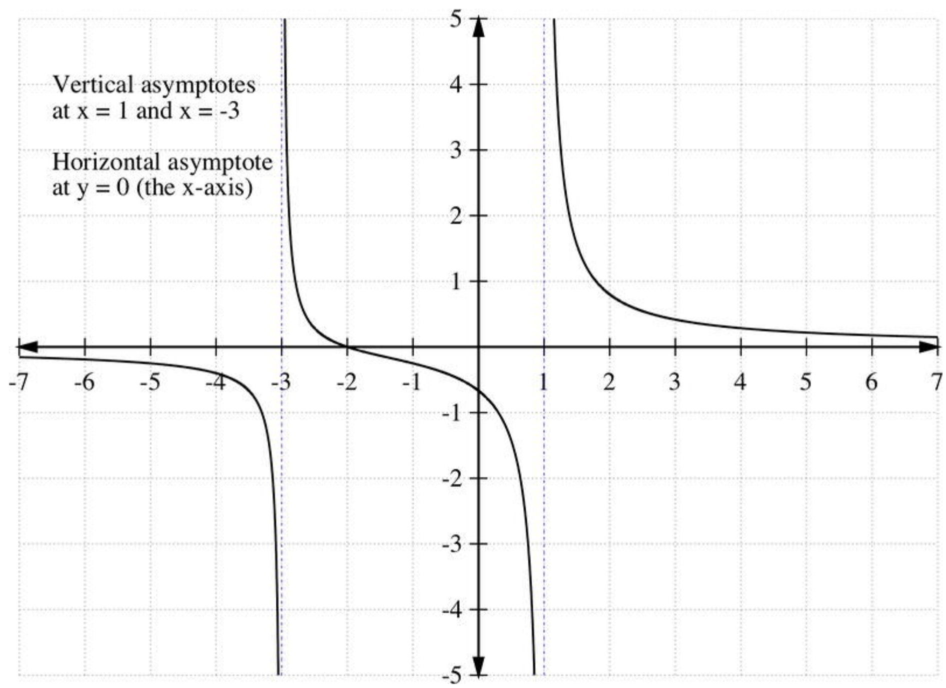
Therefore, the domain of this function is all real numbers except $x = 0$. In set notation, the domain is $\{x | x \neq 0\}$.

Example 2

What is the domain of $f(x) = \frac{x+2}{(x-1)(x+3)}$?

Solution:

The domain is all real numbers except those that cause the denominator to become zero: $x = 1$ and at $x = -3$.



Example 3

Find the vertical and horizontal asymptotes of

$$f(x) = \frac{2x^3 - 2x^2 + 5}{3x^3 - 81}.$$

Solution:

Step 1: To find the vertical asymptote(s), set the denominator to zero and then solve for x:

$$\begin{aligned} 3x^3 - 81 &= 0 \\ 3x^3 &= 81 \\ x^3 &= 27 \\ x &= \sqrt[3]{27} \\ x &= 3. \end{aligned}$$

Thus, the graph has a vertical asymptote at $x = 3$.

Step 2: To find the horizontal asymptote, consider the leading terms:

$$\frac{2x^3}{3x^3}.$$

Notice that the degree of the numerator and the denominator are the same and therefore the horizontal asymptote is the ratio of the coefficients,

$$\frac{2x^3}{3x^3} = \frac{2}{3}.$$

So the horizontal asymptote is at $y = \frac{2}{3}$.

Example 4

Find the asymptotes of

$$f(x) = \frac{3x-2}{4x^4-9}.$$

Solution:

Step 1: To find the vertical asymptote(s), set the denominator equal to 0 and solve for x:

$$\begin{aligned}
4x^4 - 9 &= 0\\
x^4 &= \frac{9}{4}\\
x &= \sqrt[4]{\frac{9}{4}}\\
x &= \left(\left(\frac{9}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\\
x &= \pm\frac{\sqrt{3}}{\sqrt{2}}\\
x &= \pm\frac{\sqrt{6}}{2}.
\end{aligned}$$

Thus, the graph has a vertical asymptote at $x = \frac{\sqrt{6}}{2}$ and $x = -\frac{\sqrt{6}}{2}$.

Step 2: To find the horizontal asymptote, consider the leading terms:

$$\frac{3x}{4x^4}.$$

Notice that the degree of the numerator is less than the degree of the denominator. Therefore, the horizontal asymptote is at $y = 0$, so the $x-$axis plays the role of the horizontal asymptote.

Example 5

Find the horizontal and vertical asymptotes of the rational function:

$$g(x) = \frac{2x^4 - 9}{3x - 2}.$$

Solution:

Step 1: To find the vertical asymptote(s), set the denominator equal to zero and solve for x:

$$3x - 2 = 0.$$

Thus, the graph has a vertical asymptote at $x = \frac{2}{3}$.

Step 2: To find the horizontal asymptote, consider the leading terms:

$$\frac{2x^4}{3x}.$$

Here, the degree of the numerator is larger than the degree of the denominator. Thus, there is no horizontal asymptote.

Example 6

Graph

$$T(x) = \frac{2x + 1}{x - 1}.$$

Solution:

To graph T, there are four important items you need to find: the $y-$intercept, the $x-$intercept, the vertical asymptote, and the horizontal asymptote.

The $y-$intercept can be found by finding $y = T(0)$:

$$y = T(0) = \frac{2(0) + 1}{0 - 1} = -1.$$

Thus the $y-$intercept is at point (0, -1).

The $x-$intercept can be found by setting $y = T(x) = 0$:

$$\frac{2x + 1}{x - 1} = 0.$$

Note that a reduced fraction $\frac{a}{b} = 0$ if and only if $a = 0$. Solve

$$\begin{aligned} 2x + 1 &= 0 \\ x &= \frac{-1}{2}. \end{aligned}$$

In general, for a rational function, $\frac{P(x)}{Q(x)}$, set $P(x) = 0$ to find the $x-$intercept for any rational function. Make sure the $x-$value is in the domain of the function. Note that if both $P(x) = 0$ and $Q(x) = 0$ for the same value of x, then the graph has a hole in it instead of a vertical asymptote at that value, if the multiplicity of that factor is the same in the numerator and denominator.

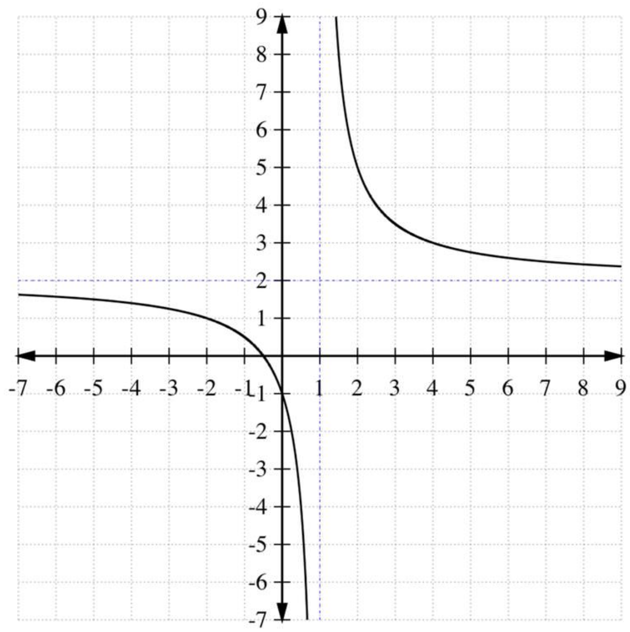
Next, the vertical asymptote: set $Q(x) = 0$:

$$\begin{aligned} x - 1 &= 0 \\ x &= 1. \end{aligned}$$

And the horizontal asymptote:

$$\frac{2x+1}{x-1} \rightarrow \frac{2x}{x} = 2.$$

Therefore, the vertical asymptote is at $x = 1$, and the horizontal asymptote is at $y = 2$. From this information, and calculating the value of a few other points if needed, sketch the graph:



Example 7

Graph

$$g(x) = \frac{2x^2+1}{2x^2-3x}.$$

Solution:

Step 1: The $y-$intercept is

$$y = g(0) = \frac{1}{0} = \text{undefined},$$

so there is no $y-$intercept.

Step 2: The $x-$intercept can be found by setting the numerator to zero:

$$\begin{aligned} 2x^2+1 &= 0 \\ 2x^2 &= -1 \\ x^2 &= \frac{-1}{2} \\ x &= \sqrt{\frac{-1}{2}}. \end{aligned}$$

Notice this equation has no real solution. Therefore, there is no $x-$intercept.

Step 3: The vertical asymptote can be found by setting the denominator to zero:

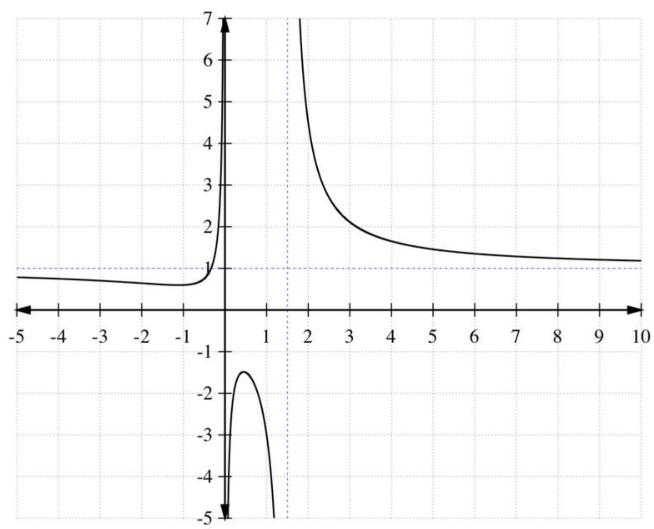
$$\begin{aligned} 2x^2-3x &= 0 \\ x(2x-3) &= 0. \end{aligned}$$

The two solutions are $x = 0$ and $x = \frac{3}{2}$, and these are the vertical asymptotes.

Step 4: Finally, the horizontal asymptote is found by analyzing the leading terms:

$$\frac{2x^2+1}{2x^2-3x} \to \frac{2x^2}{2x^2} = 1.$$

That is, $y = 1$ is a horizontal asymptote. After finding a few points on the graph, the sketch of the graph of $g(x)$ is created:



Example 8

Recall the problem from the Introduction. The study time function was $T(p) = \frac{18p}{100-p}$. How does the administration's change from 75% to 82% mastery affect student study time?

Solution:

Step 1: Calculate by evaluating at the given values:

$$T(75) = \frac{18(75)}{100-75} = 54$$
$$T(82) = \frac{18(82)}{100-82} = 82.$$

Step 2: When the administration increases the rate from 75% to 82%, the study time increases from 54 hours to 82 hours.

Example 9

Graph

$$g(x) = \frac{x^2-1}{x-2}.$$

Solution:

Step 1: Find your y-intercept.

$$g(0) = \frac{(0)^2-1}{0-2} = \frac{1}{2}$$

Step 2: Find your x-intercept, which occurs when y is zero, so the numerator is zero.

$$\begin{aligned} 0 &= x^2-1 \\ 0 &= (x-1)(x+1) \\ x &= \pm 1 \end{aligned}$$

Since neither of the above cause the denominator to be zero, both are valid intercepts.

Step 3: Observe that the vertical asymptote is at $x = 2$. Notice the degree of the numerator is greater than the degree of the denominator. This function has an oblique asymptote. To identify it, change the form of the rational expression using long division. Any rational functions can be written as

$$\frac{f(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)},$$

where $f(x)$ is the original numerator, $D(x)$ is the divisor, $Q(x)$ is the quotient, and $R(x)$ is the remainder.

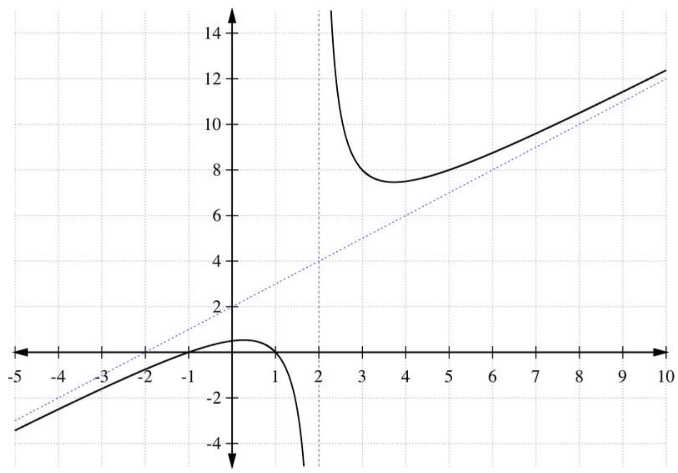
Step 4: The long division:

$$\begin{array}{r} x+2 \\ x-2\,\overline{)x^2+0x-1} \\ -\underline{(x^2-2x)}\;\downarrow \\ 2x-1 \\ -\underline{(2x-4)} \\ 3 \end{array}$$

allows the function $g(x)$ to be rewritten as

$$g(x) = \frac{x^2-1}{x-2} = x+2+\frac{3}{x-2}.$$

Step 5: As $x \to \infty$ or $x \to -\infty$, the graph of $g(x) = x+2+\frac{3}{x-2}$ gets closer and closer to the line $y = x+2$. To see this, let $x = 1,000,000$. Then the remainder of this rational function becomes $\frac{3}{999,998} \approx 0$, and $g(x)$ is nearly the same as $x+2$. This behavior is denoted with a dotted line, an oblique asymptote.



Example 10

Graph

$$f(x) = \frac{x^2-x-2}{x-1}.$$

Solution:

Step 1: Factor the numerator :

$$f(x) = \frac{x^2-x-2}{x-1} = \frac{(x-2)(x+1)}{x-1}.$$

The vertical asymptote is $x = 1$ since it would make the denominator zero and its factor doesn't cancel out with any factors in the numerator.

Step 2: Since the degree of the polynomial in the numerator is larger than the degree of the polynomial in the denominator, there is no horizontal asymptote. Instead, divide x^2-x-2 by $x-1$ to yield:

$$f(x) = x - \frac{2}{x-1},$$

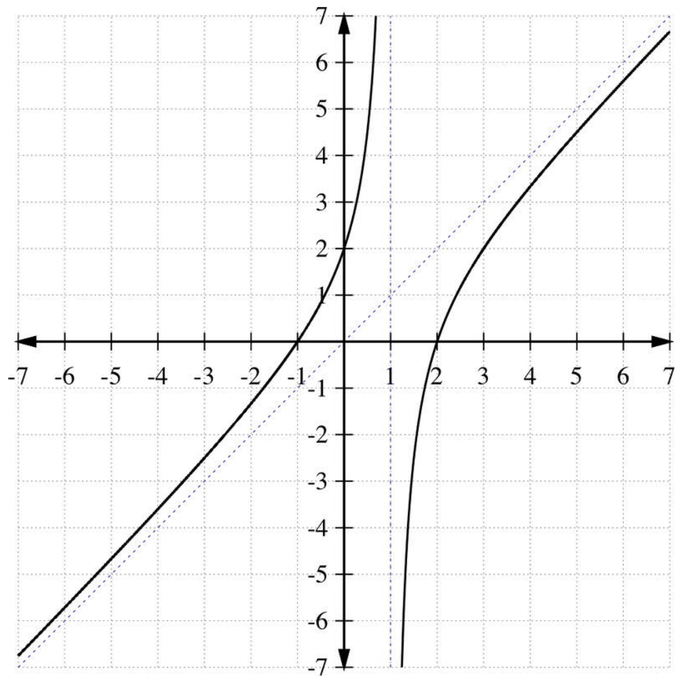
which indicates an oblique asymptote at $y = x$.

Step 3: Notice that the $x-$intercepts are at $x = 2$ and $x = -1$.

Step 4: Find the y-intercept.

$$f(0) = \frac{0^2 - 0 - 2}{0 - 1} = 2$$

Step 5: Draw the asymptote lines, plot the intercepts, and sketch the graph of the rational function:

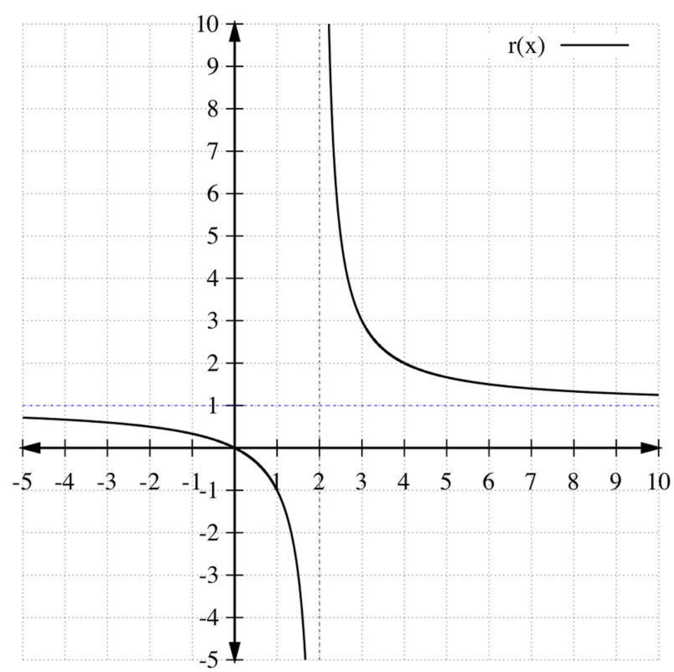


Example 11

A rational function $r(x)$ is shown in the figure below. Use the graph of $r(x)$ to sketch a graph of: a) $r(x) - 3$,

b) $r(-x)$,

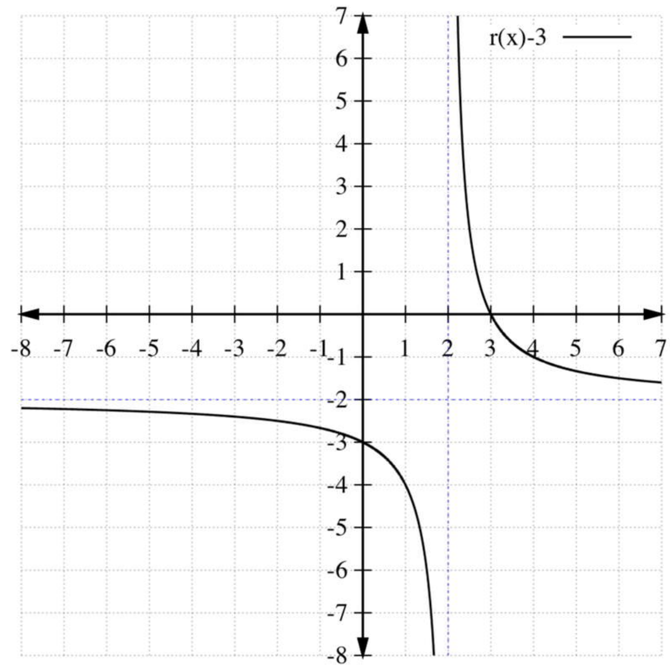
c) $r(3 - x)$.



Solutions:

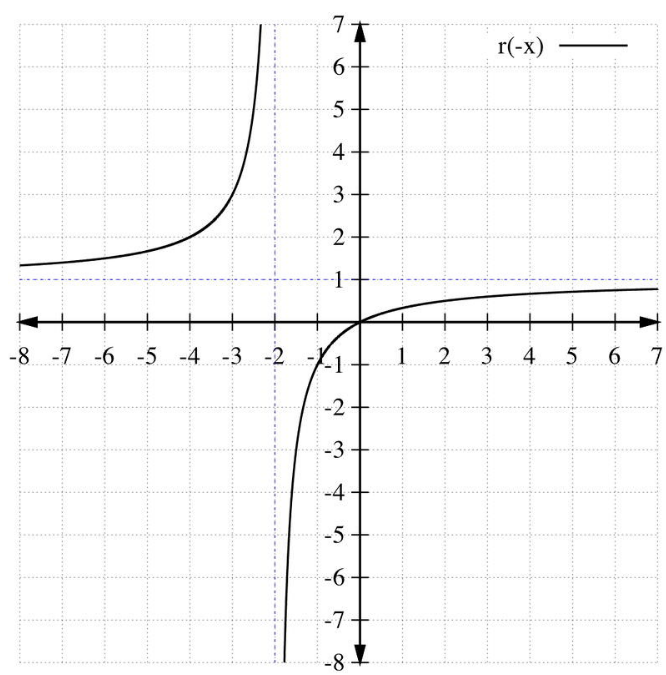
a) $r(x)-3$

The horizontal asymptote moves down by 3 units:



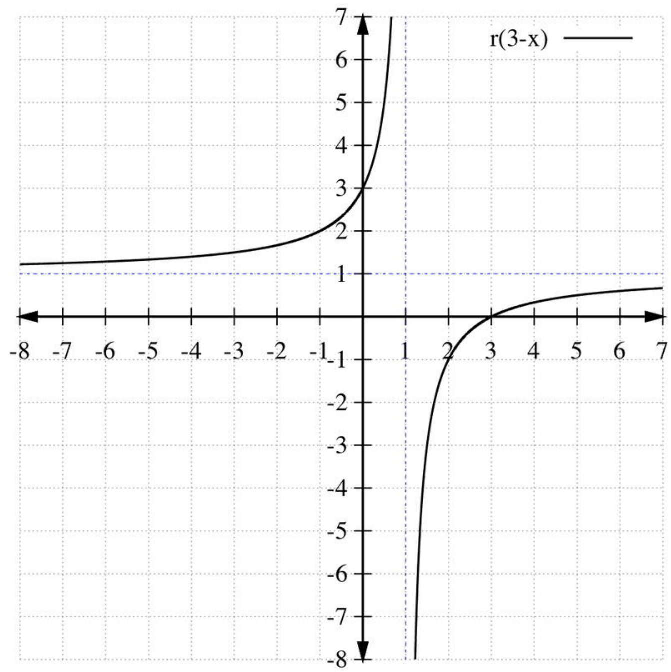
b) $f(-x)$

The function is reflected about the y-axis so the vertical asymptote is also reflected:



c) $r(3-x) = r(-(x-3))$

First graph $r(-x)$, and then shift that graph 3 units to the right to get $r(-(x-3))$. The new vertical asymptote is $x=1$.



Summary

- Asymptotes are critical to analyze rational functions.
- The vertical asymptote is found by setting the denominator of the function equal to zero if the rational function is in reduced form (all common factors have been cancelled). Then solve for x.
- The horizontal asymptote is found by creating a fraction with the dominant terms of the numerator and denominator, and comparing the degrees. There are three possibilities:
  - If the degree of the numerator is smaller than the degree of the denominator, then the horizontal asymptote equals $y=0$, the $x-$axis itself.
  - If the degree of the denominator and the numerator are the same, then the horizontal asymptote equals the ratio of the leading coefficients.
  - If the degree of the numerator is larger than the degree of the denominator, then there is no horizontal asymptote.
- If the degree of the numerator is 1 greater than the degree of the denominator, rewrite the function by using long division to reveal the oblique (slant) asymptote.

Review

Find all asymptotes of the following functions:

1. $y=\frac{x-2}{x^2+6x+8}$
2. $y=\frac{x^2-4}{x+5}$
3. $y=\frac{x^2}{x-3}$
4. Find the x-intercepts of the function in number 2.
5. Find the x-intercepts of the function in number 3.

Graph the following functions. Find any intercepts and asymptotes.

6. $y = \frac{x+1}{x^2-x-12}$
7. $f(x) = \frac{x^2+3x-10}{x-3}$
8. $y = \frac{-8x^3-8x^2+2x+8}{x+2}$
9. $g(x) = \frac{2x^2-2}{3x+5}$
10. $y = \frac{-2x^3+2x^2+5x+2}{(x-2)(x+7)}$
11. $f(x) = \frac{x^2+x-30}{2x^3-5x^2-4x+3}$
12. $y = \frac{7x^3+2x^2-7x-3}{x^3}$
13. $f(x) = \frac{2x+5}{x^2+5x-6}$
14. $g(x) = \frac{-x^2+3x+4}{2x-6}$
15. Determine the slant asymptote of $y = \frac{3x^2-x-10}{3x+5}$. Now, graph this function. Is there really a slant asymptote? Can you explain your results?
16. In physics, Boyle's Law states that the product of the pressure P of a gas and the volume V of the container is always a constant. That is,

$$PV = \text{constant}.$$

Suppose the constant is equal to 4000 $Pa \cdot m^3$ (Pascal cubic meters). So

$$P = \frac{4000}{V},$$

where the pressure is measured in Pascals and the volume is in meters cubed. Sketch the graph of the equation for $V > 0$.

Review (Answers)

Please see Appendix.

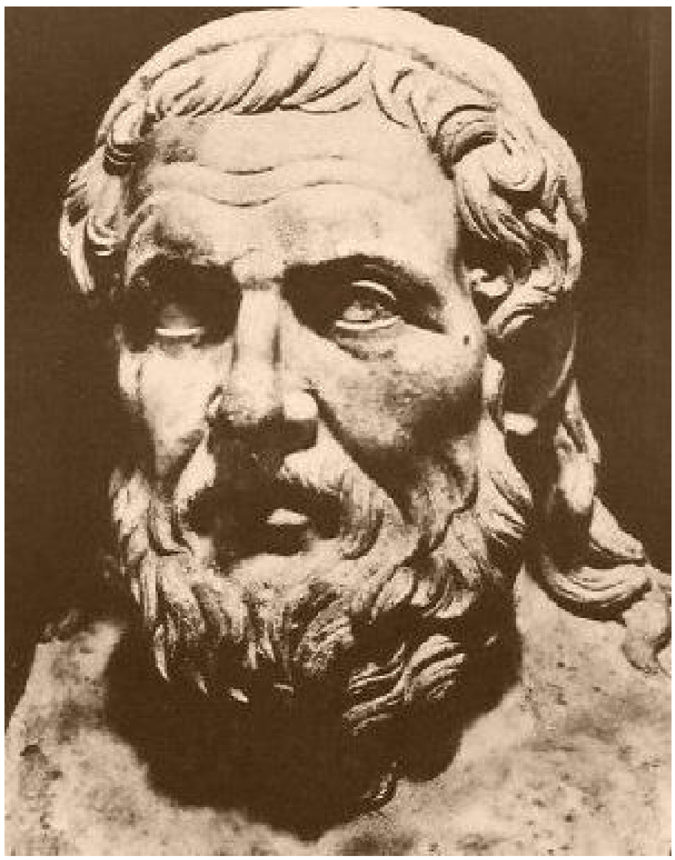
3.9 Analysis of Rational Functions

Learning Objectives

Learn to analyze rational functions by identifying removable discontinuities and to graph rational functions with these discontinuities.

Introduction

The word asymptote is derived from the Greek word *asymptotos,* which translates to "not falling together."



It was coined around the 3rd century B.C. by Greek mathematician Apollonius of Perga (above) in his work on conic sections, when he discovered lines that approached but did not intersect the curves he was studying.

Removable Discontinuity

Rational functions are defined as $r(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

The end behavior of a rational function can often be identified by the horizontal or oblique asymptote. That is, as the values of x get very large or very small, the graph of the rational function will approach the horizontal or oblique asymptote. Some rational functions have discontinuities, either instead of or along with any vertical asymptote. These are often characterized as removable discontinuities or holes.

Removable Discontinuity (or hole)

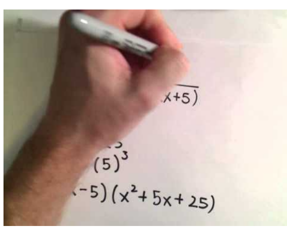
A point where a function is discontinuous or undefined.

To determine the location of a hole on the graph of a rational function:

Step 1: Determine all of the factors of the polynomials in the numerator and the denominator of the given rational function.

Step 2: Determine any sets of common factors in the numerator and denominator. At this particular x-value, the common factors would create the undefined fraction $\frac{0}{0}$. Instead, these common factors should be canceled.

Step 3: Set each canceled factor equal to zero and solve for x. The result is the point(s) where the function is undefined, creating one or more points of discontinuity.



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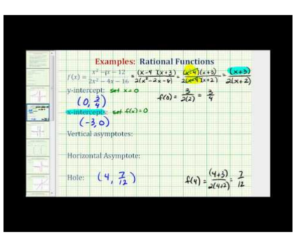
A rational function may have a vertical, a horizontal, and/or an oblique asymptote. A vertical asymptote is found by setting the denominator of a rational function in reduced form equal to zero and solving for x. A horizontal asymptote is found by creating a fraction with the leading terms of the numerator and denominator, and comparing the degrees. If the degree of the numerator is smaller than the degree of the denominator, then the horizontal asymptote equals $y = 0$. If the degree of the numerator and denominator are the same, then the horizontal asymptote equals the ratio of the leading coefficients. If the degree of the numerator is larger than the degree of the denominator, then there is no horizontal asymptote. If the degree of the numerator is 1 greater than the degree of the denominator, rewrite the function by using long division to reveal the oblique asymptote.



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Examples

Example 1

Consider the following rational function: $\frac{x^2+2x-35}{x+7}$. Find all restrictions on the domain and any vertical asymptotes.

Solution:

Step 1: Factor the numerator and reduce:

$$f(x) = \frac{(x-5)(x+7)}{x+7}$$
$$f(x) = \frac{(x-5)\cancel{(x+7)}}{\cancel{x+7}}.$$

Step 2: There is no vertical asymptote for this function, but rather a hole in the graph at $x=-7$, so the domain is the set of all real numbers except $x=-7$.

Example 2

Find the restrictions on the domain and any asymptotes for the function:

$$h(x) = \frac{3x}{x^2-25}$$

Solution:

Step 1: Set the denominator equal to 0:

$$\begin{aligned} x^2-25 &= 0 \\ x^2 &= 25 \\ x &= \pm 5 \end{aligned}$$

Step 2: The domain of $h(x)$ is the set all real numbers x with the restriction $x \neq \pm 5$. Since these values don't make the numerator equal to zero, $h(x)$ has two vertical asymptotes, one at $x=5$ and one at $x=-5$.

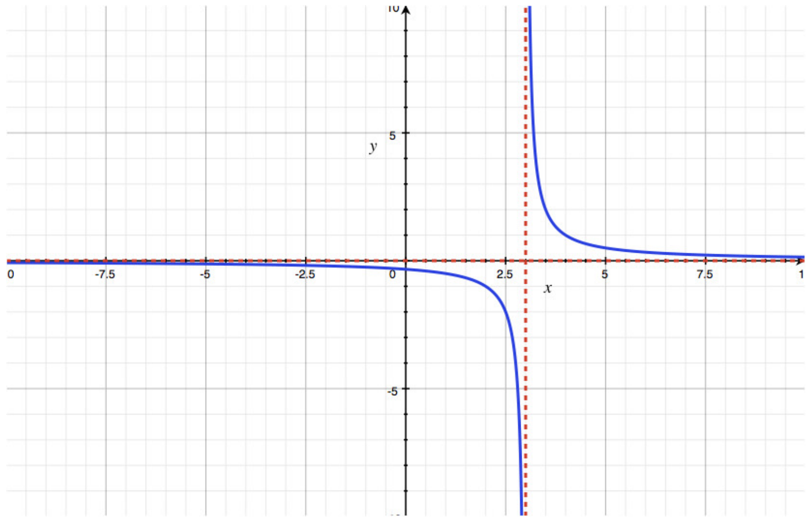
Step 3: Since the degree of the numerator is smaller than the degree of the denominator, the horizontal asymptote is at $y=0$.

Example 3

Graph $f(x) = \frac{1}{x-3}$ on the window [-10,10] by [-10,10], using online graphing software or a calculator.

Solution:

1. When graphing a rational function by entering the function in the $Y=$ screen, remember that you need to use parentheses to group the numerator and denominator of the rational function.
2. Vertical asymptotes are sometimes graphed as vertical lines.
3. Graphs of rational functions can be difficult to interpret if the window settings are not chosen carefully.



$y = \frac{1}{x-3}$ showing vertical line at $x = 3$.

Note that $f(x)$ is undefined and has a vertical asymptote at $x = 3$, but some graphing calculators draw the graph with a vertical line at $x = 3$. One way to "fix" this problem is to press **MODE** and select the option "Dot" rather than "Connected."

Example 4

What are the asymptotes of the function below?

$$k(x) = \frac{x^2}{x^2+5}$$

Solution:

Set the denominator equal to zero:

$$\begin{aligned} x^2+5 &= 0 \\ x^2 &= -5 \end{aligned}$$

There are no real solutions, so there are no vertical asymptotes. There is a horizontal asymptote at $x = 1$.

Example 5

Find the vertical asymptotes of

$$g(x) = \frac{x^3}{x^2+1}.$$

Solution:

Set the denominator equal to zero:

$$\begin{aligned} x^2+1 &= 0 \\ x^2 &= -1 \\ x &= \pm\sqrt{-1}. \end{aligned}$$

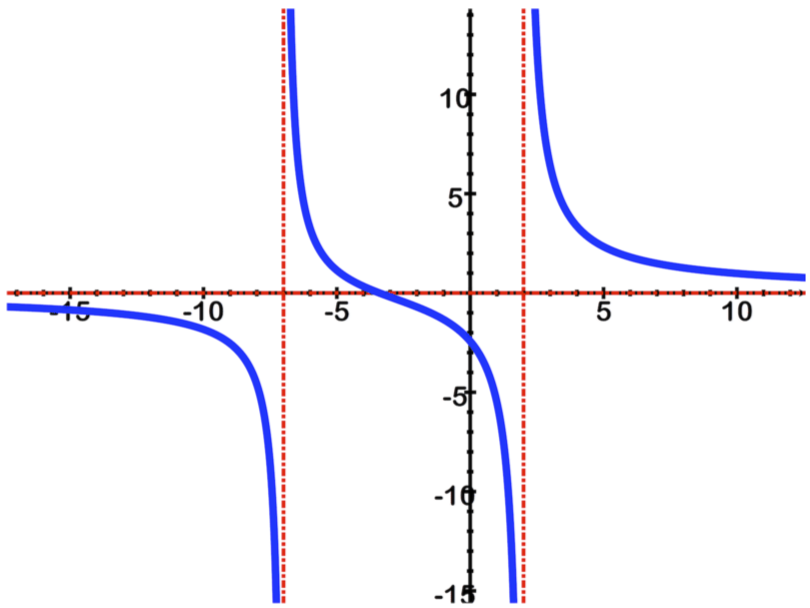
There are no real solutions, so there are no vertical asymptotes and no restrictions on the domain of this function.

Example 6

Graph the function and find any asymptotes using technology: $f(x) = \frac{6}{x-2} + \frac{4}{x+7}$

Solution:

Using online graphing software or a graphing calculator, type the given equation. The graph should look like:



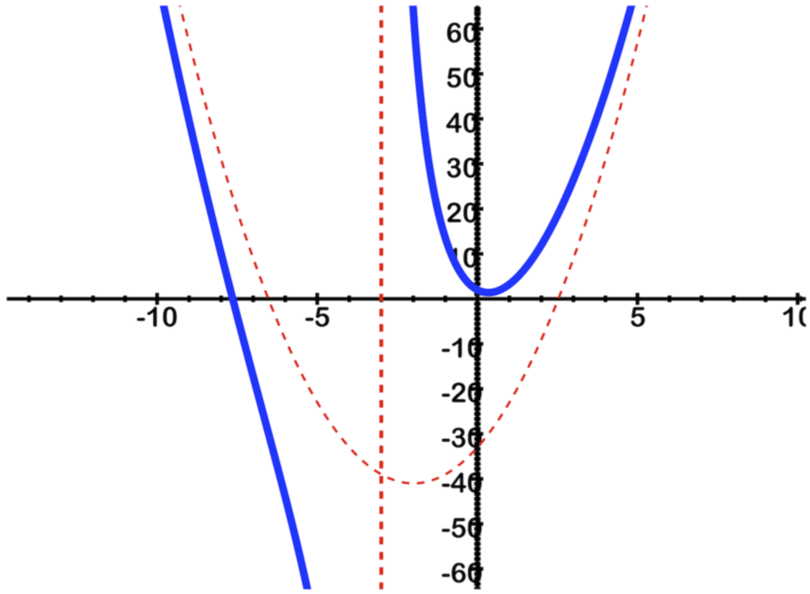
There are asymptotes at $x = 2$ and $x = -7$ and $y = 0$.

Example 7

Using technology, find all intercepts, asymptotes, and removable discontinuities, and graph: $\frac{2x^3+14x^2-9x+6}{x+3}$.

Solution:

The graph should look like the one below. The intercepts are at (approximately) $(-7.64, 0)$ and (exactly) $(0, 2)$. There is a vertical asymptote at $x = -3$. There are no horizontal asymptotes, but if you divided the numerator by the denominator, a curved asymptote can be found: $y = 2x^2 + 8x - 33$.



Example 8

Find any removable discontinuities and graph $\frac{6x^2+21x+9}{4x^2-1}$.

Solution:

Step 1: The function simplifies to

$$\frac{3(2x^2+7x+3)}{4(x^2-\frac{1}{4})} \to \frac{3(2x+1)(x+3)}{4(x-\frac{1}{2})(x+\frac{1}{2})}.$$

Note that the factors $2x+1$ and $x+\frac{1}{2}$ indicate the same common factor. Alternatively, you could have factored this to make this more evident:

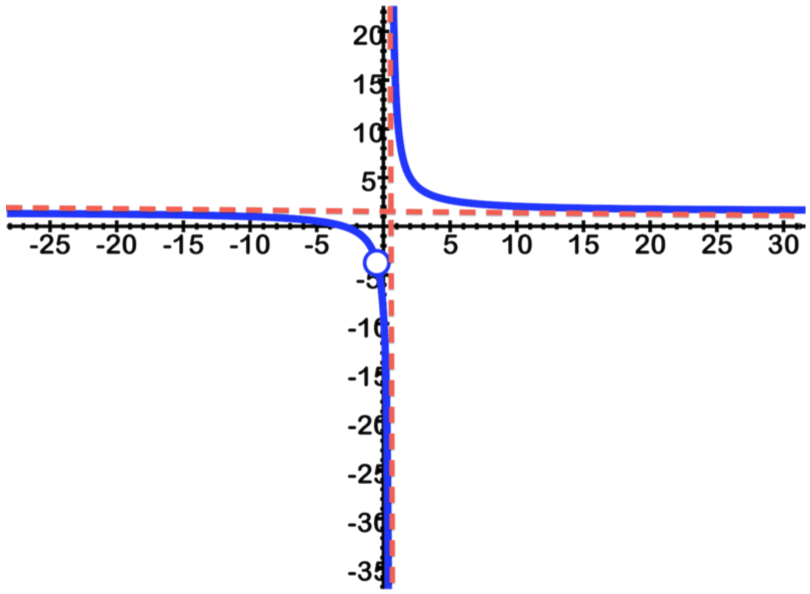
$$\frac{3(2x^2+7x+3)}{4x^2-1} \to \frac{3(2x+1)(x+3)}{(2x-1)(2x+1)}.$$

Since the factor cancels, there is a hole at $x=-\frac{1}{2}$. The other factor in the denominator, which does not cancel, means that there is a vertical asymptote at $x=\frac{1}{2}$.

Step 2: There is a horizontal asymptote at $y=\frac{3}{2}$ since the degree of the numerator and denominator are equal.

Step 3: The intercepts are $(-3,0)$ and $(0,-9)$, which you can find by setting the numerator (after cancelling the matching factor) to 0 and by plugging in 0 and evaluating the function.

Step 4: The graph should look like:



Summary

- Rational functions have two types of discontinuities:
  - Infinite—at vertical asymptotes
  - Removable—factors that cancel
- When using technology, asymptotes can be obscured. Algebraic techniques help to identify the asymptotes.

Review

For problems 1-5, factor the numerator and denominator, then set the denominator equal to zero and solve to find restrictions on the domain:

1. $y = \frac{x^2+3x-10}{x-2}$
2. $f(x) = \frac{x^2+2x-24}{x-4}$
3. $f(x) = \frac{x^2-12x+32}{x-4}$
4. $y = \frac{x^2+\frac{21}{20}}{x+\frac{4}{5}}$
5. $y = \frac{x^2+13x+42}{x+7}$

For each problem below, input the function into your graphing software or calculator carefully and accurately. Record any asymptotes or holes and record x and y intercepts. Finally, either copy and print or sketch the image of the graphed function.

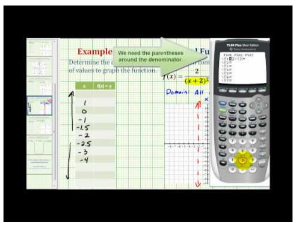
6. $y = \frac{x^3+5x^2+3x+7}{x-1}$
7. $y = \frac{9x^2+6}{x}$
8. $f(x) = \frac{x-7}{2x^2-11x-21}$
9. $f(x) = \frac{5x^3-9x^2-7x+1}{x^2-4}$
10. $y = \frac{x^2+x-30}{x+6}$
11. $y = \frac{4x^3+2x^2+7}{(x+2)^2}$
12. $f(x) = \frac{x^3-2x^2-3x}{x^2-5x+6}$

13. $f(x) = \frac{-6x^3+8x^2+7}{x^2}$
14. $y = \frac{-5x^2-2x-5}{x^2+2}$
15. $y = \frac{x-1}{x^3-2}$

Answers for Review

Please see the Appendix.

Resources



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3.10 Polynomial and Rational Inequalities

Learning Objectives

Learn how to graph and solve quadratic, polynomial, and rational inequalities.

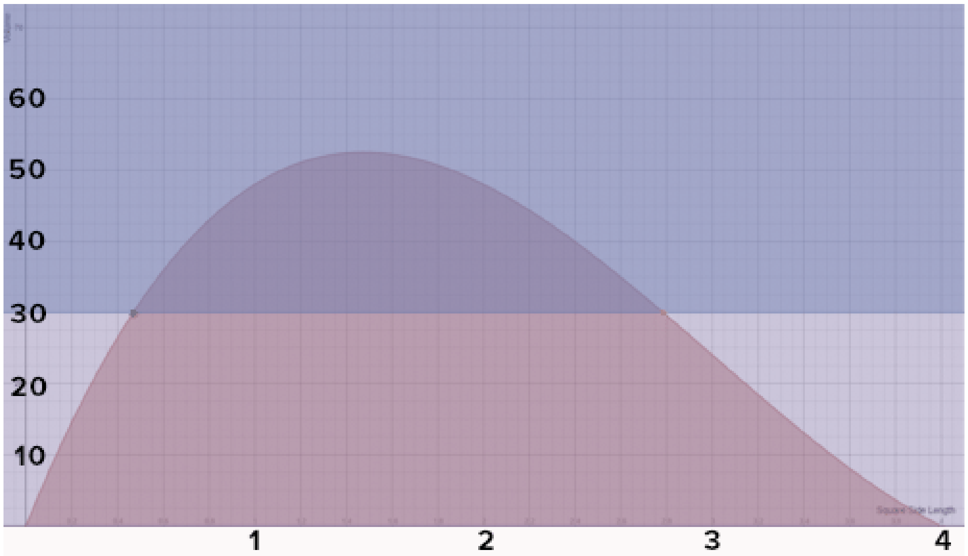
Introduction

A manufacturer needs to determine how to cut squares from a rectangular piece of cardboard, which is 8 inches by 10 inches, to create an open box. The volume function is

$$V(x) = x(10-2x)(8-2x) = 4x^3 - 36x^2 + 80x,$$

where x is the length of the side of the square.

In another section, we discussed the problem of the volume of this box. Now the manufacturer is purchasing equipment to cut the squares from the cardboard. The mathematical task is to determine a range of values for the side of the square so that the volume is 30 cubic inches or greater. The graph below shows the volume of the box as a function of its height, and helps to visualize the question:



Although the graph helps to visualize the problem, algebraic methods help to identify solutions when they are not easily seen. The techniques to answer the question are below.

Quadratic Inequalities

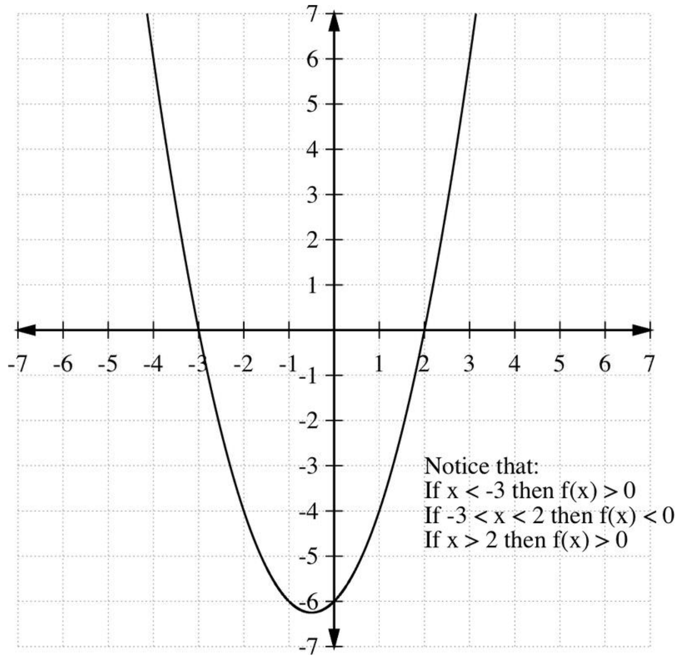
Quadratic inequalities are inequalities that have one of the following forms:

$$ax^2+bx+c>0$$

or

$$ax^2+bx+c<0.$$

These inequalities can be solved by extending the techniques to solve quadratic equations. For example, consider the graph of the equation $y=x^2+x-6$.



Notice that the curve intersects the $x-$axis at -3 and 2, since:

$$x^2+x-6=0$$
$$(x+3)(x-2)=0$$
$$x+3=0 \text{ or } x-2=0$$
$$x=-3 \text{ or } x=2$$

From graph, we notice the following:

- If $x<-3$ then $f(x)>0$.
- If $-3<x<2$, then $f(x)<0$.
- If $x>2$, then $f(x)>0$.

Therefore, $x^2+x-6>0$ whenever $x<-3$ or $x>2$, and $x^2+x-6<0$ when $-3<x<2$.

Solve Polynomial Inequalities

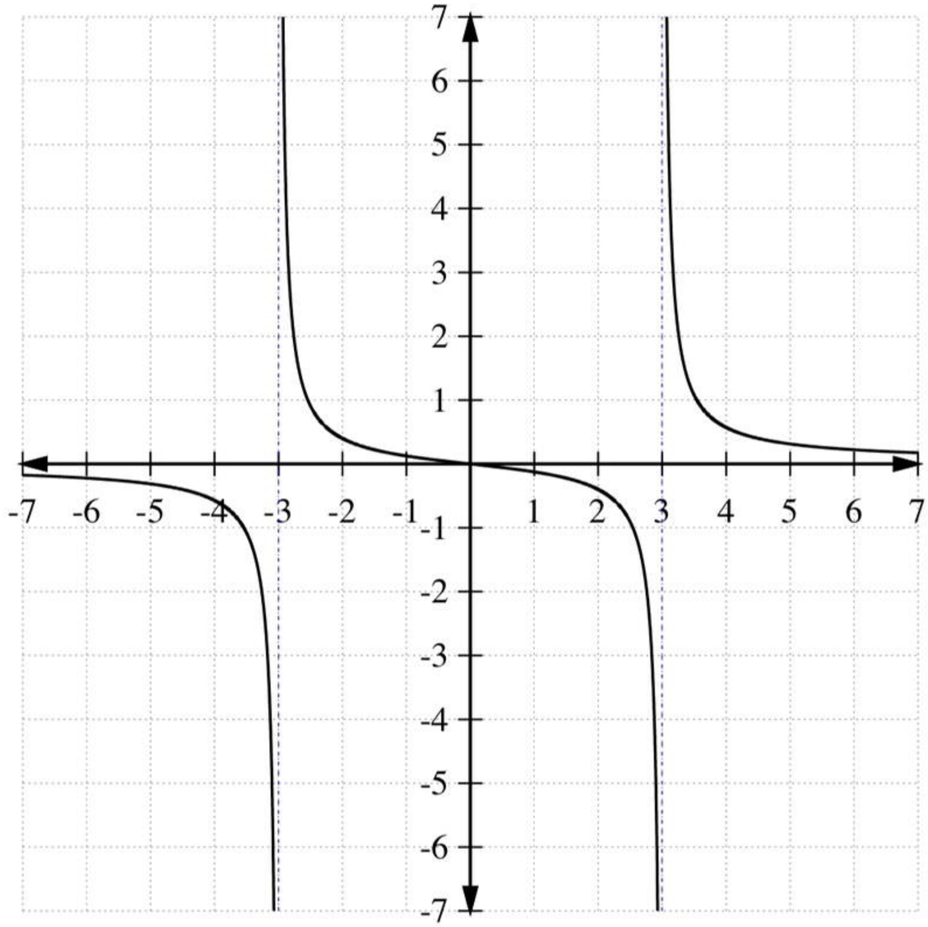
Solving polynomial inequalities is very similar to solving quadratic inequalities. The basic steps are the same:

1. Set up the inequality in the form $p(x)>0$ (or $p(x)<0, p(x)\leq 0, p(x)\geq 0$).
2. Find the solutions to the equation $p(x)=0$.
3. Divide the number line into intervals based on the solutions to $p(x)=0$.
4. Use test points to find solution sets to the equation.

Solve Rational Inequalities

The method outlined above also works for solving inequalities involving rational functions. Recall that for rational functions, you can find the roots (or zeros) by setting the numerator equal to zero.

However, one step is added to the process of solving rational inequalities, because a rational function can also change signs at its vertical asymptotes or at a hole in the graph. For instance, look at the graph of the function $r(x) = \frac{x}{x^2-9}$ below:



To help solve the inequality $\frac{x}{x^2-9} > 0$, use the following points: $x = 0, x = 3$, and $x = -3$. $x = 0$ is the solution of setting the numerator equal to 0, which gives the only root of the function. $x = \pm 3$ are the vertical asymptotes, since these values are not in the domain of the function.

Using the graph or test points, we can build the table:

TABLE 3.7:

Interval	Test Point	Positive/Negative?	Part of Solution set?
$(-\infty, -3)$	-4	-	no
(-3, 0)	-2	+	yes
(0, 3)	2	-	no
$(3, +\infty)$	4	+	yes

Thus, the solutions to $\frac{x}{x^2-9} > 0$ are $(-3, 0) \cup (3, +\infty)$.

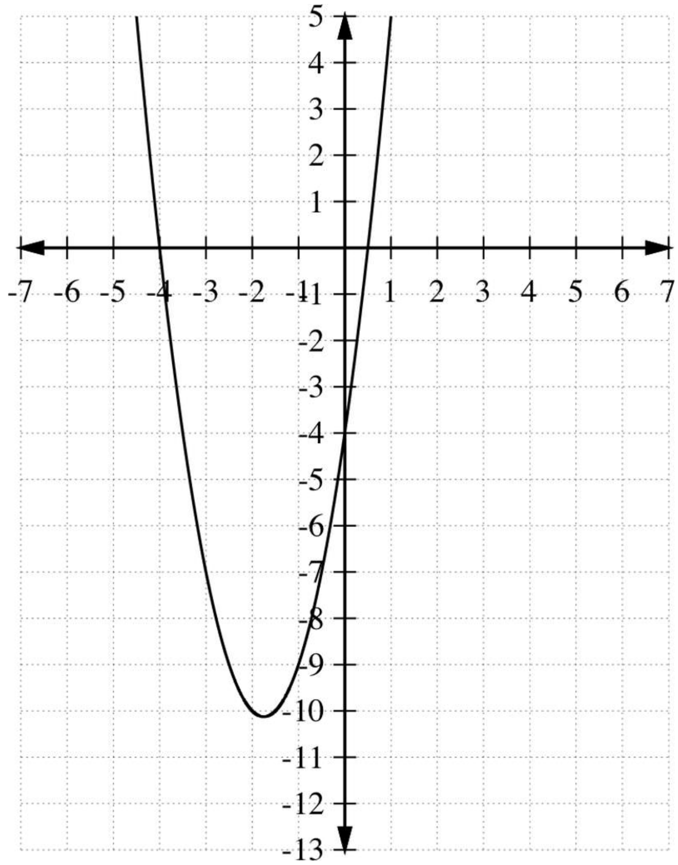
Examples

Example 1

What is the solution set of the inequality $2x^2 + 7x - 4 < 0$?

Solution:

Step 1: One option is to graph the function $f(x) = 2x^2 + 7x - 4$ and look for the the values of x such that the inequality $f(x) < 0$ is true.



Step 2: From graph, $2x^2 + 7x - 4 < 0$ only if

$$-4 < x < \frac{1}{2}.$$

Step 3: So the solution set is $x \in \left(-4, \frac{1}{2}\right)$ or, in set builder notation, $\left\{x | -4 < x < \frac{1}{2}\right\}$.

Although the method of graphing to find the solution set of an inequality is easy to follow, an algebraic method also can be used. The algebraic method involves finding the $x-$intercepts of the graph and then dividing the $x-$axis into intervals separated by the $x-$intercepts. The example below illustrates this method.

Example 2

Find the solution set of the quadratic inequality $x^2 + 2x - 8 > 0$.

Solution:

Step 1: Solve

$$x^2 + 2x - 8 = 0$$
$$(x+4)(x-2) = 0.$$

The two solutions to this quadratic equation are $x = -4$ and $x = 2$. Thus, the zeros of the function $f(x) = x^2 + 2x - 8$ are -4 and 2.

Step 3: These points divide the $x-$axis into three intervals: $(-\infty, -4), (-4, 2)$, and $(2, \infty)$. Choose a test point from each interval and substitute it into $f(x)$ to determine its sign (negative or positive). This procedure can be simplified by making the table shown below.

TABLE 3.8:

Interval	Test Point	$x^2 + 2x - 8$ Positive/Negative?	Part of Solution set?
$(-\infty, -4)$	-5	+	yes
(-4, 2)	1	-	no
$(2, +\infty)$	3	+	yes

From the table, it is clear that $x^2 + 2x - 8 > 0$ if and only if $x < -4$ or $x > 2$. The solution set can also be written as

$$(-\infty, -4) \cup (2, +\infty).$$

Example 3

A pool has a length 10 meters more than twice the width. Find the possible widths such that the area of the surface of the pool cannot exceed 100 square meters.

Solution:

Step 1: Let W be the width of the pool and L its length. Thus,

$$L = 10 + 2W.$$

Step 2: The surface area of a pool is

$$\begin{aligned} A &= LW \\ &= (10 + 2W)(W) \\ &= 10W + 2W^2. \end{aligned}$$

Step 3: The area cannot exceed 100 m^2. So

$$10W + 2W^2 \leq 100.$$

Step 4: Solve for 0:

$$2W^2 + 10W - 100 \leq 0.$$

Step 3: Divide both sides by 2:

$$W^2 + 5W - 50 \leq 0.$$

Step 4: Determine the roots for $W^2 + 5W - 50 = 0$,

$$W^2 + 5W - 50 = (W + 10)(W - 5).$$

Step 5: The partition points are 5 and -10, which form three intervals. Since the width must be positive, ignore -10. So there are only two intervals to consider, (0, 5] and [5, ∞).

The maximal allowable area is 100 m^2 . By testing points, you can find that the width must be $0 < W \leq 5$.

Example 4

Solve $x^3 - 3x^2 + 2x \geq 0$.

Solution:

Step 1:

$$\begin{aligned} x^3 - 3x^2 + 2x &= 0 \\ x(x^2 - 3x + 2) &= 0 \\ x(x-2)(x-1) &= 0 \end{aligned}$$

Step 2: The zeros are at $x = 0,\ x = 1$, and $x = 2$.

Step 3:

TABLE 3.9:

Interval	Test Point	Positive/Negative?	Part of Solution set?
$(-\infty, 0)$	-5	-	no
(0, 1)	$\frac{1}{2}$	+	yes
(1, 2)	$\frac{3}{2}$	-	no
$(2, +\infty)$	3	+	yes

Notice that this inequality is greater than or equal to zero, so we include the zeros in the solution set. Therefore, the solutions are $[0, 1] \cup [2, \infty)$.

Example 5

Solve $6x^4+5x^2<25$.

Solution:

Step 1: Rewrite the inequality to $6x^4+5x^2-25<0$.

Step 2: Solve the equation $6x^4+5x^2-25=0$.

$$6x^4+5x^2-25=0$$
$$(3x^2-5)(2x^2+5)=0$$

Step 3: The 1st factor yields the solutions $x=\pm\sqrt{\frac{5}{3}}$, and there are no real solutions for the 2nd factor.

TABLE 3.10:

Interval	Test Point	Positive/Negative?	Part of Solution set?
$\left(-\infty,-\sqrt{\frac{5}{3}}\right)$	-3	+	no
$\left(-\sqrt{\frac{5}{3}},\sqrt{\frac{5}{3}}\right)$	0	-	yes
$\left(\sqrt{\frac{5}{3}},+\infty\right)$	3	+	no

Finally, the solution set is $\left(-\sqrt{\frac{5}{3}},\sqrt{\frac{5}{3}}\right)$.

Example 6

Find the solution set of the inequality

$$\frac{4x-12}{3x-2}<0.$$

Solution:

Step 1: From the numerator, we solve $4x-12=0$ or $x=3$. In the denominator, solve $3x-2=0$, and we find the critical point $x=\frac{2}{3}$.

Step 2: Make the table:

TABLE 3.11:

Interval	Test Point	Positive/Negative?	Part of Solution set?
$\left(-\infty,\frac{2}{3}\right)$	0	+	no
$\left(\frac{2}{3},3\right)$	1	-	yes
$(3,+\infty)$	5	+	no

Step 3: Therefore, the solution set includes the numbers in the interval $\left(\frac{2}{3},3\right)$. Or in set-builder notation, the solution is $\left\{x|\frac{2}{3}<x<3\right\}$.

Example 7

Solve the inequality $\frac{3x+2}{x^2} > 3$.

Solution:

Step 1: Rewrite the inequality to get zero on one side:

$$\begin{aligned}\frac{3x+2}{x^2} &> 3\\ \frac{3x+2}{x^2} - 3 &> 0\\ \frac{-3x^2+3x+2}{x^2} &> 0\\ \frac{3x^2-3x-2}{x^2} &< 0\end{aligned}$$

Notice in the last step, both sides were multiplied by -1, so the direction of the inequality changed.

Step 2: The numerator cannot be factored, so use the quadratic formula to solve $3x^2 - 3x - 2 = 0$.

$$\begin{aligned}x &= \frac{3 \pm \sqrt{9-4(3)(-2)}}{6}\\ x &= \frac{3 \pm \sqrt{33}}{6}\end{aligned}$$

Step 3: So the two zeros of the rational function are $x = \frac{3+\sqrt{33}}{6} \approx 1.457$ and $x = \frac{3-\sqrt{33}}{6} \approx -0.457$. Be careful to remember the point at $x = 0$ due to the x^2 factor in the denominator of the rational expression. Using these values, construct the table:

TABLE 3.12:

Interval	**Test Point**	Is $\frac{3x^2-3x-2}{x^2}$ **Positive/Negative?**	**Part of Solution set?**
$\left(-\infty, \frac{3-\sqrt{33}}{6}\right)$	-1	+	no
$\left(\frac{3-\sqrt{33}}{6}, 0\right)$	$-\frac{1}{4}$	-	yes
$\left(0, \frac{3+\sqrt{33}}{6}\right)$	1	-	yes
$\left(\frac{3+\sqrt{33}}{6}, +\infty\right)$	2	+	no

The final solution set is $\left(\frac{3-\sqrt{33}}{6}, 0\right) \cup \left(0, \frac{3+\sqrt{33}}{6}\right)$.

Notice that at $x = 0$, the rational function is undefined, so 0 cannot satisfy the inequality.

Example 8

The McNeil Surf Company makes wetsuits. For a given number of wetsuits x, McNeil's profit, in dollars, is given by the function $P(x) = -0.01x^2 + 25x - 3000$.

a) If the manager of McNeil wants the profit to stay above $9,000, what is the minimum and maximum number of wetsuits the company can manufacture to maintain that level of profit?

Solution:

Set up the inequality:

$$-0.01x^2+25x-3000>9000$$
$$-0.01x^2+25x-12000>0.$$

With a calculator, graph the function $Y_1=-0.01x^2+25x-12000$. Set the **WINDOW:**

$Xmin=-1000, Xmax=4000, Xscl=500, Ymin=-5000, Ymax=5000, Yscl=1000\ xres=1$

Use the **CALC** menu **(2ND TRACE)**, and select the option **ZEROS**, to find $x=647.920$ and $x=1852.080$. By inspecting the graph, the solution set to the inequality $-0.01x^2+25x-12000>0$ is $x\in[648,1852]$.

b) What is the maximum profit McNeil can make?

Solution:

Keeping the same graph open, use **CALC MAXIMUM** to solve for the maximum profit. The maximum is at (1250, 3625), indicating that the maximum profit is $12,625 when 1,250 wetsuits are produced.

Example 9

Return to the open box problem. Use the graph provided to determine a range of values for the side of the square, so that the volume is 30 cubic inches or greater.

Solution:

When the dimensions of the square cut from the corners (the height of the open box) are 0.5 in by 0.5 in, the volume is greater than 30 cubic inches. Estimating from the graph, cutting out squares with sides between about 0.5 and 2.8 would create a box with volume 30 cubic inches or greater.

Summary

The method for solving polynomial and rational inequalities is the same:

- Rewrite the inequality so that the sign of the polynomial can be tested—that is, 0 is isolated.
- Find the zeros and points to divide the domain into intervals.
- Use test points in each interval to see which intervals satisfy the inequality.
- Build the solution set from the table of intervals and test points.

Review

Find the solution set of the inequality without using a calculator:

1. $x^2+2x-3\leq 0$
2. $3x^2-7x+2>0$
3. $-6x^2-13x+5\geq 0$
4. $\frac{5x-1}{x-2}>0$
5. $\frac{1-x}{x}<1$

6. $4x^3 - 4x^2 - 3x > 0$
7. $\frac{x^4}{4} - x^2 < 0$
8. $4x^3 - 8x^2 - x + 2 \geq 0$
9. $\frac{n^3-2n^2-n+2}{n^3+3n^2+4n+12} < 0$
10. $\frac{n^3+3n^2-4n-12}{n^3-5n^2+4n-20} \leq 0$
11. $\frac{2n^3+5n^2-18n-45}{3n^3-n^2+27n-9} \geq 0$
12. $\frac{12n^3+16n^2-3n-4}{8n^3+12n^2+10n+15} > 0$
    Use a calculator to solve the inequalities below. Round your answer to three places after the decimal.
13. $-9.8t^2 + 357.6t \geq 0$
14. $x^3 - 5x + 7 \leq -4x^2 + 18$
15. $\frac{x^2-2x}{x-5} > x^2 - 25$
16. Simplify and graph $f(x) > \frac{9x^2-4}{3x+2}$.
17. The total resistance of two electronics components wired in parallel is given by $\frac{R_1 R_2}{R_1+R_2}$, where R_1 and R_2 are the individual resistances (in ohms) of the two components.
    a. If the resistance of R_1 is 20 ohms, what is the maximum resistance of R_2 if the total resistance must be less than 15 ohms?
    b. What is the maximum theoretical resistance of this circuit? How do you know?
18. A rectangular lot of land has a length that is 7 meters more than twice its width. If the area of the lot is greater than 60 square meters, what are the possible values of the widths of the lot?

Review (Answers)

Please see the Appendix.

Applications, Technological Tools

Solving quadratic, polynomial, and rational inequalities is much easier with a calculator. Two specific functions that a TI-83/84 calculator provides to help solve rational inequalities are:

1. Using the calculator to graph a function and using the **CALC** menu to identify its roots.
2. Using the table function to substitute test values into the function.

3.11 Project: Power, Polynomial, and Rational Functions

Pollen is necessary for plants to produce fruit, nuts, and seeds, both for our enjoyment and to enable the plants to reproduce. In the process of pollination, a yellowish powder is transported from plant to plant by wind, insects, or even people, either accidentally or purposely. Yet many people are sensitive to pollen. In fact, during the springtime, when airborne pollens are at their highest levels, some people suffer. The level of discomfort depends on the geographic location and type of pollen in the environment. Pollen sensitivity causes inflammation and irritation in the nasal passage, itchy eyes, fatigue, and other symptoms.

The Asthma and Allergy Foundation of America and its research partner, IMS Health, produce a website, pollen.com, which monitors pollen levels in many locations throughout the year.

This dataset was taken from the pollen.com website:

TABLE 3.13:

Day	Date	Pollen Level
1	3/3/2016	10.7
2	3/4/2016	9.3
3	3/5/2016	8.9
4	3/6/2016	8.4
5	3/7/2016	7.6
6	3/8/2016	6.7
7	3/9/2016	2.3
8	3/10/2016	3.1
9	3/11/2016	4
10	3/12/2016	8.2
11	3/13/2016	10.9
12	3/14/2016	10.9

TABLE 3.13: (continued)

13	3/15/2016	10.6
14	3/16/2016	10
15	3/17/2016	9.3
16	3/18/2016	7.6
17	3/19/2016	10.8
18	3/20/2016	10
19	3/21/2016	8.4
20	3/22/2016	9.1
21	3/23/2016	9.7

1. Plot the data and label the axes.
2. During which time periods did the pollen level decrease? Increase?
3. On which days was the pollen level highest? Lowest? What were those extreme levels?
4. Using either a graphing calculator or a website, such as Desmos.com, create a regression analysis to fit a polynomial function to the data. Desmos.com allows the user to create an account to store work. It also offers a step-by-step tutorial to guide the user through a regression. The steps are:
   a. Create a new graph.
   b. Add a table to the graph. Enter the above dataset. Use the day number for x and the allergy index for y.
   c. Add a regression equation in the form $y_1 \quad ax_1{}^2 + bx_1 + c$. This creates a quadratic function that best fits the data.
   d. Change the scale of the axes until both the data and graph are visible.
   e. Export the graph to an image file and add it to your report.
5. Create a 2nd regression model. Use a cubic function, $y_1 \quad ax_1^3 + bx_1^2 + cx_1 + d$. Export an image and add it to your report.
6. Compare the two regression models.
   a. Consider intervals where each increases or decreases, and the extreme values. Do these correspond with the data?
   b. Which model best fits the data?
   c. Why would a straight line not fit the data?

Reference:

Asthma and Allergy Foundation of America, N.p., n.d. Web. 02 Apr. 2016.

3.12 Summary: Power, Polynomial, and Rational Functions

Chapter Summary

In this chapter, we learned about:

Power Functions

- A power function is a function of the form $f(x) = ax^n$, where $a \neq 0$ and n is a real number.
- If n is even, then the power function is even.
- If n is odd, then the power function is odd.

Polynomial Functions

- A polynomial function has the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$.
- A quadratic function is a special type of polynomial function. Its graph is a parabola that opens up if $a > 0$, and down if $a < 0$. Quadratic functions can be written in the following forms:
  - Standard form: $f(x) = ax^2 + bx + c$
  - Vertex form: $f(x) = a(x-h)^2 + k$
  - Factored form: $f(x) = a(x-r_1)(x-r_2)$

Rational Functions

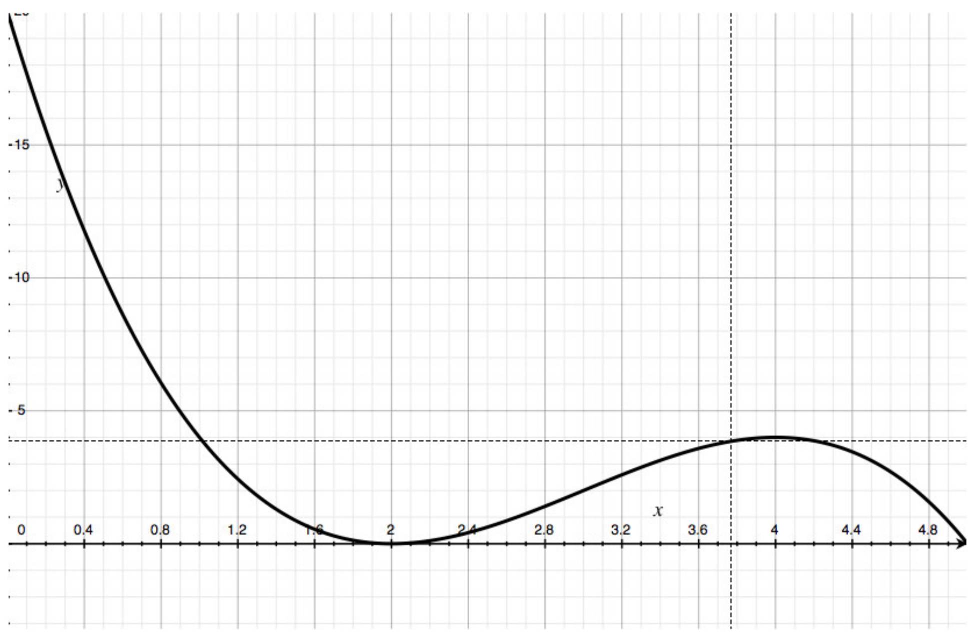
- A rational function has the form $r(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x) \neq 0$ are polynomials.
- The vertical asymptote is found by setting $Q(x) = 0$ and solving for x, assuming $P(x)$ and $Q(x)$ have no common factors.
- The horizontal asymptote is found by this method:
  - If the degree of the numerator is smaller than the degree of the denominator, then the horizontal asymptote is $y = 0$.
  - If the degree of the denominator and the numerator are the same, then the horizontal asymptote equals the ratio of the leading coefficients.
  - If the degree of the numerator is larger than the degree of the denominator, then there is no horizontal asymptote. If the degree is one larger than the denominator, you can use long division to find the oblique asymptote.
- Holes occur in the graph when the same factor exists in the numerator and denominator.
- Find x-intercepts by setting the numerator equal to zero. Solve for any value that doesn't also make the denominator zero.
- Find y-intercepts by plugging in 0 for x.

Polynomial and Rational Inequalities

- Isolate the zero and solve for zeros of the function.
- Divide the domain into intervals and use test points to help you build the solution set.

Chapter Application Problem

The chief operating officer of a young media company prepared some reports for a presentation to investors. The graph of the production cost data in thousands is below:



The portion of the costs displayed would be modeled by a polynomial function. From the appearance of the graph, the polynomial would:

- Have domain [0,5] and range [0,20].
- Be continuous on its domain.
- Decrease on [0, 2] and on [4,5].
- Increase on [2,4].
- Have zeros when 2,000 and 5,000 units are produced.
- Have maximum costs when 4,000 units are produced.

In this chapter, we showed that polynomials are appropriate to model this type of graph because they:

- Are defined and continuous everywhere.
- Have intervals where they are strictly increasing or decreasing.
- Have zeros that can all be found, though some zeros may not be real numbers.
- Have relative maximum and minimum values.

For the graph of production costs, what would be the minimum degree of this polynomial? Does it have a maximum degree?

The function that was found to best fit the production cost data is $C(x) = -x^3 + 9x^2 - 24x + 20$ on [0,5].

The minimum degree for this polynomial is 3, because the end behavior shows the degree is odd. There is no maximum degree, but it must be odd.

The production costs were maximum when the production level was at 4,000 units. If the costs kept increasing, the company would have an increasingly difficult time ensuring profitability of the company.

Some situations do exhibit continual strong growth rather than the ebb and flow of polynomials. Usually this type of growth in not sustainable in the real world. Hypothetical models for strong consistent growth are helpful to study in theory, just as we studied polynomials in this chapter.

Review

Try the following cumulative review problems to practice the concepts in this chapter:

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/195063

3.13 References

1. Heather/Cast a Line. https://www.flickr.com/photos/58754750@N08/5465845813/in/photolist-9jZTwV-hMkg5t-6qLHvm-7RudEa-guKpXQ-F9qvS-5mMsu9-23Gx89-5ZBsJu-F9qvw-brGD5f-cMR3tb-6bRSW6-54hCPU-4QYDbV-6b7KU7-a7pxsq-4AAniS-4JoLj3-4a5Z1h-tmQYDf-JEpWh-naiWMr-3ZkfbH-ciWAXY-a9diiJ-9V8FX-fuTnx4-4HqB5x-5iaick-4tCzx-egntok-qdGEyV-egntac-og5Hii-oA8krS-9vzWeA-e8GSSZ-FEJ1r-4JGMWk-cpFauW-c4fmb-66iFav-6rpg8P-6j2rrt-7UDraF-6j2rrV-bX5vD7-SRkBgE-sjYzTV .
2. CK-12 Foundation. Linear Fit Pollen Count.
3. CK-12 Foundation. .
4. Anders Sandberg. https://www.flickr.com/photos/arenamontanus/2932164542 .
5. CK-12 Foundation. Height of Ball.
6. . Example 3 Graph.
7. CK-12. Example 1 Graph .
8. CK-12. Example 4 Graph .
9. Krzysztof Urbanowicz. https://www.flickr.com/photos/kurbanowicz/12758383654 .
10. CK-12 Foundation. .
11. Kentucky National Guard Public Affairs Office. https://www.flickr.com/photos/kyngpao/9723274227/in/photolist-fPdjF6-8YiUeR-as7VnN-fPdjdk-3Nhobs-7krEZb-divWio-diwKer-8A51rH-8WJm9u-8ybvkm-7ad3n9-as7Vz5-7krEMd-as7Vf1-8y8piz-as7ViU-8ybvR3-p2NNPj-as5imM-as5iiZ-7knLAB-5Bi2Kj-as5izP-5Hhykc-d9r71A-as7VkN-7knLGX-as7VBy-5AyD9w-p2NNKS-divYhW-8AfaK1-ayGjYB-d9r7A9-QrrMnU-8ybsHW-as7Vsw-phgB8w-QrrJ2A-REkPrm-Rx7k6X-RzMztt-Qm2ciZ-ks6JGP-jsyrMb-6sTSLa-RWF1nm-3zm4sg-72u6S .
12. CK-12. Power Function Graphs .
13. CK-12. Example 1 Graph .
14. CK-12. Example 1.a. Graph .
15. CK-12. Example 1.b. Graph .
16. CK-12. Example 1. c. Graph .
17. CK-12. x-intercepts .
18. . Polynomial End Behavior a.
19. . Polynomial End Behavior b.
20. . Polynomial End Behavior c.
21. . Polynomial End Behavior d.
22. CK-12. Example 3 Graph .
23. CK-12. Example 4 .
24. CK-12. Example 5 Graph .
25. Paula Evans. Example 6 Graph .
26. Jasmine Wong. https://commons.wikimedia.org/wiki/File:Computer_in_tuition_centre.JPG .
27. Frans Hals. https://commons.wikimedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg .
28. CK-12. Example 3 Graph .
29. CK-12. Example 3 Graph Zoomed .
30. CK-12. Example 6 Graph .
31. z287marc. https://www.flickr.com/photos/z287marc/3192570544 .
32. CK-12. Example 1 Graph .
33. CK-12. Example 5 Graph .
34. Paula Evans. Intermediate Value Theorem Illustration .
35. Paula Evans. IVT Illustration Step 2 .

36. Paula Evans. IVT Illustration Step 3 .
37. CK-12. Practice1 .
38. CK-12. Practice2 .
39. CK-12. Practice3 .
40. CK-12. Practice4 .
41. CK-12. Practice5 .
42. Lead Beyond. https://www.flickr.com/photos/32493043@N00/3788069391 .
43. Unknown. http://www.daviddarling.info/encyclopedia/A/Apollonius.html .
44. . Example 3 Graph.
45. CK-12. Example 66 Graph .
46. CK-12. Example 7 Graph .
47. CK-12. Example 8 Graph .
48. David Hilowitz. https://www.flickr.com/photos/12176869@N00/26470112585/ .
49. Paula Evans. Volume of Box Inequality Graph .
50. CK-12. Quadratic Inequality Graph Illustration .
51. . Rational Inequality Graphical Illustration.
52. CK-12. Example 1 Graph .
53. sfbaywalk. https://www.flickr.com/photos/9681213@N06/8049871664 .
54. Ben Crowe. https://www.flickr.com/photos/croweb/2860626128/in/photolist-5mMsu9-23Gx89-5ZBsJu-F9qvw-brGD5f-cMR3tb-6bRSW6-6qLHvm-54hCPU-7RudEa-4QYDbV-6b7KU7-a7pxsq-4AAniS-guKpXQ-ps2F22-4JoLj3-4a5Z1h-tmQYDf-JEpWh-9V8FX-fuTnx4-4HqB5x-5iaick-4tCzx-egntok-egntac-qdGEyV-og5Hii-naiWMr-3ZkfbH-9vzWeA-oA8krS-ciWAXY-e8GSSZ-FEJ1r-4JGMWk-cpFauW-c4fmb-6rpg8P-6j2rrt-7UDraF-66iFav-6j2rrV-62s9TA-4bKqFC-bX5vD7-5u4z2L-7xegER-7RxYEu .
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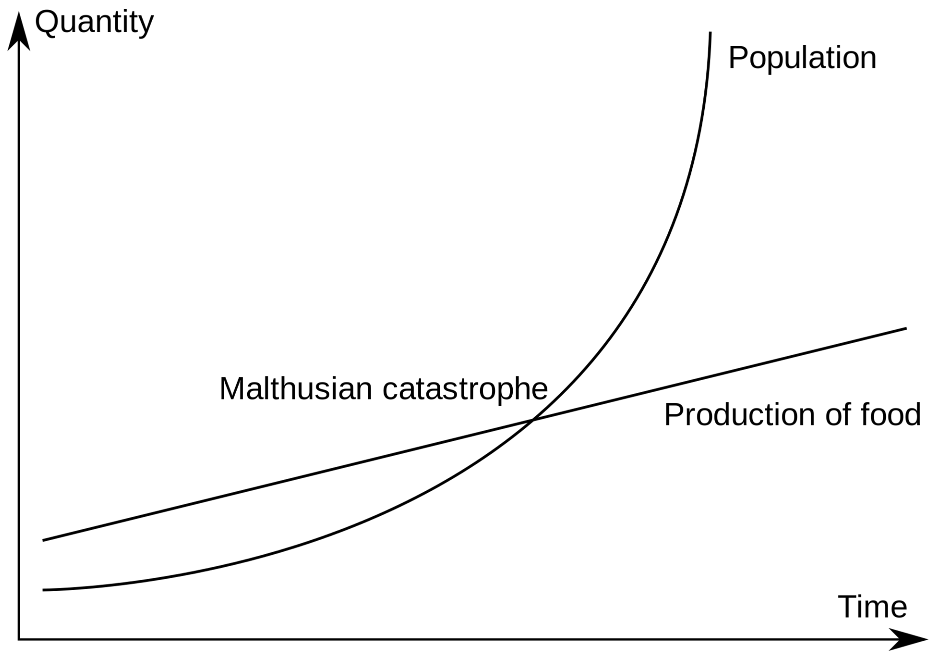
CHAPTER 4 Exponential and Logarithmic Functions

Chapter Outline

4.1 Introduction: Exponential and Logarithmic Functions

Introduction

Thomas Malthus, an 18th-century researcher, studied the growth of human populations. He observed that human population grows exponentially, which he described as geometrically. He also observed that the resources that humans needed to survive, such as food, were more limited than the growing human population. He concluded that if the human population grew without limitation, it would only be a matter of time before the world's population would be too large to feed itself, causing what has been called a "Malthusian catastrophe."



Malthus's model is commonly called the natural growth model or exponential growth model. In this chapter we will examine the exponential growth model and its inverse function, the logarithmic model. We will also look at special applications of these function families, including compounded interest and the population logistic model.

4.2 Graphing and Evaluating Exponential Functions

Learning Objectives

Learn to find equations of exponential functions and graph them.

Introduction

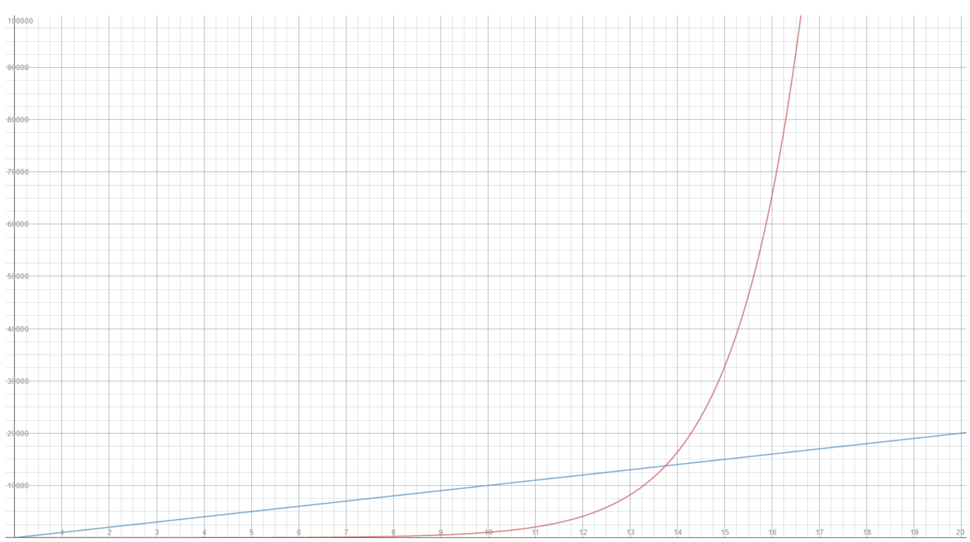
An old legend recorded in 1260 B.C. by Ibn Khallikan, a Kurdish historian, relates a story about how the inventor of chess was asked to name his own reward for creating the game for the king. One version of the story says the man asked the king to choose one of two methods of payment:

1. One million grains of rice for each square of the chessboard.
2. A single grain of rice for the 1st square of the chessboard, two grains of rice for the 2nd square, four grains of rice for the 3rd, and so on. The king would simply double these few grains of rice for each successive square of the chessboard.

The king proudly chose the 2nd option, clearly unfamiliar with the power of compounded interest. The mathematical models for each method of payment are as follows:

1. $P_1(x) = 1{,}000{,}000x$
2. $P_2(x) = 2^x$

The graph below compares the payment methods. Clearly, one method of payment requires a much higher payment than the other, although this is not apparent until the 3rd row of the chessboard.



The 2nd option was the wrong choice. By the 32nd square, the payment was $P_2(32) = 2^{32} = 4{,}294{,}967{,}296$, more than 4 billion grains of rice, for a single square. This is an exponential function, the next class of functions to study. Clearly, an important characteristic is the strong growth rate.

Exponential Functions

Exponential functions take the form $f(x) = a \cdot b^x$, where $a \neq 0$ and $b \neq 1,\ b > 0$ are constants. The constant a is the starting amount when $x = 0$. The constant b can be any number except 1, and tells the story about the growth. If the y-value is doubling for each unit increase in x, then bIf the y-value is halving for each unit increase in x (which would be decay), then b is $\frac{1}{2}$. If the y-value is increasing by 6% for each unit increase in x, then b is 1.06.

Exponential Function

An exponential function f with base b and initial value a is defined by:
$f(x) = a \cdot b^x$ or $y = a \cdot b^x$, where $a \neq 0,\ b > 0,\ b \neq 1$ and x is any real number.

Evaluating Exponential Functions

Consider the function $f(x) = 2^x$.

- If $x = 3,\ f(3) = 2^3 = 8$.
- If $x = 0$, $f(0) = 2^0 = 1$.
- If $x = -3,\ f(-3) = 2^{-3} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$.
- If $x = -10,\ f(-10) = 2^{-10} = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$.

Graph of the Exponential Function

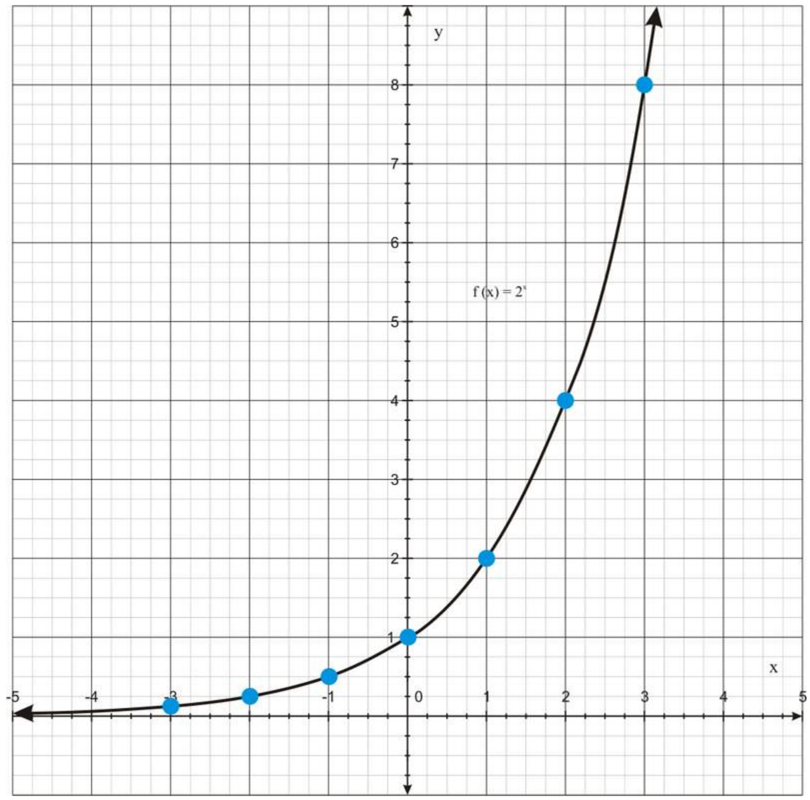
Exponential functions can be graphed by inputing x-values into the function and plotting the resulting points, or simply by using a graphing calculator.

Properties of an Exponential Function of the form $y = ab^x,\ a > 0$:

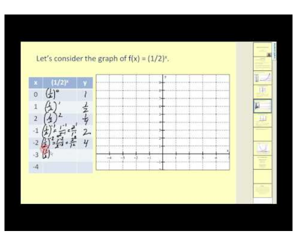
- b is the value of the base. Within the function, as the x-value increases by 1, the y-value is multiplied by the common ratio.
- If $b > 1$, then the curve will be increasing and represent exponential growth.
- If $0 < b < 1$, then the curve will be decreasing and represent exponential decay.
- Every exponential function of the form $y = ab^x$ will pass through the point $(0, a)$. a will be the y-intercept of the function, or its value at $x = 0$ also called the initial value.
- Every exponential function of the form $y = ab^x$ will have the domain and range:

$$D = \{x \mid x \in R\} \text{ and } R = \{y \mid y > 0,\ y \in R\}.$$

Let's now consider the graph of $f(x) = 2^x$:



Notice that as x approaches ∞, the function grows without bound. That is, $f(x) \to \infty$. However, if x approaches $-\infty$, the function values get closer and closer to 0.



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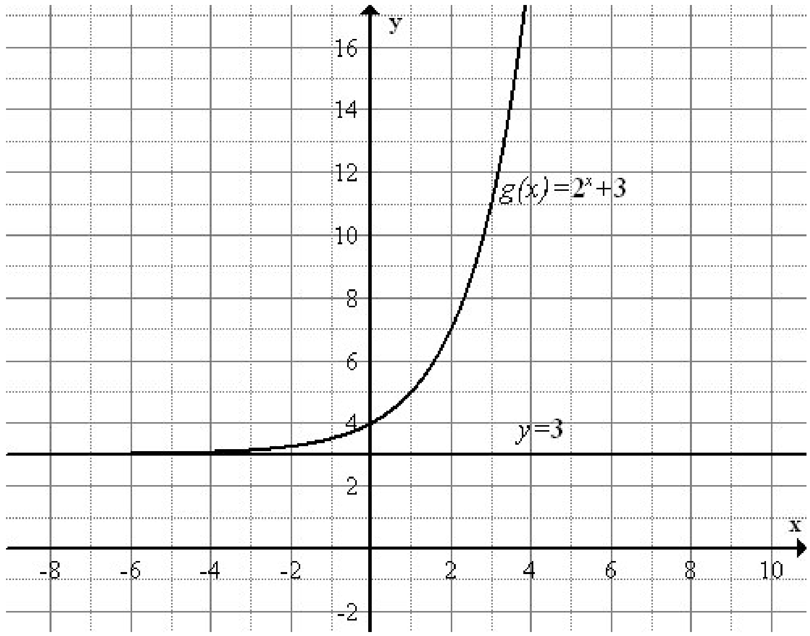
Graphing Exponential Functions Using Transformations

Graph $g(x) = 2^x + 3$ using transformations. Clearly, it has a vertical shift up 3 units, so the horizontal asymptote of the function is the line $y = 3$. The graph of this function and the horizontal asymptote are shown below:

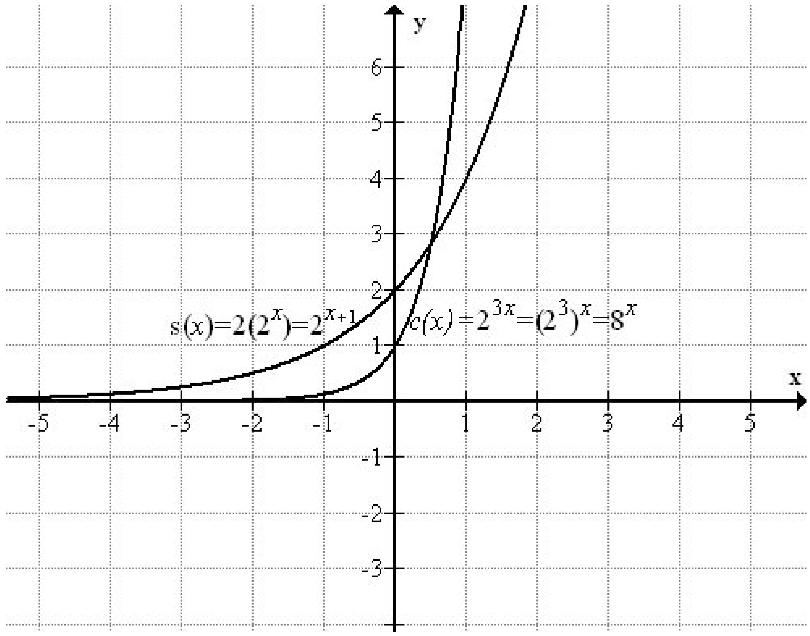


From the study of transformation of functions, we recognize the graph of $g(x) = 2^x + 3$ as a vertical shift of the graph of $f(x) = 2^x$.The table below summarizes the different kinds of transformations of $f(x) = 2^x$:

TABLE 4.1:

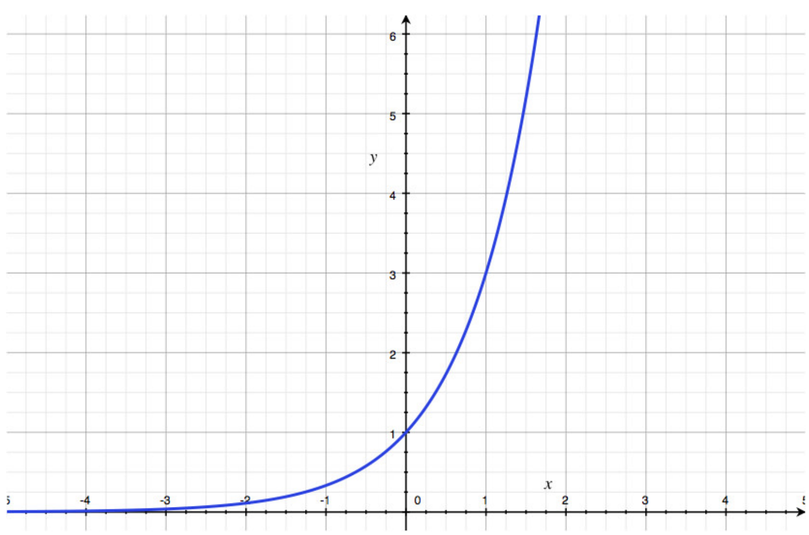
Equation	Relationship to $f(x)$=2^x	Range
$g(x) = \frac{2^x}{2^a} = 2^{x-a}$, for $a > 0$	Obtain a graph of g by shifting the graph of $f(x)$ right a units.	y >0
$g(x) = 2^a \cdot 2^x = 2^{a+x}$, for $a > 0$	Obtain a graph of g by shifting the graph of $f(x)$ left a units.	y >0
$g(x) = 2^x + a$, for $a > 0$	Obtain a graph of g by shifting the graph of $f(x)$ up a units.	y >a
$g(x) = 2^x - a$, for $a > 0$	Obtain a graph of g by shifting the graph of $f(x)$ down a units.	y >a
$g(x) = a(2^x)$, for $a > 0$	Obtain a graph of g by vertically stretching the graph of $f(x)$ by a factor of a.	y >0
$g(x) = 2^{ax}$, for $a > 0$	Obtain a graph of g by horizontally compressing the graph of $f(x)$ by a factor of a.	y >0
$g(x) = -2^x$	Obtain a graph of g by reflecting the graph of $f(x)$ over the x-axis.	y <0
$g(x) = 2^{-x}$	Obtain a graph of g by reflecting the graph of $f(x)$ over the y-axis.	y >0

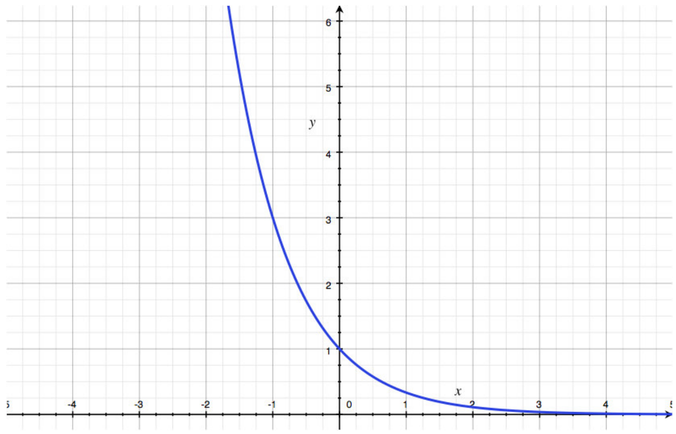
Consider the function $s(x) = 2(2^x)$ and $c(x) = 2^{3x}$. The 1st function represents a vertical stretch of $f(x) = 2^x$ by a factor of 2. The 2nd function represents a horizontal compression of $f(x) = 2^x$ by a factor of 3. The function $c(x)$ is actually the same as another parent function: $c(x) = 2^{3x} = (2^3)^x = 8^x$. The function $s(x)$ is actually the same as a shift of $f(x) = 2^x$, $s(x) = 2(2^x) = 2 \times 2^x = 2^{x+1}$.The graphs of s and c are shown below. Notice that the graph of c has a y -intercept of 1, while the graph of s has a y -intercept of 2.



Exponential Function Properties

Exponential functions can model either growth or decay. The following graphs demonstrate the general trajectory of an exponential graph. The first graph displays exponential growth and the second graph displays exponential decay.





There are attributes common to each type of exponential function. The table below lists the differences in exponential growth when compared with exponential decay. In the growth functions, the base will be greater than 1 while in decay functions the base will be between 0 and 1. Both growth and decay have matching domain, range, intercepts, and asymptotes. The end behavior is opposite, however. When analyzing exponential functions, we can describe each characteristic as related to the function.

TABLE 4.2: Exponential Function Propoerties

Type	Growth	Decay
Function	$f(x) = a \cdot b^x, b > 1$	$f(x) = a \cdot b^x, 0 < 1$
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$(0, \infty)$	$(0, \infty)$
Intercepts	no x-intercept, $(0, a)$	no x-intercept, $(0, a)$
Horizontal Asymptote	$y = 0$	$y = 0$
End behavior f(x) $\to 0$ as $x \to -\infty$ f(x) $\to \infty$ as $x \to -\infty$	$f(x) \to \infty$ as $x \to \infty$ $f(x) \to 0$ as $x \to \infty$	
Increasing/Decreasing	Increasing: $(-\infty, \infty)$	Decreasing: $(-\infty, \infty)$

Exponential growth and decay are used in measuring financial applications, populations of people, bacteria, and any other quantity that grows or decays at a constant rate. One example of exponential decay is carbon dating. Radioactive isotopes such as carbon-14 decay very slowly. It takes about 5,730 years for half of the initial number of carbon-14 molecules in an environment to decay. This is how scientists can date artifacts of ancient humans.

Solving Exponential Equations

Solving an exponential equation means determining the value of x for a given function value. For example, if we have the equation $2^x = 8$, the solution to the equation is the value of x that makes the equation a true statement. Here, the solution is $x = 3$, as $2^3 = 8$.

Consider a slightly more complicated equation, $3(2^{x+1}) = 24$. Solve this equation by writing both sides of the equation as a power of 2:

$$3(2^{x+1}) = 24$$
$$\frac{3(2^{x+1})}{3} = \frac{24}{3}$$
$$2^{x+1} = 8$$

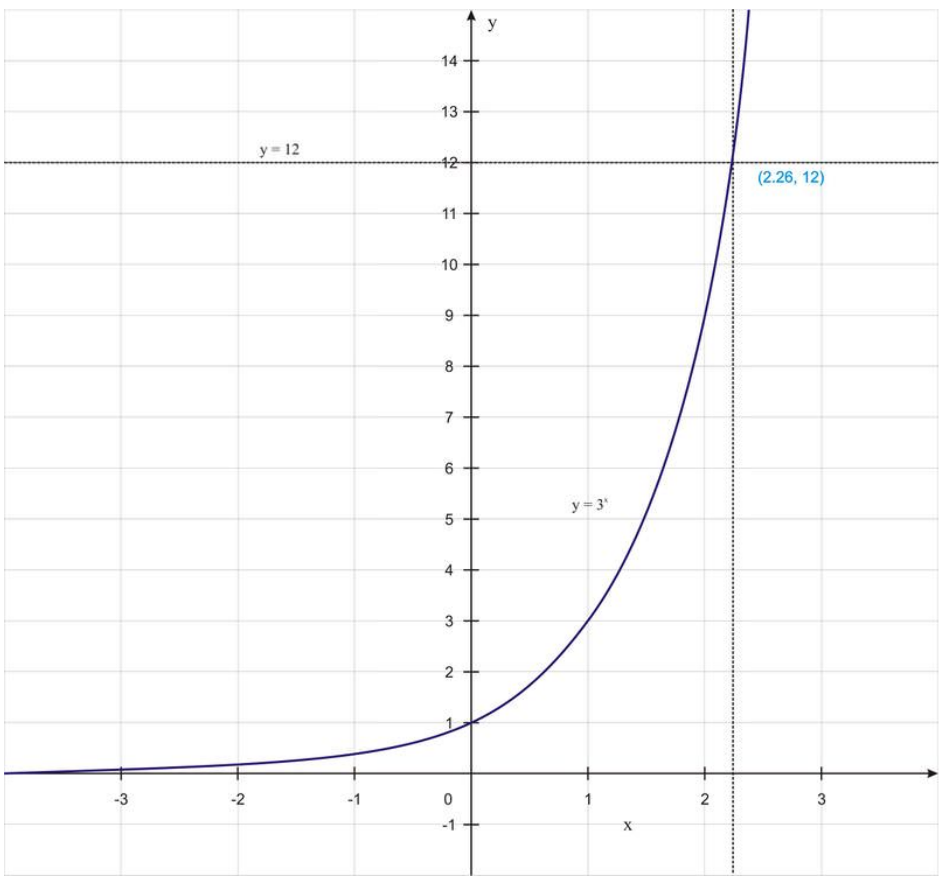
$2^{x+1} = 2^3.$

To solve the equation now, recall a property of exponents: if $b^x = b^y$, then $x = y$. That is, if two powers of the same base are equal, the exponents must be equal. This property tells us how to solve

$$\begin{aligned} 2^{x+1} &= 2^3 \\ \Rightarrow x+1 &= 3 \\ x &= 2. \end{aligned}$$

In this example, it was possible to algebraically solve the equation because both sides of the equation were written as a power of the same exponent. But what if that is not possible?

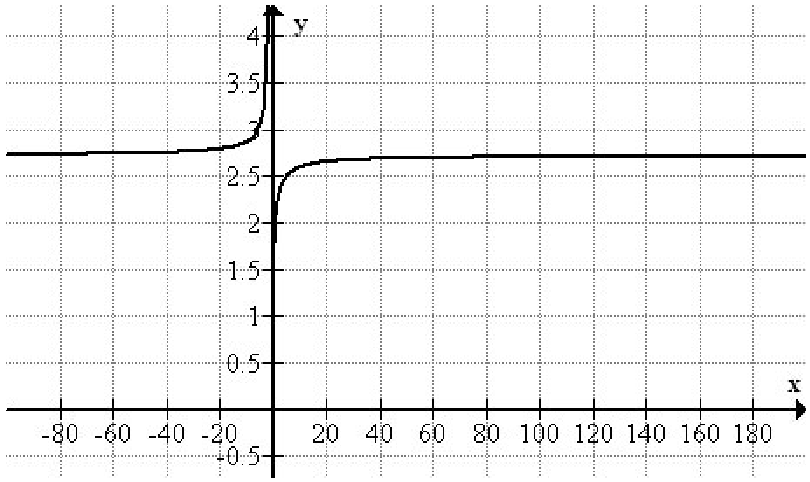
Consider, for example, the equation $3^x = 12$. It is possible to find the solution to the equation $3^x = 12$ by finding the intersection of $y = 3^x$ and the horizontal line $y = 12$. Using a graphing calculator's intersection capability, we find that the solution is approximately $x = 2.26$.



The Number e and The Function $y = e^x$

In geometry, we encountered the number π. The number e is much like π. First, both are irrational numbers: they cannot be expressed as a quotient of integers. Second, both numbers are transcendental: they are not the solution of any polynomial with rational coefficients.

Like π, mathematicians found e to be a natural constant in the world. One way to "discover" e is to consider the end behavior of the function $f(x) = \left(1 + \frac{1}{x}\right)^x$. The graph of this function is shown below:



Notice that as x approaches infinity, the graph of the function approaches a horizontal asymptote around $y = 2.7$.

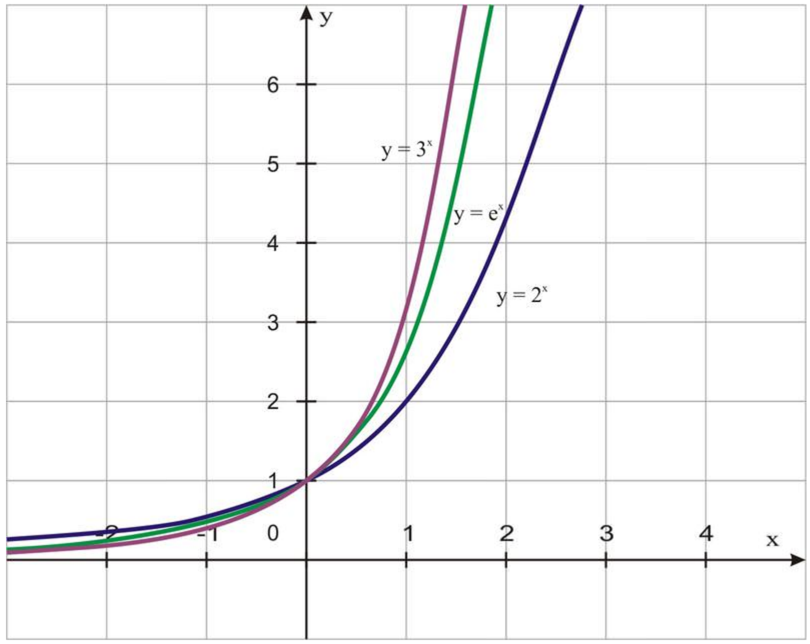
TABLE 4.3:

x	y
0	(not defined)
1	2
2	2.25
5	2.48832
10	2.5937424601
50	2.69158802907
100	2.70481382942
1,000	2.71692393224
5,000	2.7180100501
10,000	2.71814592683
50,000	2.7181825464614

Around $x = 100$, the function values pass 2.7 as x increases, but they will never reach 2.8. Since e is the y-value as x approaches infinity, we see that the irrational number e is approximately 2.718.

The number e is used as the base of functions that can be used to model situations involving growth or decay, such as population growth in the absence of predators or resource restrictions.

The graph below shows $y = e^x$, along with $y = 2^x$ and $y = 3^x$:



The graph of $y = e^x$ (in green) has the same shape as the graphs of the other exponential functions. It sits between the graphs of the other two functions. Notice that the graph is closer to $y = 3^x$ than to $y = 2^x$. All three graphs have the same y-intercept, (0, 1).

Examples

Example 1

Identify which of the functions below are exponential functions and which are not.

Since exponential functions are of the form $y = a \cdot b^x$, analyze each function to determine whether they include a constant as the base and variable exponent.

a) $y = x^6$

Solution:

$y = x^6$ is not an exponential function because x is not in the exponent.

b) $y = 5^x$

Solution:

$y = 5^x$ is an exponential function.

c) $y = 1^x$

Solution:

$y = 1^x$ is not a true exponential function because y is always 1, which is a constant function.

d) $y = x^x$

Solution:

$y = x^x$ is not an exponential function because x is both the base and power of the exponent.

e) $y = x^{\frac{1}{2}}$

Solution:

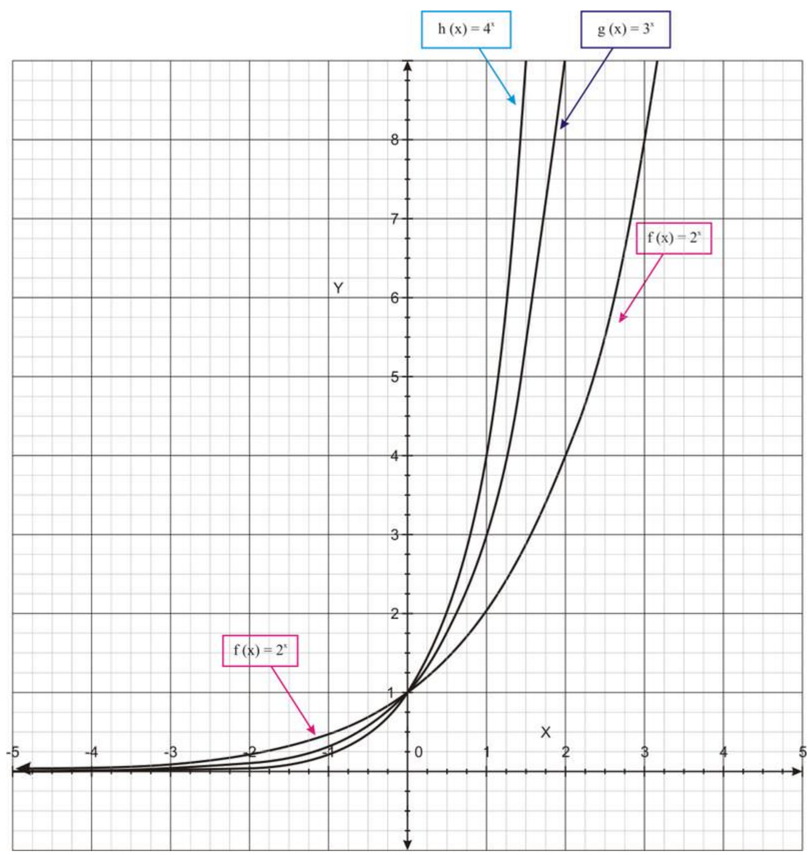
$y = x^{\frac{1}{2}}$ is not an exponential function because the base is not a constant and the exponent is not a variable.

Example 2

Use a graphing utility to graph $f(x) = 2^x$, $g(x) = 3^x$, and $h(x) = 4^x$. How are the graphs the same, and how are they different?

Solution:

$f(x) = 2^x$, $g(x) = 3^x$, and $h(x) = 4^x$ are shown together below.



The graphs of the three functions have the same overall shape: they have the same end behavior, and they all contain the point (0, 1). The difference lies in their rate of growth. Notice that for positive x-values, $h(x) = 4^x$ grows the fastest and $f(x) = 2^x$ grows the slowest. The function values for $h(x) = 4^x$ are highest, and the function values for $f(x) = 2^x$ are the lowest for any given positive value of x. For negative x -values, the relationship changes: $f(x) = 2^x$ has the highest function values of the three functions.

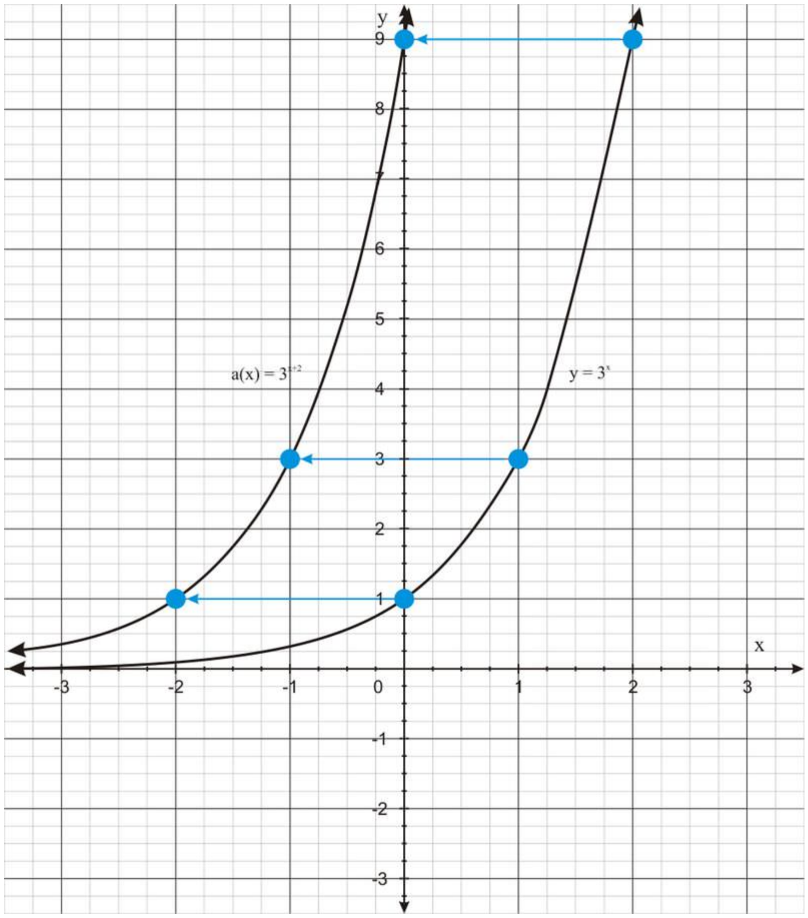
Example 3

Use transformations to graph the functions a) $a(x) = 3^{x+2}$ and b) $b(x) = -3^x + 4$.

Solution:

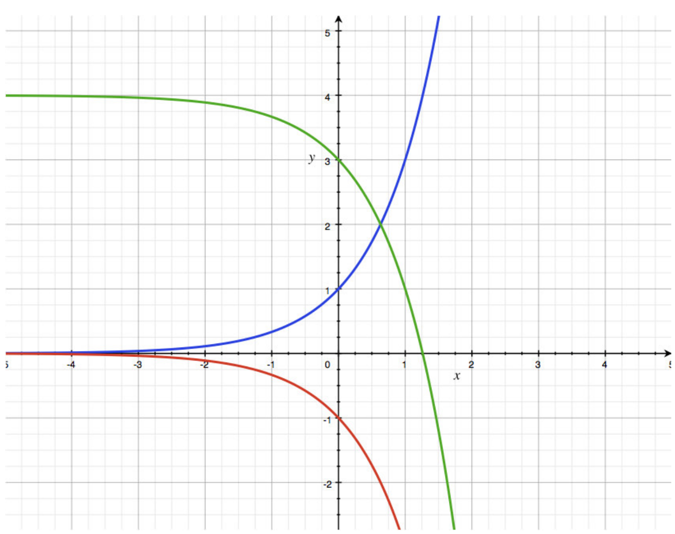
a) $a(x) = 3^{x+2}$

This graph represents a shift of $y = 3^x$ two units to the left. The graph below shows this relationship between the graphs of these two functions:



b) $b(x) = -3^x + 4$

This graph represents a reflection over the x -axis and a vertical shift of 4 units. Produce a graph of $b(x)$ using three steps: sketch $y = 3^x$, reflect the graph over the x-axis, and then shift the graph up 4 units. The graph below shows this process:



Example 4

Recall the question from the Introduction. The king, when choosing the 2nd payment option, could not possibly have delivered the last payment. The number of grains of rice on the last square would have been almost 10 quintillion (million million million). That is more rice than is produced in the world in an entire year.

Solution:

$P_2(63) = 2^{63} = 9,223,372,036,854,775,808$ grains of rice.

Example 5

Solve the following equations algebraically or graphically:

a) $5^{6x+10} = 25^{x-1}$

Solution:

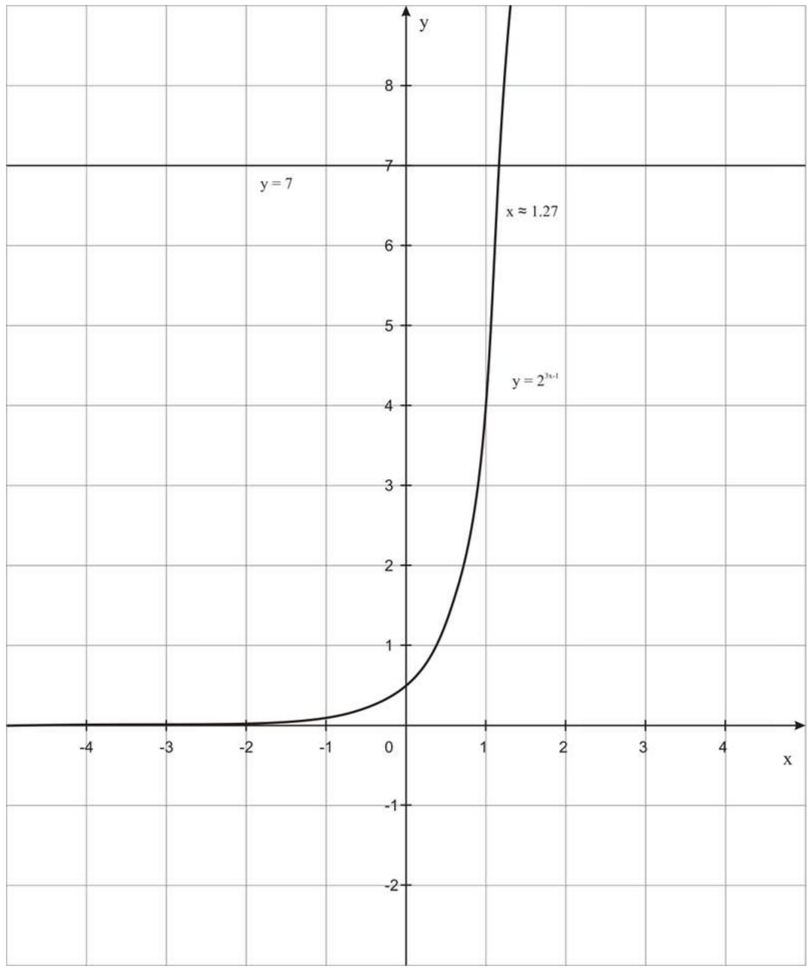
Since both sides of the equation can be written as a power of the same base, use algebra to solve for x:

$$\begin{aligned}
5^{6x+10} &= 25^{x-1} \\
5^{6x+10} &= 5^{2(x-1)} \\
6x + 10 &= 2(x-1) \\
6x + 10 &= 2x - 2 \\
4x &= -12 \\
x &= -3
\end{aligned}$$

b) $2^{3x-1} = 7$

Solution:

Since both sides of the equation cannot be written as a power of the same base, graphically solve for x. Graph the function $y = 2^{3x-1}$, and find the point where the graph intersects the horizontal line $y = 7$.

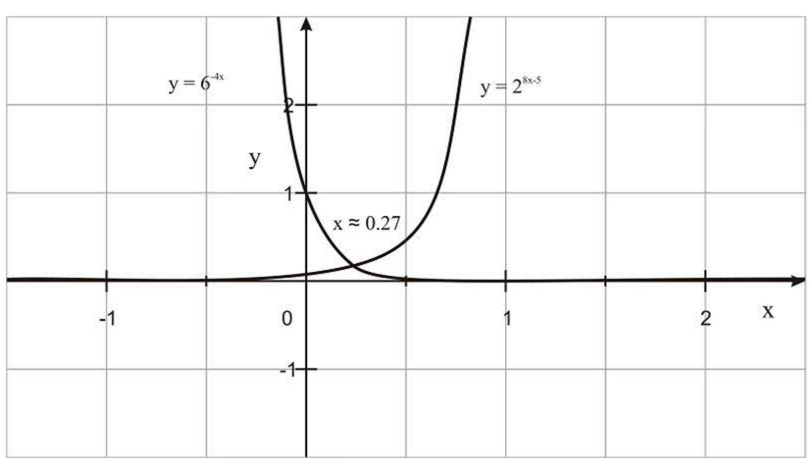


The solution is $x \approx 1.27$.

c) $6^{-4x} = 2^{8x-5}$

Solution:

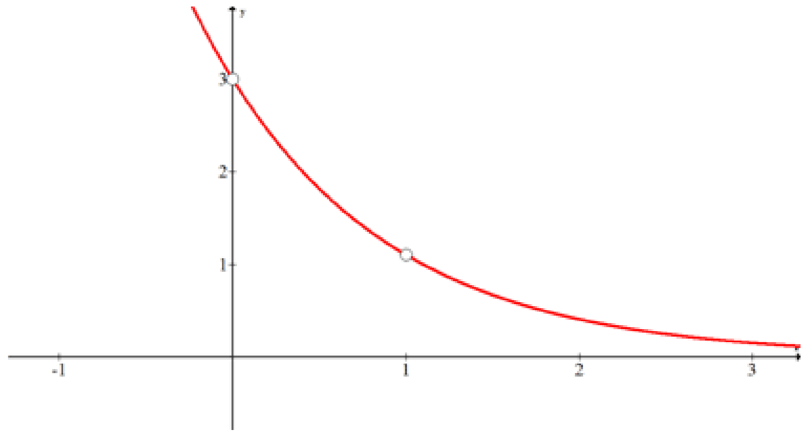
Since both sides of the equations cannot be written as a power of the same base, graphically solve for x. Graph the functions $y = 6^{-4x}$ and $y = 2^{8x-5}$, and find their intersection point.



The solution is approximately $x \approx 0.27$.

Example 6

Write the exponential function that passes through the following points: $(0,3), \left(1, \frac{3}{e}\right)$.



Solution:

Determine the initial value. From our properties of exponential functions, we know that every exponential function of the form $y = ab^x$ will pass through the point $(0, a)$. a will always be the y-intercept of the function, or its value at $x = 0$. Therefore, the initial value is $a = 3$.

Now you can find b using the 2nd point.

$$f(1) = 3b^1 = \frac{3}{e}$$
$$b = \frac{1}{e}$$

so that $f(x) = 3\left(\frac{1}{e}\right)^x = 3e^{-x}$.

Example 7

A mummified animal is found preserved on the slopes of an ice-covered mountain. After testing, a scientist measures that 75% of the initial amount of the carbon-14 sample has decayed. How long ago was this animal alive?

Solution:

Let x denote the time measured in half-life units. Since $100\% - 75\% = 25\%$, there is 0.25 of the carbon-14 remaining.

$$\left(\frac{1}{2}\right)^x = 0.25$$
$$\left(\frac{1}{2}\right)^x = \left(\frac{1}{4}\right)$$
$$\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^2$$
$$x = 2$$

Thus, $x = 2$. This does not mean that two years ago the animal was alive. It means that two half-lives ago the animal was alive.

Since the half-life for carbon-14 is 5,730 years, $5{,}730 \cdot 2 = 11{,}460$. Thus, to the nearest thousand, this animal died approximately 11,000 years ago.

Summary

- Exponential functions take the form $f(x) = a \cdot b^x$, where $a \neq 0$ and $b \neq 1,\ b > 0$ are constants.
- For $b > 1$, the exponential function grows without bound as x approaches ∞. As x approaches $-\infty$, the function values get closer and close to 0. For $0 < 1$, these limits are reversed.
- Every exponential function of the form $y = ab^x$ will pass through the point $(0,a)$, where a will be the y-intercept of the function or its value at $x = 0$.
- Exponential equations can be solved by:
  - Writing both sides as the same base to a power and equating exponents.
  - Graphing as two separate equations to determine the intersection point.
- If $b > 1$, then the curve will be increasing and represent exponential growth.
- If $0 < b < 1$, then the curve will be decreasing and represent exponential decay.

Review

Write the exponential function that passes through the following points:

1. (0, 5) and (1, 25)

2. (0, 2) and (1, 8)

3. (0, 16) and (2, 144)

Sketch the graph of each function:

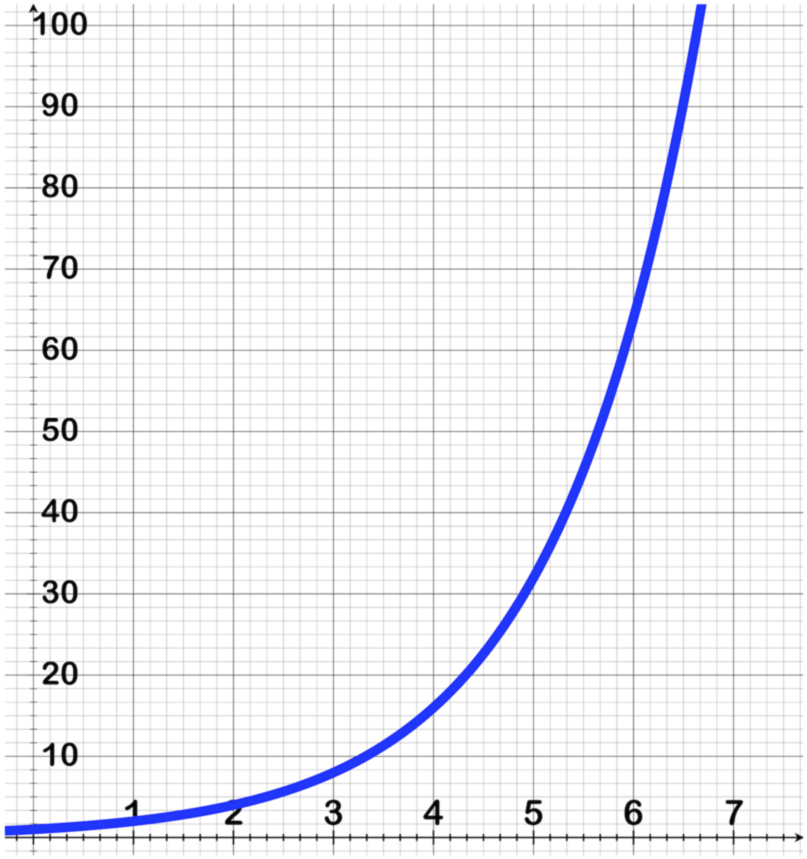
4. $y = 2 \cdot 3^x$

5. $y = 4 \cdot \frac{1}{2}^x$

6. $y = 2 \cdot \frac{1}{2}^{x+1} + 2$

Write an equation for each graph:

7.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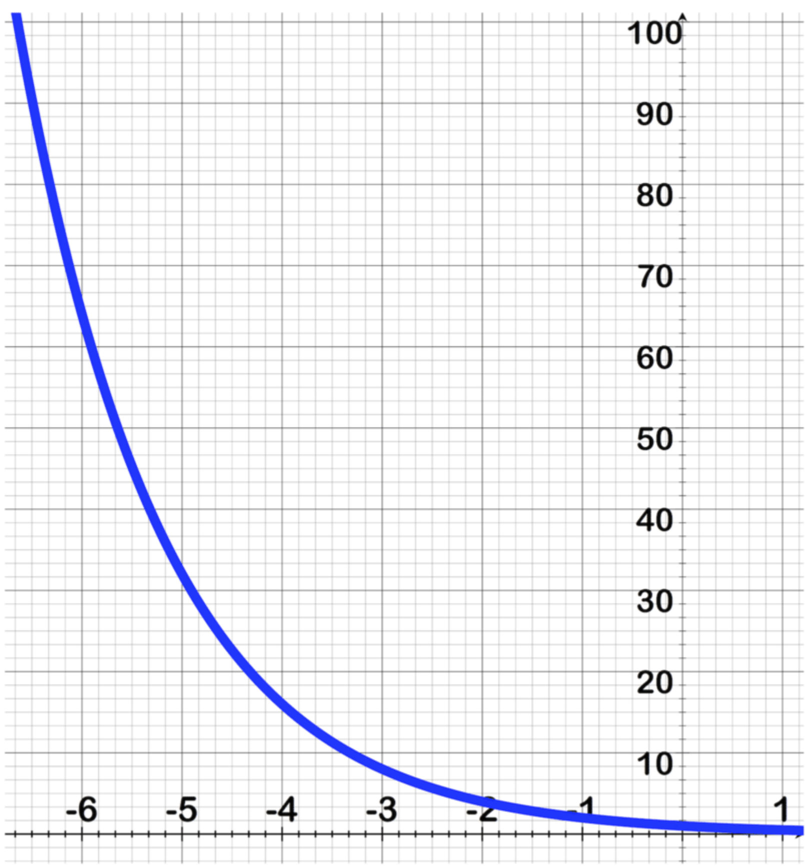
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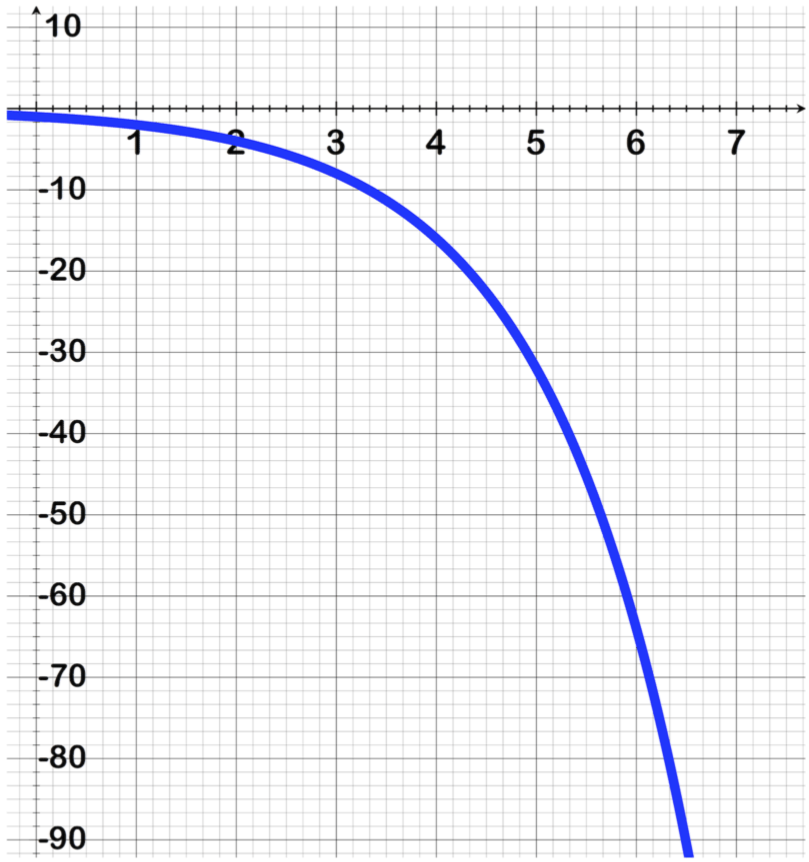


Use graphing transformation rules to sketch the graph of each function:

10. $f(x) = 3^{x-4}$

11. $f(x) = -4^x$

12. $f(x) = 3^x - 2$

13. $f(x) = -5^{x+2}$

14. $f(x) = 5^{x-4} - 3$

15. Explain why for exponential functions of the form $y = a \cdot b^x$, the y-intercept is always the value of a.

16. Suppose 40 rabbits are released on an island. The rabbits mate once every four months and produce an average of four offspring, which also produce more offspring four months later. Assume half the population is female. Estimate the number of rabbits on the island in three years if their population grows exponentially.

17. The half-life of a particular element is 2.552 minutes. How much of an 10-gram sample would be left after eight minutes?

Review (Answers)

Please see the Appendix.

4.3 Graphing and Evaluating Logarithmic Functions

Learning Objectives

Learn to graph a logarithmic function.

Introduction

Consider the emergency personnel who were handling the aftermath of Hurricane Fran. t days after the hurricane, the number of power outages could be modeled by the exponential function $f(t) = 1{,}274{,}100(0.6343)^t$, $t \geq 0$. This function is helpful, but it would be more beneficial to determine the number of days the emergency personnel need to report, given the number of power outages. For this calculation, the inverse function of the given exponential function must be used. This inverse function is

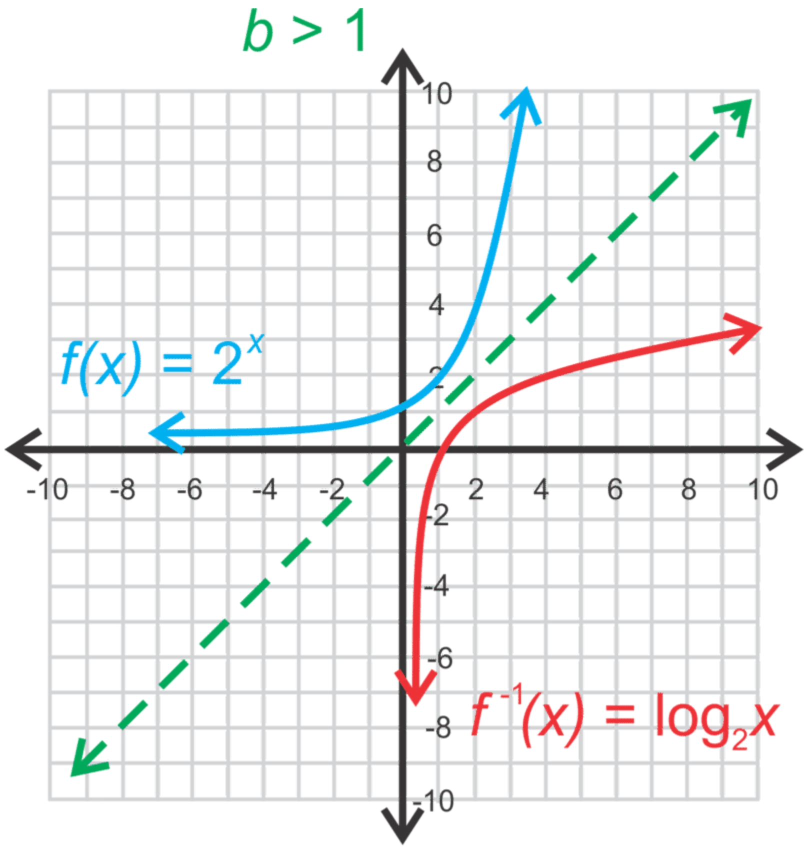
$$T(n) = \frac{\ln n - \ln 1{,}274{,}100}{\ln 0.6343},$$

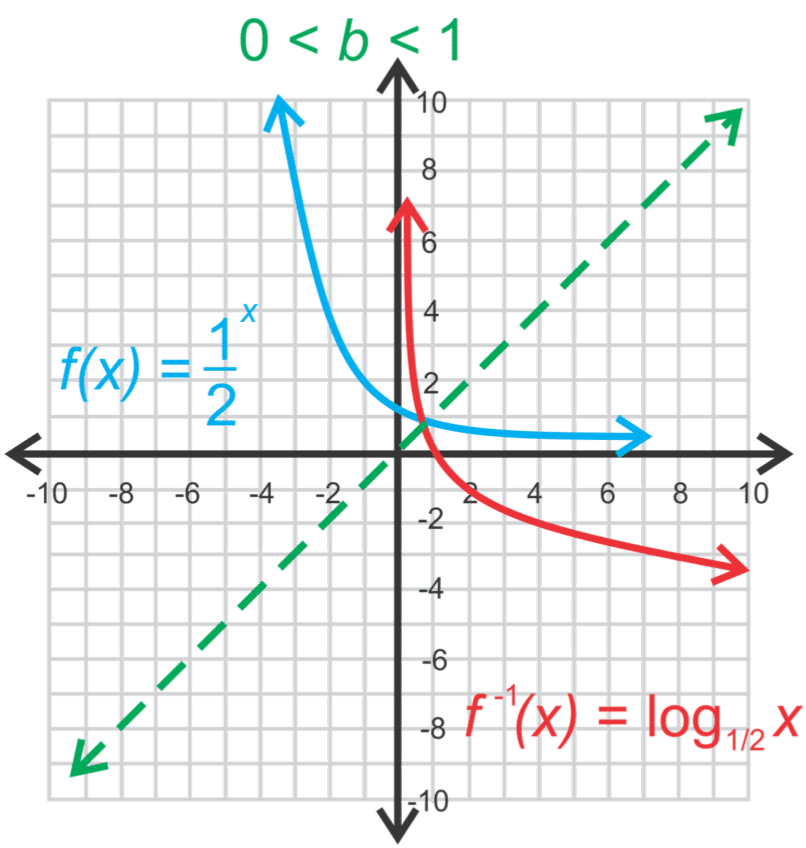
where $n > 0$ is the number of power outages, and $T(n)$ is the total number of days. If the number of power outages reaches 5,000, determine the number of days the emergency personnel need to report after the hurricane.

Inverse of an Exponential Function

Functions are inverses of each other when their graphs are mirror images over the line $y = x$. Consider $f(x) = b^x$, where $b > 1$ or $0 < 1$. When the function's graph is reflected over the line $y = x$, the resulting curve is the graph of $f^{-1}(x) = \log_b x$, where log represents a logarithm function.

The graphs below illustrate an example of the exponential function with base 2 and its inverse. The first graph illustrates $f(x) = 2^x$ and its inverse, $f^{-1}(x) = \log_2 x$. The second graph illustrates $f(x) = \left(\frac{1}{2}\right)^x$ and its inverse, $f^{-1}(x) = \log_{\frac{1}{2}} x$. The base in the 1st graph below is 2, and in the 2nd graph below is $\frac{1}{2}$, to display two ways the inverse functions are graphed in the coordinate plane. Note that the base b can be any number greater than 0 except 1.


b > 1
f(x) = 2^x
f^-1(x) = log_2 x
10
8
6
4
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
2
4
6
8
10


0 < b < 1
f(x) = 1/2^x
f^-1(x) = log_1/2 x
10
8
6
4
2
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
2
4
6
8
10

Inverse of the exponential function

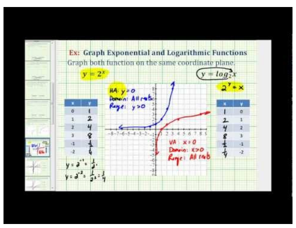
The inverse of the function $f(x) = b^x$ is $f^{-1}(x) = \log_b x$.

The general form of a logarithmic function is $f(x) = a\log_b x$. If $a = 1$, there are three key points on the graph of the logarithmic function: $\left(\frac{1}{b}, -1\right)$, $(1,0)$, and $(b,1)$.

When the base is 10, the inverse function is $f(x) = \log_{10} x$. This logarithmic function is called the common logarithm and is written as $f(x) = \log x$. When the base is e, the inverse function is $f(x) = \log_e x$. This logarithmic function is called the natural logarithm and is written as $f(x) = \ln x$.

The logarithmic function can also undergo transformations: $f(x) = a\log_b(x-h)+k$.

Since an exponential function has a horizontal asymptote, the logarithm will have a vertical asymptote at $x = h$. The domain is $x > h$ and the range is all real numbers. If $b > 1$ and $a > 0$, the graph increases on (h, ∞). If $0 < b < 1$ and $a > 0$, the graph decreases on (h, ∞).



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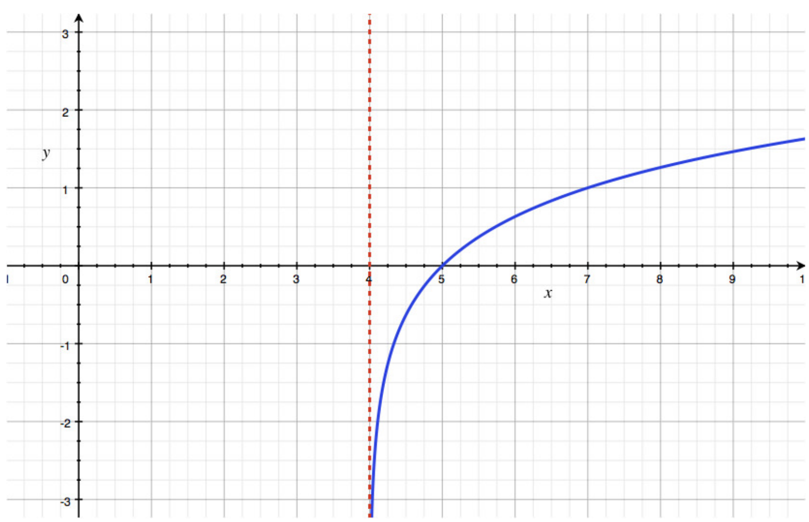
Example 1

Graph $y = \log_3(x-4)$. State the domain and range.

Solution:

The vertical asymptote is at $x = 4$. Since $3 > 1$, the graph is increasing on the interval $(4, \infty)$. The domain is $x > 4$, and the range is all real numbers.

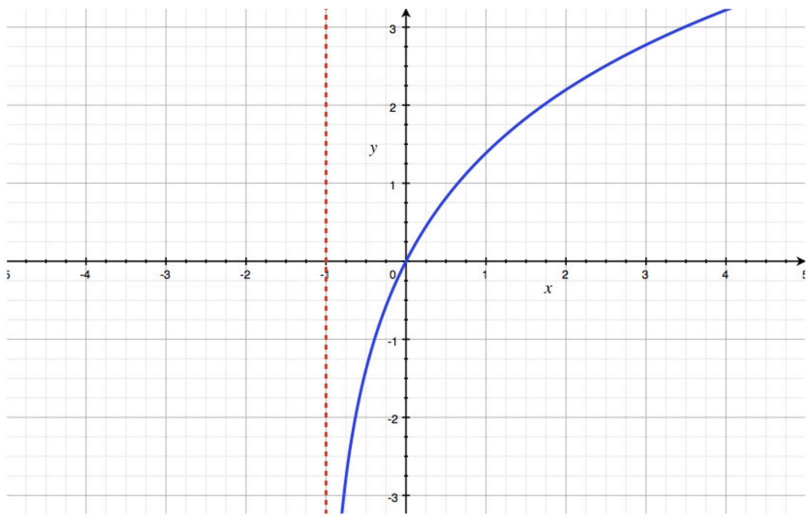


Example 2

Graph $f(x) = 2\ln(x+1)$.

Solution:

- The vertical asymptote is at $x = -1$.
- Since $e > 1$, the graph is increasing on the interval $(-1, \infty)$.
- The domain is $x > -1$, and the range is all real numbers.



Example 3

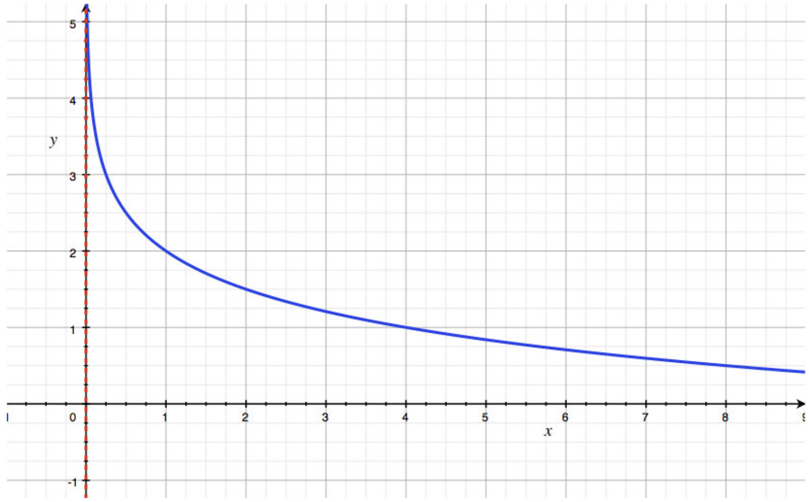
Find three key points for the function $y = \log_{\frac{1}{4}} x + 2$. Then graph the function.

Solution:

- There is a vertical asymptote at $x = 0$.
- Since $0 < \frac{1}{4} < 1$,the function is decreasing on $(0, \infty)$.
- Use point transformation to find some key points. The graph has a vertical shift up 2:

TABLE 4.4:

$(x_{old}, y_{old}) = \left(x, \log_{\frac{1}{4}} x\right)$	$(x_{old}, y_{old}) \to (x_{new}, y_{new}) = (x_{old}, y_{old} + 2)$
$\left(\frac{1}{4}, 1\right)$	$\left(\frac{1}{4}, 3\right)$
$(1, 0)$	$(1, 2)$
$(4, -1)$	$(4, 1)$

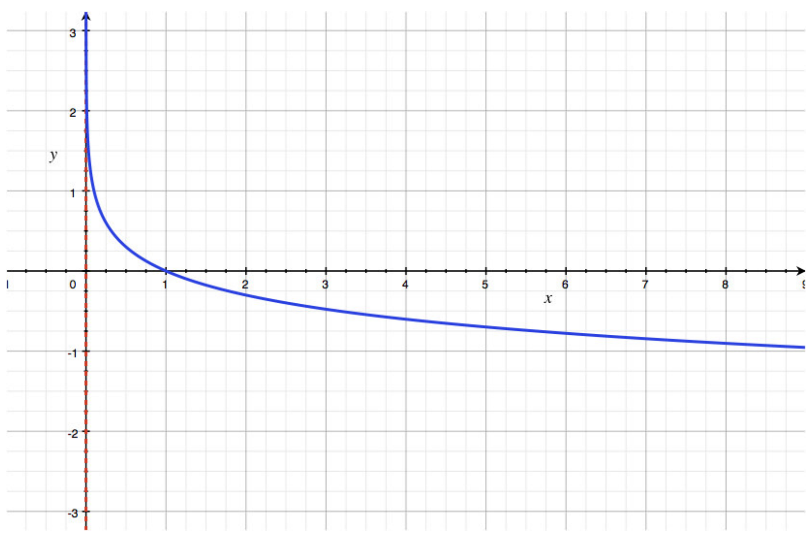


Example 4

Graph $y = -\log x$. Find the domain and range.

Solution:

- Since $10 > 1$ and this graph is reflected across the x-axis, the function is decreasing.
- The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$.



Example 5

Is $(-2, -1)$ on the graph of $f(x) = \log_{\frac{1}{2}}(x+4)$?

Solution:

Plug (-2, -1) into $f(x) = \log_{\frac{1}{2}}(x+4)$ to see if the equation holds true:

$$\begin{aligned}
-1 &= \log_{\frac{1}{2}}(-2+4) \\
-1 &= \log_{\frac{1}{2}} 2 \\
\frac{1}{2}^{-1} &= 2 \\
2 &= 2.
\end{aligned}$$

Therefore, (-2, -1) is on the graph.

Example 6

Is (16, 1) on $y = \log(x-6)$?

Solution:

Plug in the point to the equation to see if it holds true:

$$\begin{aligned} 1 &= \log(16-6) \\ 1 &= \log_{10} 10 \\ 10^1 &= 10. \end{aligned}$$

Yes, this is true, so (16, 1) is on the graph.

Example 7

Recall the problem from the Introduction. Determine the number of days the emergency personnel need to report after Hurricane Fran, if the number of power outages reaches 5,000. The inverse function for this problem is:

$$T(n) = \frac{\ln n - \ln 1{,}274{,}100}{\ln 0.6343} \approx -2.1967 \ln n + 30.8803$$

To determine the number of days after the hurricane that the outages will reach 5,000, calculate:

$$T(5{,}000) = \frac{\ln 5{,}000 - \ln 1{,}274{,}100}{\ln 0.6343} \approx 12 \text{ days}$$

Summary

- If $f(x) = b^x$, where $b > 1$ or $0 < 1$, then $f^{-1}(x) = \log_b x$.
- When $b = 10$, the inverse function is called the common logarithm and is written as $f(x) = \log x$.
- When $b = e$, the inverse function is called the natural logarithm and is written as $f(x) = \ln x$.
- The logarithmic function can also undergo transformations: $f(x) = a \log_b (x-h) + k$.

Review

Graph the following logarithmic functions without using a calculator by reflecting the corresponding exponential function. State the equation of the asymptote, the domain, and the range of each function:

1. $y = \log_5 x$
2. $y = \log_2 (x+1)$
3. $y = \log(x) - 4$
4. $y = \log_{\frac{1}{3}} (x-1) + 3$
5. $y = -\log_{\frac{1}{2}} (x+3) - 5$
6. $y = \log_4 (2-x) + 2$

Graph the following logarithmic functions using your graphing device:

7. $y = \ln(x+6) - 1$
8. $y = -\ln(x-1) + 2$
9. $y = \log(1-x) + 3$
10. $y = \log(x+2) - 4$

11. **Bonus:** How would you graph $y = \log_4 x$ on the graphing device? Graph the function.
12. **Bonus:** Graph $y = \log_{\frac{3}{4}} x$ on the graphing device.
13. Is (3, 8) on the graph of $y = \log_3(2x-3) + 7$?
14. Is (9, -2) on the graph of $y = \log_{\frac{1}{4}}(x-5)$?
15. Is (4, 5) on the graph of $y = 5\log_2(8-x)$?

Review (Answers)

Please see the Appendix.

4.4 Properties of Logs

Learning Objectives

Learn how to combine and expand logarithms using properties.

Introduction

Consider the exponential function $y = 2^x$. Its inverse is $y = \log_2 x$.To progress further with this class of functions, we need to clarify what a log expression represents. Consider the following:

TABLE 4.5:

Function Domain: x	**Function Range:** $y = 2^x$	**Inverse Domain:** $y = 2^x$	**Inverse Range:** x
-2	$2^{-2} = \frac{1}{4}$	$2^{-2} = \frac{1}{4}$	-2
-1	$2^{-1} = \frac{1}{2}$	$2^{-1} = \frac{1}{2}$	-1
0	$2^0 = 1$	$2^0 = 1$	0
1	$2^1 = 2$	$2^1 = 2$	1
2	$2^2 = 4$	$2^2 = 4$	2

Using the columns regarding the inverse function, we can rewrite the table as follows:

TABLE 4.6:

x	$y = \log_2 x$
$\frac{1}{4}$	$\log_2 \frac{1}{4} = -2$
$\frac{1}{2}$	$\log_2 \frac{1}{2} = -1$
1	$\log_2 1 = 0$
2	$\log_2 2 = 1$
4	$\log_2 4 = 2$

This table illustrates the meaning of the logarithm. The logarithm is an exponent. The expression $\log_2 4 = 2$ can be read: "The power (log) to which 2 is taken (subscript) to get 4 (argument) is 2." Practice reading the right column of the last table using that language.

Logarithms

The definition of a logarithm can be expressed in following two equivalent equations:

$a = b^x \iff x = \log_b a.$

The exponential equation is read "b to the x is a." The logarithmic equation is read "log base b of a is x."

Properties of Logarithms

There are several properties of logarithms that are used as tools to solve equations with exponential and logarithmic expressions. Each of these can be easily established by translating the expression into words.

Properties of Logarithms

For $b > 0$,

1. $\log_b 1 = 0$

Verification: Rewrite the equation using the definition $b^0 = 1$, which is a more familiar statement.

2. $\log_b b = 1$

Verification: Rewrite the equation using the definition $b^1 = b$, which is a more familiar statement.

3. $\log_b (b^x) = x$

Verification: The equation can be rewritten using the definition $b^x = b^x$.

Consider the function $f(x) = \log_b x$. Then the inverse function is $f^{-1}(x) = b^x$. Since the two functions are inverses, $\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = \log_b f^{-1}(x) = \log_b b^x$. Recall that for inverse functions, $\left(f \circ f^{-1}\right)(x) = x$. Thus, $\log_b b^x = x$.

4. $b^{\log_b x} = x,\ x > 0$

Verification: The inverse function argument works the same way as the previous property, $\left(f^{-1} \circ f\right)(x) = b^{\log_b x} = x$.

Along with the properties of logarithms, there are rules for logarithms and exponents that detail how to handle arithmetic operators.

Rules of Logarithms & Exponents

For $b,\ x,$ and $y > 0$:

Addition or Multiplication of Logarithms

Logarithmic form: $\log_b(x \cdot y) = \log_b x + \log_b y$

Exponential form: $b^w \cdot b^z = b^{w+z}$

Subtraction or Division of Logarithms

Logarithmic form: $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

Exponential form: $\frac{b^w}{b^z} = b^{w-z}$

Power to a Power

Logarithmic form: $\log_b (x^n) = n \cdot \log_b x$

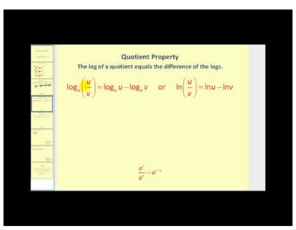
Exponential form: $(b^w)^n = b^{w \cdot n}$

Change of Base

$\log_b x = \frac{\log_a x}{\log_a b}$

$\log_{\frac{1}{b}} x = -\log_b x$

$\log_{a^n} x = \frac{1}{n} \log_a x$



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Examples

Example 1

Simplify the following expressions:

a) $\log_4 64$

Solution:

$\log_4 64 = \log_4 4^3 = 3 \cdot \log_4 4 = 3 \cdot 1 = 3$

b) $\log_{\frac{1}{2}} 32$

Solution:

$\log_{\frac{1}{2}} 32$
$-\log_2 32$
-5

c) $\log_3 3^5$

Solution:

$\log_3 3^5 = 5 \cdot \log_3 3 = 5$

d) $\log_2 128$

Solution:

$\log_2 128 = \log_2 2^7 = 7$

Example 2

Write the expression as a single logarithm:

$$\log_2 12 + \log_2 6^{\frac{1}{2}} - \log_2 24$$

Solution:

The expression with the same base is

$$\begin{aligned}\log_2 12 + \log_2 6^{\frac{1}{2}} - \log_2 24 &= \log_2 \left(\frac{12 \cdot \sqrt{6}}{24}\right)\\ &= \log_2 \left(\frac{\sqrt{6}}{2}\right).\end{aligned}$$

Example 3

Simplify the expression $2\log_{12} 144^{-4}$ using properties of logarithms.

Solution:

$2\log_{12} 144^{-4} = -8 \cdot \log_{12} 12^2 = -16 \cdot \log_{12} 12 = -16(1) = -16$

Example 4

Prove the following log identity:

$\log_a b = \frac{1}{\log_b a}$

Solution:

Let the left side of the equation be equal to x. Rewrite in exponential form in order to manipulate the equation to solve for a. Then rewrite back in logarithmic form until you get the expression from the left side of the equation.

$$\log_a b = x$$
$$a^x = b$$
$$a = b^{\frac{1}{x}}$$
$$\log_b a = \log_b b^{\frac{1}{x}} = \frac{1}{x}$$
$$x = \frac{1}{\log_b a}$$

Therefore, $\frac{1}{\log_b a} = \log_a b$.

Example 5

Rewrite the following expression as a single logarithm:

$\ln e - \ln 4x + 2(e^{\ln x} \cdot \ln 5)$.

Solution:

$$\begin{aligned}
\ln e - \ln 4x + 2(e^{\ln x} \cdot \ln 5) &= \ln\left(\frac{e}{4x}\right) + 2x \cdot \ln 5 \\
&= \ln\left(\frac{e}{4x}\right) + \ln(5^{2x}) \\
&= \ln\left(\frac{e \cdot 5^{2x}}{4x}\right)
\end{aligned}$$

Example 6

True or false?

$(\log_3 4x) \cdot (\log_3 5y) = \log_3(4x + 5y)$

Solution:

False! The logarithm of a product is the sum of logs. However, it is not true that the product of logs is the log of a sum.

Example 7

Simplify $\log_2 48 - \log_4 36$.

Solution:

Use the change of base formula when one of the bases is an exponent of the other:

$$\begin{aligned}
\log_2 48 - \log_4 36 &= \frac{\log_4 48}{\log_4 2} - \log_4 36 \\
&= \frac{\log_4 48}{\log_4 4^{\frac{1}{2}}} - \log_4 36 \\
&= \frac{\log_4 48}{\frac{1}{2}} - \log_4 36 \\
&= 2\log_4 48 - \log_4 36 \\
&= \log_4 48^2 - \log_4 36 \\
&= \log_4 \frac{48^2}{36} \\
&= \log_4 64 \\
&= 3
\end{aligned}$$

Summary

- Definition of logarithm: $b^x = a \leftrightarrow \log_b a = x$
- Properties:
  1. $\log_b 1 = 0$
  2. $\log_b b = 1$
  3. $\log_b (b^x) = x$
  4. $b^{\log_b x} = x$
- Rules:
  1. $\log_b (x \cdot y) = \log_b x + \log_b y$
  2. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$
  3. $\log_b (x^n) = \log_b (x)^n = n \cdot \log_b x$
  4. $\log_b x = \frac{\log_a x}{\log_a b}$
  5. $\log_{\frac{1}{b}} x = -\log_b x$
  6. $\log_{a^n} x = \frac{1}{n} \log_a x$

Review

Decide whether each of the statements below is true or false. Explain.

1. $\frac{\log x}{\log y} = \log \left(\frac{x}{y}\right)$

2. $(\log x)^n = n \log x$

3. $\log x + \log y = \log xy$

Rewrite each of the following expressions as a single logarithm:

4. $\log 4x + \log(2x+4)$

5. $5\log x + \log x$

6. $4\log_2 x + \frac{1}{2}\log_2 9 - \log_2 y$

7. $6\log_3 z^2 + \frac{1}{4}\log_3 y^8 - 2\log_3 z^4 y$

Expand the expression as much as possible:

8. $\log_4\left(\frac{2x^3}{5}\right)$

9. $\ln\left(\frac{4xy^2}{15}\right)$

10. $\log\left(\frac{x^2(yz)^3}{3}\right)$

Translate from exponential form to logarithmic form:

11. $2^{x+1} + 4 = 14$

Translate from logarithmic form to exponential form:

12. $\log_2(x-1) = 12$

Prove the following properties of logarithms:

13. $\log_{b^n} x = \frac{1}{n}\log_b x$

14. $\log_{b^n} x^n = \log_b x$

15. $\log_{\frac{1}{b}} \frac{1}{x} = \log_b x$

Review (Answers)

Please see the Appendix.

4.5 Solving Exponential Equations Using Logs

Learning Objectives

Learn to apply algebraic techniques associated with logs to solve exponential equations.

Introduction

A small town was established in 1950, and the population is given by

$$P(t) = 2{,}000(1.05)^t,$$

where t is the number of years since 1950. The mayor would like to know when the population reached 20,000. In other words, a solution to the following equation is needed:

$$20{,}000 = 2{,}000(1.05)^t.$$

The difficulty is that the variable is in the exponent. We will explore algebraic techniques to handle that issue.

Exponential Equation Solution Techniques

To solve an exponential equation:

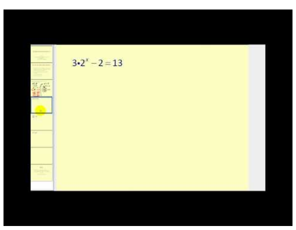
1. Isolate the exponential part of the equation. If there are two exponential parts, then rewrite so there is a single exponent on each side of the equation.

2. Take the logarithm of each side of the equation.
3. Solve for the variable.
4. Check your solution.

A common technique for solving equations with variables in exponents is to take the log of both sides of the equation. The properties of logs can be used to simplify and solve the equation.

Properties of Using Logarithms

$$\log a^x = x \cdot \log a$$
$$\ln e^x = x$$
$$\log 10^x = x$$



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Examples

Example 1

The amount of time it will take to have \$9,000 in a savings account, paying 6% annual compound interest, if \$300 is deposited at the end of each year, satisfies the equation

$$9{,}000 = 300 \cdot \frac{(1.06)^t - 1}{0.06}.$$

This type of investment is called an annuity. Solve the preceding equation for t.

Solution:

$$\begin{aligned}
30 &= \frac{(1.06)^t - 1}{0.06} \\
1.8 &= (1.06)^t - 1 \\
2.8 &= 1.06^t \\
\ln 2.8 &= \ln(1.06^t) = t \cdot \ln(1.06) \\
t &= \frac{\ln 2.8}{\ln 1.06} \approx 17.67 \ years
\end{aligned}$$

Example 2

Solve the following equation for x: $16^x = 25$.

Solution:

Take the log of both sides. Use log properties and a calculator to approximate the solution:

$$\begin{aligned}
16^x &= 25 \\
\log 16^x &= \log 25 \\
x \log 16 &= \log 25 \\
x &= \frac{\log 25}{\log 16} \\
x &\approx 1.16
\end{aligned}$$

Example 3

Solve the following equation for all possible values of x: $(\log_2 x)^2 - \log_2\left(x^7\right) = -12$.

Solution:

Step 1: Identify that this is a quadratic log problem, because the logarithmic term is squared in the 1st term. Use a substitution to examine each layer of the problem.

Step 2: Let $u = \log_2 x$.

$$\begin{aligned}
(\log_2 x)^2 - 7\log_2 x + 12 &= 0 \\
u^2 - 7u + 12 &= 0 \\
(u-3)(u-4) &= 0 \\
u &= 3, 4
\end{aligned}$$

Step 3: Now, substitute back and solve for x in each case.

$$\begin{aligned}
\log_2 x = 3 &\iff x = 2^3 = 8 \\
\log_2 x = 4 &\iff x = 2^4 = 16
\end{aligned}$$

Example 4

Return to the mayor's question from the Introduction. When will the small town reach a population of 20,000, as modeled by the equation $20{,}000 = 2{,}000(1.05)^t$?

$$\begin{aligned} 20{,}000 &= 2{,}000(1.05)^t \\ 10 &= (1.05)^t \\ \log 10 &= \log (1.05)^t \\ \log(10) &= t \cdot \log 1.05 \\ t &= \frac{\log(10)}{\log(1.05)} \\ t &\approx 47.19 \text{ years} \end{aligned}$$

The population will reach 20,000 in 1997, because 1997 is 47 years after 1950.

Example 5

List all possible values of x for the following equation: $(x+1)^{x-4} - 1 = 0$.

Solution:

$$\begin{aligned} (x+1)^{x-4} - 1 &= 0 \\ (x+1)^{x-4} &= 1 && \text{Add 1.} \\ \log (x+1)^{x-4} &= \log 1 && \text{Take the log of both sides.} \\ (x-4) \cdot \log (x+1) &= 0 && \text{Use properties of logs.} \\ x-4 = 0 \text{ or } \log (x+1) &= 0 \\ x-4 = 0 \text{ or } 10^0 &= x+1 \\ x-4 = 0 \text{ or } 1-1 &= x \\ x &= 4, 0 \end{aligned}$$

Recall that you can only take the log of a positive argument. What if $x+1$ is negative 1 but raised to an even power? Notice that when $x = -2$,

$$(-2+1)^{-2-4} - 1 = (-1)^{-6} - 1 = \frac{1}{(-1)^6} - 1 = 0,$$

so it is also a solution. However, $\log (-2+1) = \log (-1)$ is not possible.

Note that you shouldn't fall into the habit of assuming you can take the log of both sides and get all the solutions. This is only true when the argument is strictly positive.

Example 6

Light intensity as it travels at specific depths of water in a swimming pool can be described by the relationship between i for intensity, and d for depth in feet. What is the intensity of light at 10 feet?

$\log \left(\frac{i}{12}\right) = -0.0145 \cdot d$

Solution:

Given $d = 10$, solve for i measured in lumens.

$$\log\left(\frac{i}{12}\right) = -0.0145 \cdot d$$
$$\log\left(\frac{i}{12}\right) = -0.0145 \cdot 10$$
$$\log\left(\frac{i}{12}\right) = -0.145$$
$$\left(\frac{i}{12}\right) = 10^{-0.145}$$
$$i = 12 \cdot 10^{-0.145} \approx 8.594 \text{ lumens}$$

Example 7

Solve the following equation for all possible values of x:

$$\frac{e^x - e^{-x}}{3} = 14$$

Solution:

First solve for e^x:

$$\frac{e^x - e^{-x}}{3} = 14$$
$$e^x - e^{-x} = 42$$
$$e^x \cdot \left(e^x - e^{-x}\right) = (42) \cdot e^x$$
$$e^{2x} - 1 = 42e^x$$
$$(e^x)^2 - 42e^x - 1 = 0$$

Let $u = e^x$.

$$u^2 - 42u - 1 = 0$$
$$u = \frac{-(-42) \pm \sqrt{(-42)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{42 \pm \sqrt{1768}}{2} \approx 42.023796, -0.0237960$$

Since the range of the exponential function is greater than 0, $e^x > 0$, then $e^x \approx -0.0237960$ does not exist. Thus, -0.0237960 is extraneous, so there is only one result.

$$e^x \approx 42.023796$$
$$x \approx \ln 42.023796 \approx 3.738$$

Summary

- To solve an exponential equation:

- Isolate the exponential part of the equation. If there are two exponential parts, then rewrite so there is a single exponent on each side of the equation.
- Take the logarithm of each side of the equation.
- Solve for the variable.
- Check your solution.

Review

Solve each equation for x. If necessary, round each answer to three decimal places.

1. $4^x = 6$
2. $5^x = 2$
3. $12^{4x} = 1{,}020$
4. $7^{3x} = 2{,}400$
5. $2^{x+1} - 5 = 22$
6. $5x + 12^x = 5x + 7$
7. $2^{x+1} = 2^{2x+3}$
8. $3^{x+3} = 9^{x+1}$
9. $2^{x+4} = 5^x$
10. $13 \cdot 8^{0.2x} = 546$
11. $b^x = c + a$
12. $32^x = 0.94 - .12$

Solve each log equation by using log properties and rewriting as an exponential equation:

13. $\log_3 x + \log_3 5 = 2$
14. $2 \log x = \log 8 + \log 5 - \log 10$
15. $\log_9 x = \frac{3}{2}$

Review (Answers)

Please see the Appendix.

4.6 Solving Logarithmic Equations

Learning Objectives

Learn how to solve a logarithmic equation with any base.

Introduction

The doubling time for an investment that has earnings compounded annually is $t = \frac{\ln 2}{\ln(1+r)}$,where $r$is the interest rate in decimal form. For interest rates close to 8%, or 0.08, this formula can be accurately approximated by the formula $t = \frac{72}{100r}$. In other words, the doubling time is about the quotient obtained by dividing 72 by the annual interest rate, as a percent. Since it is fairly easy to divide numbers into 72, this rough rule of thumb shows up in business discussions as "the rule of 72." For example, when does a 4% interest rate double? Since 72/4 = 18, a 4% rate should double in about 18 years. The question to be answered in this section is that, assuming an investment must double in 2 years, what would the rate of return need to be? Rather than using the approximation, the rule of 72, find the interest rate exactly. The equation to solve is $2 = \frac{\ln 2}{\ln(1+r)}$. Solving this problem will require us to use new techniques.

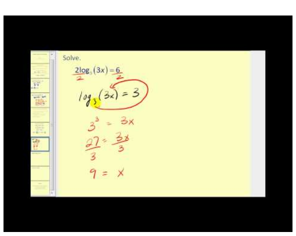
Method to Solve Logarithmic Equations

When solving simple exponential equations, we have used the definition $\log_b x = a \leftrightarrow b^a = x$ up to this point to translate the equation into an exponential one, which is most intuitive and often easier to solve. At times, more algebraic simplification is needed, and usually this involves the properties of logs as well as the one-to-one properties:

One-to-One Properties

For $b > 0$,

1. Exponential functions: $A = B$ if and only if $b^A = b^B$
2. Logarithm functions: $A = B$ if and only if $\log_b A = \log_b B$



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Learn, Play, and Explore with Logarithmic Equations: Logarithmic Equations

Examples

Example 1

Solve $\log_2(x+5) = 9$.

Solution:

There are two different ways to solve this equation.

Method 1: The 1st is to use the definition of a logarithm.

$$\begin{aligned}\log_2(x+5) &= 9\\ 2^9 &= x+5\\ 512 &= x+5\\ 507 &= x\end{aligned}$$

Method 2: Apply the 1st one-to-one property.

$$\begin{aligned}2^{\log_2(x+5)} &= 2^9\\ x+5 &= 512\\ x &= 507\end{aligned}$$

Example 2

Solve $3\ln(-x) - 5 = 10$.

Solution:

Step 1: Add 5 to both sides.

$$\begin{aligned}3\ln(-x) - 5 &= 10\\ 3\ln(-x) &= 15\end{aligned}$$

Step 2: Divide by 3 to isolate the natural log.

$$\begin{aligned}3\ln(-x) &= 15\\ \ln(-x) &= 5\end{aligned}$$

Step 3: Apply the cancellation property.

$$\begin{aligned}e^{\ln(-x)} &= e^5\\ -x &= e^5\\ x &= -e^5 \approx -148.41\end{aligned}$$

Example 3

Solve $\log 5x + \log(x-1) = 2$.

Solution:

Step 1: Condense the lefthand side using the product property.

$$\begin{aligned}\log 5x + \log(x-1) &= 2\\ \log[5x(x-1)] &= 2\\ \log(5x^2-5x) &= 2\end{aligned}$$

Step 2: Apply the cancellation property.

$$\begin{aligned}10^{\log(5x^2-5x)} &= 10^2\\ 5x^2-5x &= 100\\ 5x^2-5x-100 &= 0\\ x^2-x-20 &= 0\\ (x-5)(x+4) &= 0\\ x &= 5,-4\end{aligned}$$

Step 3: Check to see that both answers are in the domain of the equation.

$$\begin{aligned}\log 5(5) + \log(5-1) &= 2 & \log 5(-4) + \log((-4)-1) &= 2\\ \log 25 + \log 4 &= 2 & \log(-20) + \log(-5) &= 2\\ \log 100 &= 2 & &\end{aligned}$$

In the step $\log(-20) + \log(-5) = 2$, the logarithm arguments are negative. However, the domain of the log function does not include negatives. Thus, -4 is an extraneous solution, and the solution is $x = 5$.

Example 4

Return to the problem from the introduction, the doubling time for an investment which has earnings compounded annually is $t = \frac{\ln 2}{\ln(1+r)}$.

$$\begin{aligned}2 &= \frac{\ln 2}{\ln(1+r)}\\ \ln(1+r) &= \frac{\ln(2)}{2}\\ e^{\ln(1+r)} &= e^{\frac{\ln(2)}{2}} = \left(e^{\ln(2)}\right)^{\frac{1}{2}}\\ (1+r) &= 2^{\frac{1}{2}} = \sqrt{2}\\ r &= \sqrt{2}-1 \approx .414 = 41.4\%\end{aligned}$$

Thus, the interest rate required to double the investment in 2 years is about 41.4%.

Example 5

Solve the following logarithmic equation:

$$9+2\log_3 x = 23.$$

Solution:

Rewrite the logarithm and apply the 1st one-to-one property.

$$\begin{aligned}
9+2\log_3 x &= 23 \\
2\log_3 x &= 14 \\
\log_3 x &= 7 \\
3^{\log_3 x} &= 3^7 \\
x &= 2{,}187
\end{aligned}$$

Example 6

Solve the following logarithmic equation:

$$\ln(x-1)-\ln(x+1) = 8.$$

Solution:

Rewrite the lefthand side as a single logarithm, and then apply the definition of a logarithmic function.

$$\begin{aligned}
\ln(x-1)-\ln(x+1) &= 8 \\
\ln\left(\frac{x-1}{x+1}\right) &= 8 \\
e^{\ln\left(\frac{x-1}{x+1}\right)} &= e^8 \\
\frac{x-1}{x+1} &= e^8 \\
x-1 &= (x+1)e^8 \\
x-1 &= xe^8+e^8 \\
x-xe^8 &= 1+e^8 \\
x(1-e^8) &= 1+e^8 \\
x &= \frac{1+e^8}{1-e^8} \approx -1.0007
\end{aligned}$$

Since a logarithm argument must be greater than 0, $\frac{x-1}{x+1} > 0$ and $x > 1$. Thus, -1.0007 is not in the domain of this equation, so there is no solution.

Example 7

Solve the following logarithmic equation:

$$\frac{1}{2}\log_5(2x+5) = 2.$$

Solution:

Rewrite the logarithm and apply the 1st one-to-one property.

$$\begin{aligned}
\frac{1}{2}\log_5(2x+5) &= 2\\
\log_5(2x+5) &= 4\\
5^{\log_5(2x+5)} &= 5^4\\
2x+5 &= 625\\
2x &= 620\\
x &= 310
\end{aligned}$$

Summary

- To solve logarithmic equations:
  - Use one-to-one properties:
    - $A = B$ if and only if $b^A = b^B$
    - $A = B$, if and only if $\log_b A = \log_b B$.
  - Rewrite the equation as an exponential equation using $\log_b x = y \longleftrightarrow b^y = x$.

Review

Use properties of logarithms and a calculator to solve the following equations for x. Round answers to three decimal places and check for extraneous solutions.

1. $\log_2 x = 15$
2. $\log_{12} x = 2.5$
3. $\log_9(x-5) = 2$
4. $\log_7(2x+3) = 3$
5. $8\ln(3-x) = 5$
6. $4\log_3 3x - \log_3 x = 5$
7. $\log(x+5) + \log x = \log 14$
8. $2\ln x - \ln x = 0$
9. $3\log_3(x-5) = 3$
10. $\frac{2}{3}\log_3 x = 2$
11. $5\log\frac{x}{2} - 3\log\frac{1}{x} = \log 8$
12. $2\ln x^{e+2} - \ln x = 10$
13. $2\log_6 x + 1 = \log_6(5x+4)$
14. $2\log_{\frac{1}{2}} x + 2 = \log_{\frac{1}{2}}(x+10)$
15. $3\log_{\frac{2}{3}} x - \log_{\frac{2}{3}} 27 = \log_{\frac{2}{3}} 8$

Review (Answers)

Please see the Appendix.

4.7 Compound Interest

Learning Objectives

Learn to calculate compounded interest and how it applies to the real world.

Introduction

An investor is considering two investments. The 1st choice is a capital investment and pays a guaranteed 5.4% annually. The 2nd choice is a savings account that pays 5.3% interest compounded daily. The investor plans to choose the 1st investment, but is this the best choice?

Compound Interest

Nearly all investment accounts can be established so that the interest is compounded. This means the investor is paid interest on any funds in an account, even if they were earnings on the original investment. Every investor needs to understand the mathematical effects of compounding. First, the formula:

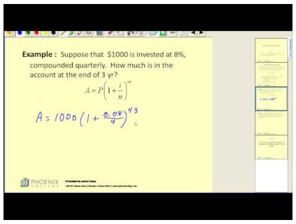
$$A(t) = p \cdot \left(1 + \frac{r}{n}\right)^{nt},$$

where

- $A(t)$ is the accumulated value of the investment in the account after t years.
- p is the principal, or amount originally invested.
- r is the annual interest rate.
- n is the number of times that the interest is compounded each year. This means the interest is paid in n equal payments throughout the year.
- t is the number of years that the money remains in the account.

If the amount of money is compounded quarterly, then $n = 4$. If the amount of money is compounded monthly, then $n = 12$. Notice that t is determined in years, so 6 months would mean $t = 0.5$.

Compound interest yields more interest because throughout the year, the principal includes both the initial principal and the interest earnings for the year.



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Continuously Compounded Interest

In finance, interest that is calculated continuously is often discussed. This type of interest is called **continuously compounded interest**. Similar to population growth and decay, an exponential function is used to simulate the continuous growth of the interest. The formula for continuously compounded interest is

$$A(t) = pe^{rt}.$$

Examples

Example 1

Use the formula for compound interest to determine the amount of money in an investment after 20 years, if \$2,000 is invested, and the interest rate is 5% compounded annually.

Solution:

$$A(20) = 2{,}000(1.05)^{20}$$
$$A(20) \approx 5{,}306.60$$

The investment will be worth \$5,306.60.

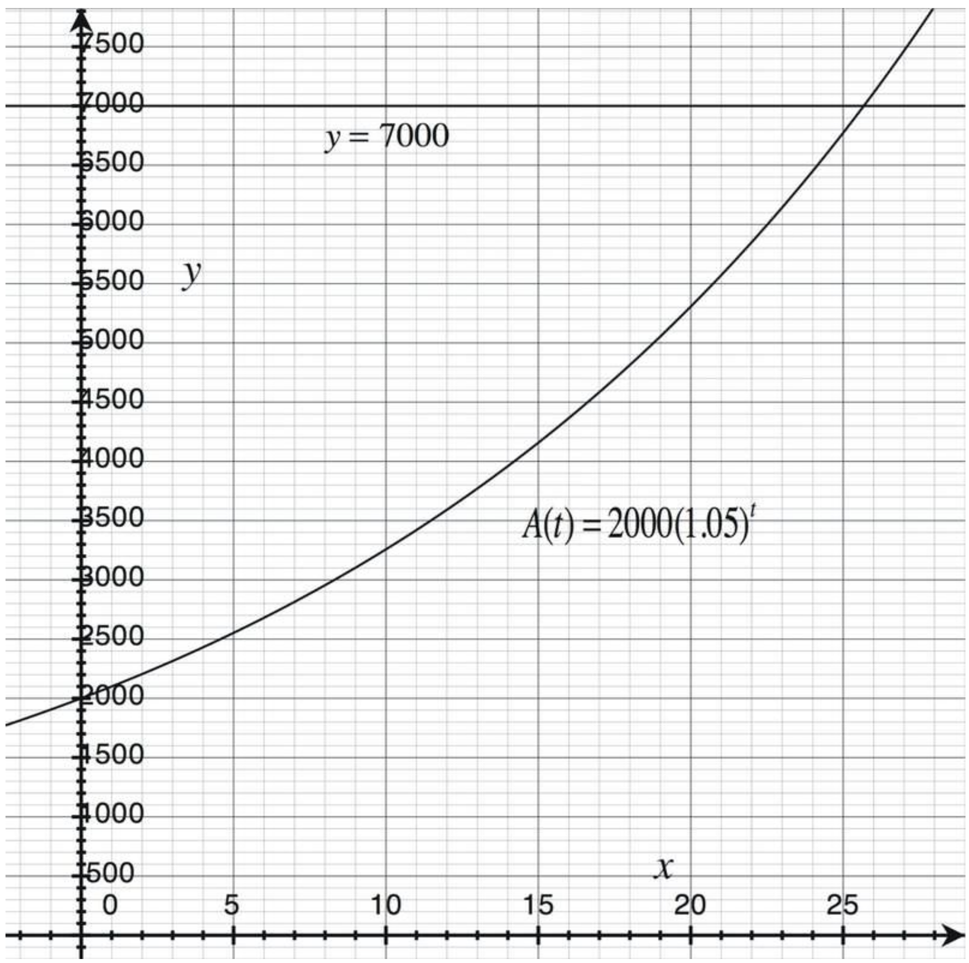
Example 2

How long will it take for \$2,000 invested at 5% compounded annually to reach \$7,000?

Solution:

$$2{,}000(1.05)^t = 7{,}000$$
$$1.05^t = 3.5$$
$$\ln 1.05^t = \ln 3.5$$
$$t \cdot \ln 1.05 = \ln 3.5$$
$$t = \frac{\ln 3.5}{\ln 1.05}$$
$$t \approx 25.677$$

This example could also be solved by graphing the function $A(t) = 2{,}000(1.05)^t$ and the horizontal line $y = 7{,}000$. The graph shows account values for a 28-year period. The intersection of the exponential function for compound interest and the horizontal line can be used to estimate the year when the account reaches $7,000.



The intersection point is approximately (25.7, 7000), so it would take almost 26 years for the investment to reach $7,000.

Example 3

What is the value of an investment after 20 years, if $2,000 is invested at the interest rate 5% compounded continuously?

Solution:

Using the formula for compound interest:

$$A(20) = 2{,}000e^{0.05(20)}$$
$$A(20) = 2{,}000e^{1}$$
$$A(20) \approx 5{,}436.56.$$

So, the value of the investment is $5,436.56.

Example 4

Return to the problem from the Introduction, concerning an investor who is considering two investments. The 1st choice pays a guaranteed 5.4% annually, and the 2nd choice pays 5.3% interest compounded daily. Which is the best choice?

Solution:

To answer the original problem, assume $20,000 is invested and calculate the values of each account after one year. The choices are:

First choice: $A(1) = 20{,}000(1+.054)^1 = 21{,}080.00.$

Second choice: $A(1) = 20{,}000\left(1+\frac{0.053}{365}\right) \approx 21{,}088.51.$

So, although the 1st choice pays 5.4% interest, it is paid only once per year. The 2nd choice pays 5.3% interest, but pays 365 times per year. The effect of repeatedly paying interest on interest is the compounding of the interest. Clearly, the investor is benefitted by more frequent compounding.

Example 5

Compare the values of the investments shown in the table. How does the compounding influence the value of the investment?

TABLE 4.7:

	Principal	r	n	t
a.	$4,000	.05	1 (annual)	8
b.	$4,000	.05	4 (quarterly)	8
c.	$4,000	.05	12 (monthly)	8
d.	$4,000	.05	365 (daily)	8
e.	$4,000	.05	8,760 (hourly)	8

Solution:

Apply the compound interest formula. For this example, the n is the quantity that changes: $A(8) = 4{,}000\left(1+\frac{.05}{n}\right)^{8n}$.

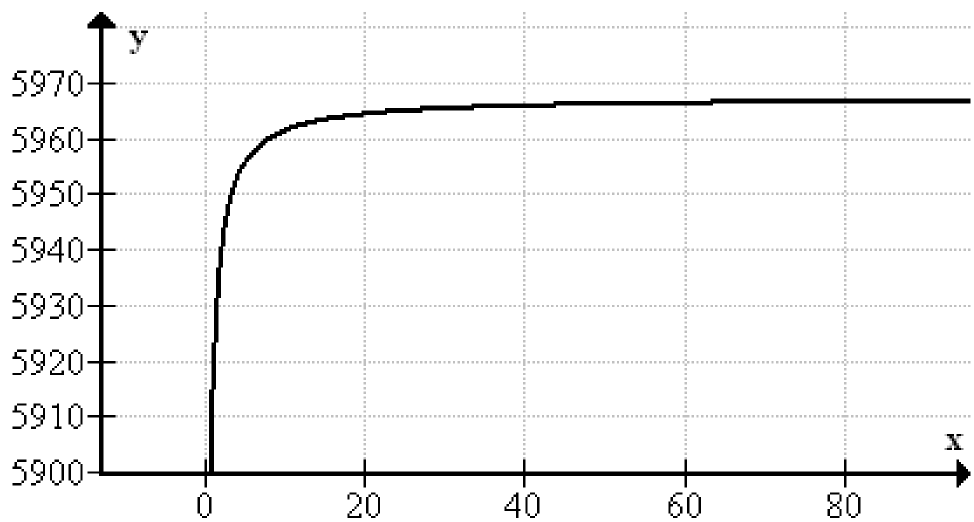
TABLE 4.8:

	Principal	r	n	t	A
a.	$4,000	.05	1 (annual)	8	$5,909.82
b.	$4,000	.05	4 (quarterly)	8	$5,952.52

TABLE 4.8: (continued)

	Principal	r	n	t	A
c.	\$4,000	.05	12 (monthly)	8	\$5,962.34
d.	\$4,000	.05	365 (daily)	8	\$5,967.14
e.	\$4,000	.05	8,760 (hourly)	8	\$5,967.29

A graph of the function $f(x) = 4{,}000\left(1+\frac{.05}{x}\right)^{8x}$, where x represents the number of compounding periods is shown below. The graph indicates that the function has a horizontal asymptote at approximately \$5,968.



This means that for the investment of \$4,000, at 5% interest for 8 years, compounding more and more frequently will never result in more than about \$5,968.

Example 6

Determine the value of each investment:

a) \$5,000 is invested in an account that pays 6% interest, compounded monthly. What is the value of the investment after 10 years?

Solution:

\$5,000, invested for 10 years at 6% interest, compounded monthly.

$$A(10) = 5{,}000\left(1+\frac{.06}{12}\right)^{12\cdot 10}$$
$$A(10) = 5{,}000(1.005)^{120}$$
$$A(10) = \$9{,}096.98$$

Thus, the initial \$5,000 investment is worth \$9,096.98 after 10 years.

b) \$6,000 is invested in an account that pays 2.5% interest, compounded quarterly. What is the value of the investment after 10 years?

Solution:

\$6,000, invested for 10 years at 2.5% interest, compounded quarterly.

Quarterly compounding means that interest is compounded 4 times per year. So, in the equation, $n = 4$.

$$\begin{aligned}
A(10) &= 6{,}000\left(1+\frac{.025}{4}\right)^{4\cdot 10}\\
A(10) &= 6{,}000(1.00625)^{40}\\
A(10) &= \$7,698.16
\end{aligned}$$

Thus, the initial \$6,000 investment is worth \$7,698.16 after 10 years.

Example 7

How long does it take for a \$2,000 investment, which earns 5% interest compounded continuously to grow to \$25,000?

Solution:

$$\begin{aligned}
25,000 &= 2{,}000e^{0.05t}\\
12.5 &= e^{0.05t}\\
\ln 12.5 &= \ln\left(e^{0.05t}\right)\\
\ln 12.5 &= 0.05t\\
t &= \frac{\ln 12.5}{0.05} \approx 50.5
\end{aligned}$$

Thus, it takes approximately 50.5 years for the investment of \$2,000 to be worth \$25,000.

Summary

- Compound interest formula: $A(t) = p \cdot \left(1+\frac{r}{n}\right)^{nt}$
- Continuously compounded interest: $A(t) = p \cdot e^{rt}$

Review

1. What is the formula for determining the value of an investment with compound interest?
2. If someone invested \$4,500, how much would the person have earned after 4 years at a compounded quarterly interest rate of 2%?
3. Kyle opened up a savings account in July. He deposited \$900. The bank pays a compounded monthly interest rate of 5% annually. What is Kyle's balance at the end of 4 years?
4. After having an account for 6 years, how much money does Roberta have in the account if her original deposit was \$11,000, and her semiannually (twice a year) compound interest rate is 8.4%?
5. Tom called his bank today to check on his savings account balance. He was surprised to learn that he has balance of \$6,600. He opened his account 8 years old with \$5,000, and his interest is compounded monthly. Based on this data, what percent interest rate has the bank been paying on the account?
6. Julie opened a 4% interest account with a bank that compounds the interest quarterly. If she deposited \$3,000 into the account at the beginning of the year, how much could she expect to have at the end of the year?

7. What is the balance on a deposit of $818, earning 5% interest compounded semiannually (twice a year) for 5 years?
8. Karen made a decent investment. After 4 years she had $3,250 in her account, and expects to have $16,250 after another 4 years with no additional deposit. Her savings account is a compounding interest account. How much was her original deposit?
9. Write an expression that correctly represents the balance on an account after 7 years, if the account was compounded yearly at a rate of 5%, with an initial balance of $1,000.
10. Kathy receives an inheritance check for $3,000 and decides to put it in a savings account so she can send her daughter to college when she gets older. Kathy finds an account that pays compounding interest annually at a rate of 14%. The balance on the account can be represented by a function, where x is the time in years. Write a function, and then use it to determine how much money will be in the account at the end of 7 years.
11. Stan is late on his car payment. The finance company charges 3% interest per month it is late. His monthly payment is $300. What is the total amount he will owe if he pays the August 1 bill on October 1? (Assume he paid his September bill on time.)

Today, you get your 1st credit card. It charges 12.49% interest on all purchases, and compounds that interest monthly. Within one day you max out the credit limit of $1,200.

12. If you pay the monthly accrued interest plus $50 towards the initial $1,200 amount every month, how much will you still owe at the end of the 1st 12 months?
13. How much will you have paid in total at the end of the year?

You are preparing for retirement. You invest $10,000 for 5 years in an account that compounds monthly at 12% per year. However, unless this money is in an IRA or other tax-free vehicle with zero inflation, you also have an annual tax payment of 30% on the earned interest.

14. How much will you have in 5 years?
15. Now, take into account that the money loses 3% spending value per year due to inflation. How much is what you have saved really worth at the end of the 5 years?

Review (Answers)

Please see the Appendix.

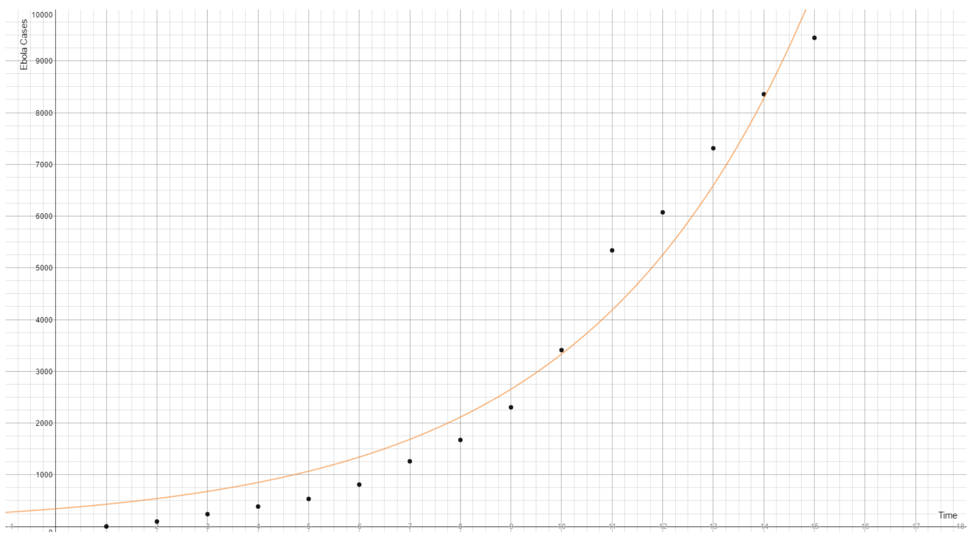
4.8 Population Growth Models and Logistic Functions

Learning Objectives

Learn to determine and interpret logistic models.

Introduction

In May of 2014, the 1st cases of the ebola virus were detected in Sierra Leone. The following graph shows the number of cases detected every other week from May 15, 2014, through December 31, 2014:



The mathematical model for the growth of the virus for this data is $P(t) = 343.001e^{0.22739t}$, where t is the number of 2-week periods since May 13, 2014, and $P(t)$ is the number of diagnosed cases of ebola in Sierra Leone. This model describes the population of the occurrences of ebola virus cases in that area.

Population Growth Model

Populations grow, at least for a short period of time, according to an exponential model. These populations can be of cells, animals, data, people, viruses, or anything that tends to grow at a fairly constant rate. The population growth model is $P(t) = P_0e^{rt}$, where P_0 is the initial population when $t = 0$, r is the growth rate, and t is the time since the population was detected.

Population Growth Model

$$P(t) = P_0e^{rt}$$

Exponential growth increases without bound, which is reasonable in the short run. However, populations usually have some type of upper bound. This can be caused by limitations on food, space, or other scarce resources. The logistic model describes population growth that is limited by its environment.

The Logistic Model

Logistic Equation

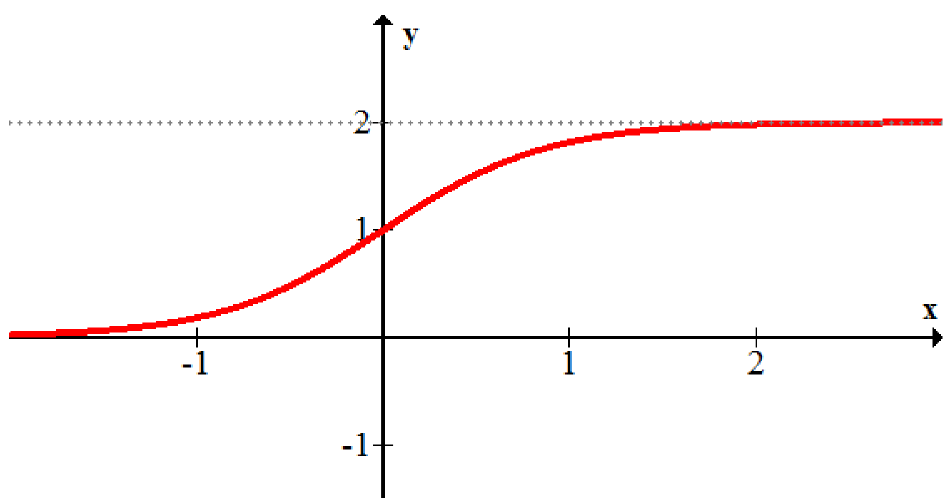
$$f(x) = \frac{c}{1 + a \cdot b^x}$$

where a, b, and c are constants and $0 < 1$.

In the logistic equation, the letters a, b, and c are constants that can be changed to match the situation being modeled. The constant c is particularly important because it is the limit of the growth, or the **carrying capacity**.

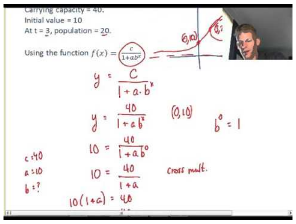
The following logistic function has a carrying capacity of 2 because $c = 2$. When graphing the function, a horizontal asymptote occurs at the horizontal line of carrying capacity, $y = 2$. Another horizontal asymptote occurs at the x-axis. If the logistic function has a vertical shift, then both horizontal asymptotes will be vertically shifted as well.

$$f(x) = \frac{2}{1 + 0.1^x}$$



An important note about the logistic function is its inflection point. Observe that at the point $(0,1)$,the graph transitions from curving up (concave up) to curving down (concave down). The inflection point occurs halfway between the two horizontal asymptotes. In this example, the inflection point occurs halfway between the carrying capacity and the x-axis.

The logistic equation can be determined from given conditions, such as the carrying capacity, the initial value, and a population at a certain time. Remember that the carrying capacity is c in the equation. The initial value is the function value when $x = 0$. When the initial value and carrying capacity are inserted into the logistic equation, a can be solved for easily because $b^0 = 1$. Once a and c are determined, b can be calculated by inserting the given population and time into the logistic equation with a and c. A specific example can be seen in the following video:



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Examples

Example 1

A rumor is spreading in an office that has 1,200 employees. Four people know the rumor when it starts, and three days later, 300 have heard the rumor. About how many people in the office know the rumor by the 4th day?

Solution:

Step 1: In a limited population, the count of people who know a rumor is an example of a situation that can be modeled using the logistic function. The population is 1,200 so this will be the carrying capacity.

Parameters to fit the model: $c = 1{,}200; (0,4); (3,300)$

Step 2: Evaluate at the point (0, 4) to solve for a.

$$\begin{aligned}\frac{1{,}200}{1+a\cdot b^0} &= 4\\ \frac{1{,}200}{1+a} &= 4\\ \frac{1{,}200}{4} &= 1+a\\ a &= 299\end{aligned}$$

Step 3: Evaluate at the point (3, 300) to solve for b.

$$\begin{aligned}\frac{1{,}200}{1+299\cdot b^3} &= 300\\ 4 &= 1+299\,b^3\\ \frac{3}{299} &= b^3\\ 0.21568 &\approx b\end{aligned}$$

The modeling equation at $x = 4$:

$$f(x) = \frac{1{,}200}{1+299\cdot 0.21568^x} \rightarrow f(4) \approx 729 \text{ people}$$

A similar growth pattern will exist with any kind of infectious disease that spreads quickly and can only infect a person or animal once.

Example 2

Long Island has roughly 8 million people. A hundred years ago, it had 2 million people. Suppose that the resources and infrastructure of the island could only support 20 million people. When will the population reach 10 million inhabitants?

Solution:

Step 1: Identify known points and the carrying capacity. (0, 8,000,000) and (-100, 2,000,000). $c = 20{,}000{,}000$. Use the 1st point to solve for a.

$$\begin{aligned} 8{,}000{,}000 &= \frac{20{,}000{,}000}{1+a\cdot b^0} \\ 8{,}000{,}000 &= \frac{20{,}000{,}000}{1+a} \\ 1+a &= \frac{20{,}000{,}000}{8{,}000{,}000} = 2.5 \\ a &= 1.5 \end{aligned}$$

Step 2: Evaluate at the 2nd point to solve for b.

$$\begin{aligned} 2{,}000{,}000 &= \frac{20{,}000{,}000}{1+1.5\cdot b^{-100}} \\ 1+1.5\cdot b^{-100} &= 10 \\ b^{-100} &= \frac{9}{1.5} \\ \frac{1}{b^{100}} &= 6 \\ \frac{1}{6} &= b^{100} \\ b &\approx 0.98224 \end{aligned}$$

Step 3: The question asks for the x value when $f(x) = 10{,}000{,}000$.

$$\begin{aligned} 10{,}000{,}000 &= \frac{20{,}000{,}000}{1+1.5\cdot(0.98224)^x} \\ 1+1.5\cdot(0.98224)^x &= 2 \\ (0.98224)^x &= \frac{2}{3} \\ x\cdot\ln(0.98224) &= \ln\left(\frac{2}{3}\right) \\ x &= \frac{\ln\left(\frac{2}{3}\right)}{\ln(0.98224)} \approx 22.63 \end{aligned}$$

This means that the predicted time from now that the population of Long Island will reach 10 million inhabitants is about 22.6 years.

Example 3

A special kind of algae is grown in giant clear plastic tanks and can be harvested to make biofuel. The algae are given plenty of food, water, and sunlight to grow rapidly, and the only limiting resource is space in the tank. The algae are harvested when 95% of the tank is full, leaving the tank 5% full of algae to reproduce and refill the tank. Currently the time between harvests is 20 days, and the payoff is 90% harvest. Would you recommend a more optimal harvest schedule?

Solution:

Identify known quantities and model the growth of the algae.

Known quantities: $(0,0.05);(20,0.95);c=1 \textit{ or } 100\%$

$$\begin{aligned} 0.05 &= \frac{1}{1+a\cdot b^0} \\ 1+a &= \frac{1}{0.05} \\ a &= 19 \end{aligned}$$

$$\begin{aligned} 0.95 &= \frac{1}{1+19\cdot b^{20}} \\ 1+19\cdot b^{20} &= \frac{1}{0.95} \\ b^{20} &= \frac{\left(\frac{1}{0.95}-1\right)}{19} \\ b &\approx 0.74495 \end{aligned}$$

The model for the algae growth is

$$f(x)=\frac{1}{1+19\cdot(0.74495)^x}.$$

The question asks about optimal harvest schedule. Currently the harvest is 90% per 20 days, or a unit rate of 4.5% per day. Shortening the time between harvests where the algae are growing the most efficiently will potentially increase this unit rate. Suppose 15% of the algae is left in the tank and harvested when it reaches 85%. How much time will that take to yield 70%?

$$\begin{aligned} 0.15 &= \frac{1}{1+19\cdot(0.74495)^x} \\ x_1 &\approx 4.10897 \\ 0.85 &= \frac{1}{1+19\cdot(0.74495)^x} \\ x_2 &\approx 15.8914 \\ 15.8914-4.10897 &\approx 11.78243 \end{aligned}$$

It takes about 12 days for the batches to yield 70% harvest, which is a unit rate of about 6% per day. This is a significant increase in efficiency. A harvest schedule that maximizes the time where the logistic curve is steepest creates the fastest overall algae growth.

Example 4

Given the following logistic model, predict the x value that will produce a height of 14 and then predict the height when x is 4.

$$f(x) = \frac{20}{1+4\cdot(0.9)^x}$$

Solution:

Step 1: Solve for x with a known height of 14.

$$\begin{aligned} 14 &= \frac{20}{1+4\cdot(0.9)^x} \\ 1+4\cdot(0.9)^x &= \frac{20}{14} \\ (0.9)^x &= \frac{\left(\frac{20}{14}-1\right)}{4} \\ x &= \log_{0.9}\left(\frac{\left(\frac{20}{14}-1\right)}{4}\right) = \frac{\ln\left(\frac{\left(\frac{20}{14}-1\right)}{4}\right)}{\ln 0.9} \approx 21.1995 \end{aligned}$$

Step 2: Substitute for $x = 4$.

$$f(x) = \frac{20}{1+4\cdot(0.9)^4} = 5.51815$$

Example 5

Determine the logistic model given $c = 12$ and the points (0, 9) and (1, 11).

Solution:

Step 1: Use the given conditions to solve for a and b.

$$\begin{aligned} 9 &= \frac{12}{1+a\cdot b^0} \\ 1+a &= \frac{12}{9} \\ a &= \frac{1}{3} \\ 11 &= \frac{12}{1+\left(\frac{1}{3}\right)\cdot b^1} \\ 1+\left(\frac{1}{3}\right)\cdot b &= \frac{12}{11} \\ b &= 0.\overline{27} = \frac{3}{11} \end{aligned}$$

Step 2: Complete the logistic equation.

$$f(x) = \frac{12}{1+\left(\frac{1}{3}\right)\cdot\left(\frac{3}{11}\right)^x}$$

Example 6

Determine the logistic model given $c = 7$ and the points (0, 2) and (3, 5).

Solution:

Step 1: Use the given conditions to solve for a and b.

$$\begin{aligned} 2 &= \frac{7}{1+a\cdot b^0} \\ 1+a &= \frac{7}{2} \\ a &= 2.5 \end{aligned}$$

$$\begin{aligned} 5 &= \frac{7}{1+(2.5)\cdot b^3} \\ 1+(2.5)\cdot b^3 &= \frac{7}{5} \\ b^3 &= 0.16 \\ b &\approx 0.5429 \end{aligned}$$

Step 2: Complete the logistic equation.

$$f(x) = \frac{7}{1+(2.5)\cdot(0.5429)^x}$$

Summary

- The population growth model is $P(t) = P_0e^{rt}$.
- The logistic equation is $f(x) = \frac{c}{1+a\cdot b^x}$.
- The logistic equation can be determined when given the carrying capacity, the initial value, and a population at a certain time by solving for the constants a, b, and c.

Review

For 1-5, determine the logistic model given the carrying capacity and two points.

1. $c = 12; (0,5); (1,7)$

2. $c = 200; (0,150); (5,180)$

3. $c = 1{,}500; (0,150); (10,1{,}000)$

4. $c = 1{,}000{,}000; (0, 100{,}000); (-40, 20{,}000)$

5. $c = 30{,}000{,}000; (-60, 10{,}000); (0, 8{,}000{,}000)$

For 6-8, use the logistic function $f(x) = \frac{32}{1+3e^{-x}}$.

6. What is the carrying capacity of the function?

7. What is the y-intercept of the function?

8. Use your answers to 6 and 7 along with at least two points on the graph to make a sketch of the function.

For 9-11, use the logistic function $g(x) = \frac{25}{1+4 \cdot 0.2^x}$.

9. What is the carrying capacity of the function?

10. What is the y-intercept of the function?

11. Use your answers to 9 and 10 along with at least two points on the graph to make a sketch of the function.

For 12-14, use the logistic function $h(x) = \frac{4}{1+2 \cdot 0.68^x}$.

12. What is the carrying capacity of the function?

13. What is the y-intercept of the function?

14. Use your answers to 12 and 13 along with at least two points on the graph to make a sketch of the function.

15. **Bonus**: Give an example of a logistic function that is decreasing (models decaying). In general, how can you tell from the equation whether the logistic function is increasing or decreasing?

Review (Answers)

Please see the Appendix.

4.9 Project: Exponential and Logarithmic Functions

Hurricane Fran hit North Carolina on the evening of September 5, 1996. Over 1 million homes and businesses were left without power. Repair crews began immediately restoring electrical service. The following table lists the number of customers without power:

TABLE 4.9:

Date	Customers without power
September 6	1,159,000
September 7	804,000
September 8	515,000
September 9	340,000
September 10	195,200
September 11	136,300
September 12	77,000
September 13	37,600

1. Use graph paper to plot the data points, using a clear scale. Sketch a best guess at an appropriate model. Several points may be off the curve, so just try to balance the number above and below the curve.
2. What kind of function seems to fit this curve?
3. What would the equation of the parent function (model) be?
4. Pick two points that are on the sketched curve, and come up with a specific function that fits it.
5. Choose a point that is not on the sketched curve. Calculate its value predicted by the model. What is the percentage error between the predicted and the actual value?
6. Is it appropriate to discuss half-life or doubling time with this data? Explain. Then find the value of the appropriate one.
7. If the recovery continued at the same pace, how long until all power was restored (less than 1,000 customers without power)?
8. Find the function whose domain is number of customers without power and range is the date when that power is restored. What is its relationship to the function created in question 4?

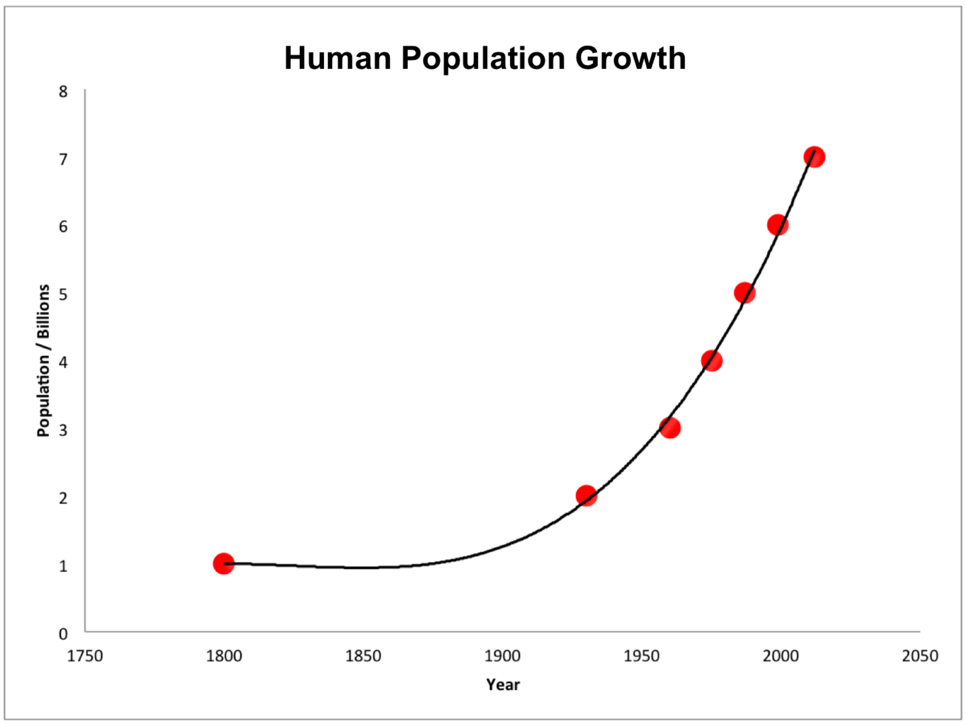
Reference:

"Hurricane Fran: September 5, 1996." *Hurricane Fran: September 5, 1996*. National Weather Service, n.d. Web. 01 Apr. 2016.

4.10 Summary: Exponential and Logarithmic Functions

Chapter Summary

Exponential and logarithmic functions help model many real-world situations, such as population growth and compounded interest. For example, a small town's population is given by $P(t) = 2{,}000 \cdot (1.5)^t$, where t is the number of years since 1950. Populations rarely grow in an unbounded fashion in the long run. The logistic model improves on exponential growth models for real-world application.



Furthermore, nearly all investment accounts use compounded interest, which is determined by the exponential function $A(t) = p \cdot \left(1 + \frac{r}{n}\right)^{nt}$.

Using the rules of exponents and properties of logarithms and solving exponential and logarithmic equations make it possible to solve these sorts of problems.

To solve an exponential equation:

- Isolate the exponential part of the equation. If there are two exponential parts, then rewrite so there is an exponential on each side of the equation.
- Take the logarithm of each side of the equation.
- Solve for the variable.
- Check your solution.

To solve logarithmic equations, use the following:

- One-to-one properties:
  - $A = B$ if and only if $b^A = b^B$
  - $A = B$ if and only if $\log_b A = \log_b B$.

- Rewrite the equation as an exponential equation using $\log_b x = y \longleftrightarrow b^y = x$.

Review

Try the following cumulative review problems to practice the concepts in this chapter:

Multimedia

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/195399

4.11 References

1. John Linnell. https://commons.wikimedia.org/wiki/File:Thomas_Robert_Malthus.jpg .
2. Paula Evans. King's Choice of Payment .
3. . Exponential Function Graph – Base 2.
4. . Strech and Shift Graph.
5. . Exponential Growth Function.
6. . Exponential Decay Function.
7. . Solve Equations Graphically.
8. . Exponential Function Base e.
9. . Compare base e to bases 2 and 3.
10. . Exponential Function Graphs – Comparing Bases.
11. . Shift Graph.
12. . Reflect and shift Graph.
13. CK-12. Example 3 .
14. U.S. Navy photo by Jim Brooks. https://commons.wikimedia.org/wiki/File:US_Navy_050709-N-0000B-001_The_Gulf_of_Mexico_spills_over_an_oceanfront_road_at_Naval_Air_Station_Key_West%5Ersquo,s_Truman_Annex_as_Hurricane_Dennis_passes_the_southernmost_tip_of_the_base_Saturday_morning.jpg .
15. CK-12. Graph Log Base Growth .
16. CK-12. Graph Log Decay .
17. . Example 1.
18. . Example 2.
19. . Example 3.
20. . Example 4.
21. Unknown. https://commons.wikimedia.org/wiki/File:Adelaide_town_hall_1950.jpg .
22. reynermedia. https://www.flickr.com/photos/89228431@N06/11322953266 .
23. CK-12. Example 2 Graph .
24. CK-12. Example 5 Graph .
25. Paula Evans. Spread of Ebola Virus .
26. CK-12. Logistic Function .

CHAPTER 5 Angles

Chapter Outline

5.1 Introduction: Angles

The photo above shows a window of a building on the campus of Princeton University in Princeton, New Jersey. Angles of rotation, arc length, and area of a sector can be used to create such a window. More specifically, what angle should be used to space out the dot design of the window? Will this angle be measured in degrees or radians? What is the length of the arc between certain dots? In this chapter, we will discuss these types of questions as you learn about angles of rotation, radians, arc length, area sector, and angular speed.

5.2 Angles of Rotation in Standard Positions

Learning Objectives

Learn how to express angles of rotation.

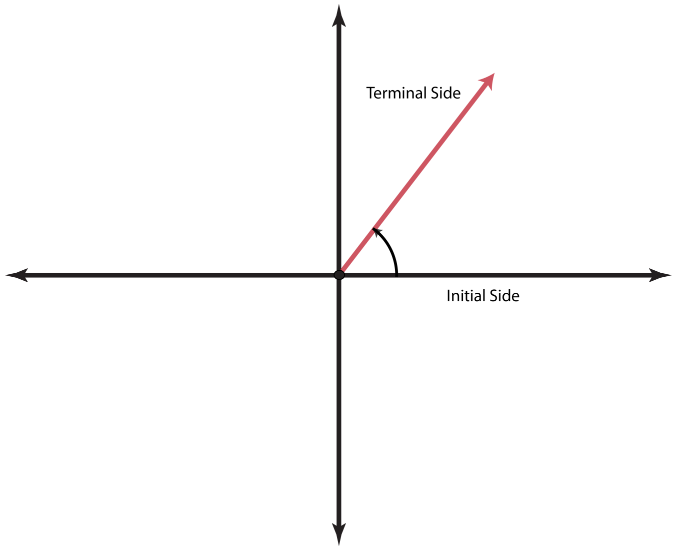
Introduction

A California vegetable grower is considering new options to irrigate crops. The grower needs to find a method that is both sustainable and economical. One possible choice is the center pivot irrigation system, a method widely used in the Great Plains region of the United States. The grower needs to consider how to convert the current drip irrigation system to a center pivot system.

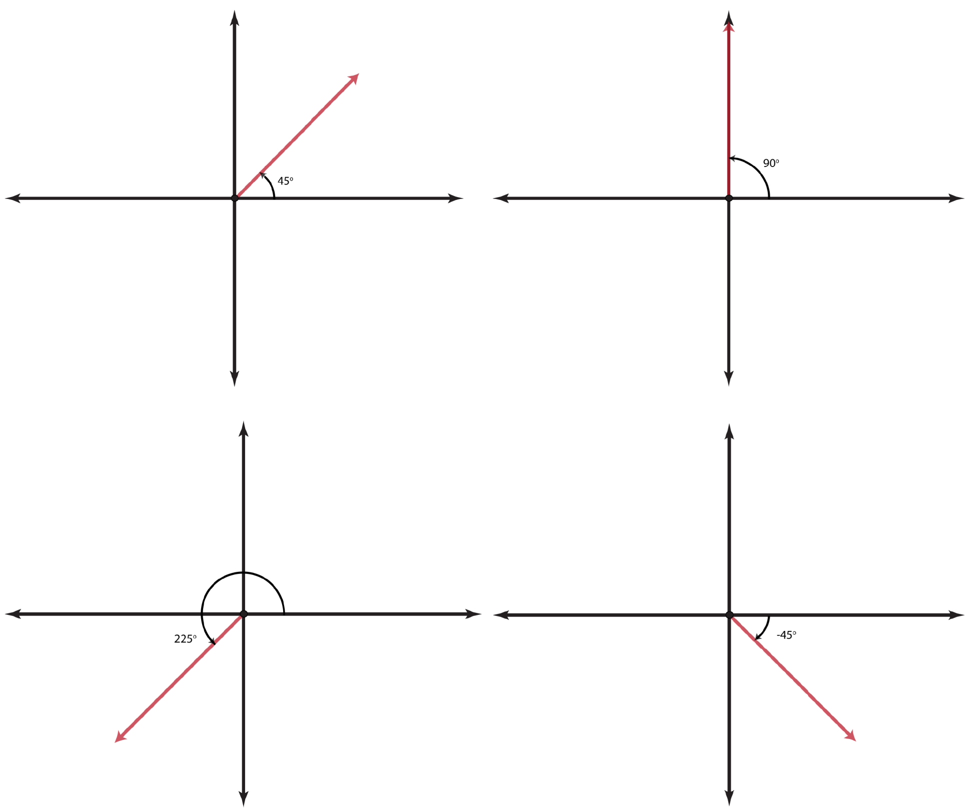
To do this, calculations for the area of the fields to be irrigated have to be converted so that the length of the irrigation pipe and the angle that it is rotated are used. Until now, the grower only used area measurements that considered linear dimensions, such as length and width. To calculate the area given the length of the irrigation pipe, the grower first has to understand angles and how they are measured.

Angles of Rotation in Standard Position

Rotation angles can be measured in degrees. You may recall from geometry that one full rotation is 360 degrees, usually written as 360°. Half a rotation is 180°, and a quarter rotation is 90°. As linear quantities x and y are represented in a standard way graphically, a standard is needed for angular measure. The figure below shows an angle in what is called **standard position**.



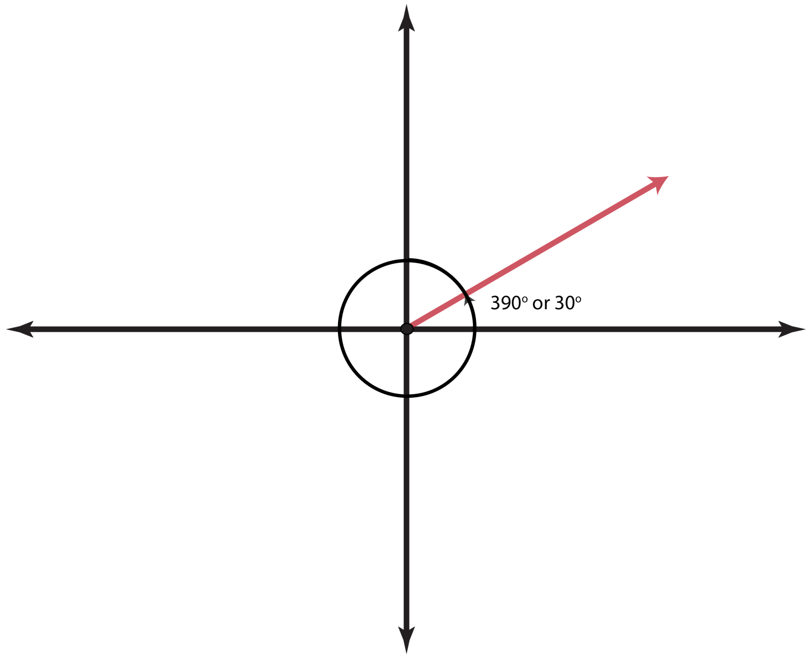
The initial side of an angle in standard position is always on the positive $x-$axis. The terminal side always meets the initial side at the origin. Notice that the rotation goes in a counterclockwise direction. This means that if the angle measures a counterclockwise rotation, the angle has a positive measure. A clockwise rotation corresponds to a negative measure. Below are several examples of angles in standard position:

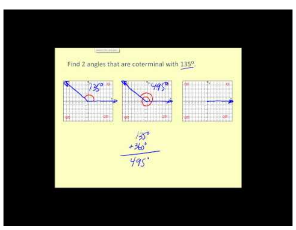


Learn, Play, and Explore with Angles of Rotation: Clock Angles

Coterminal Angles

In the example below, the angle whose measure is 30° could also be expressed as 390° or even −330°. These three measures have the same initial side and the same terminal side, so they represent the same angle. They are called **coterminal angles**.





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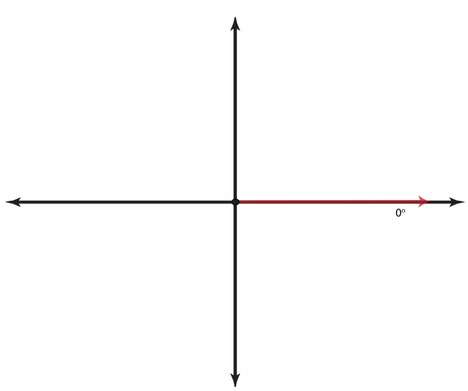
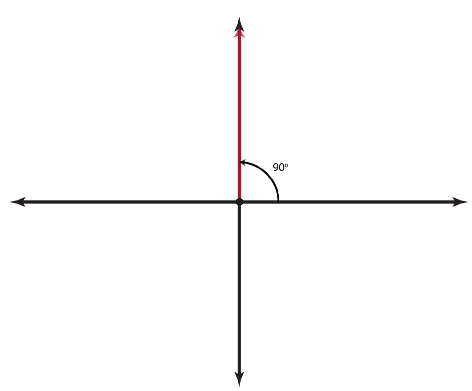
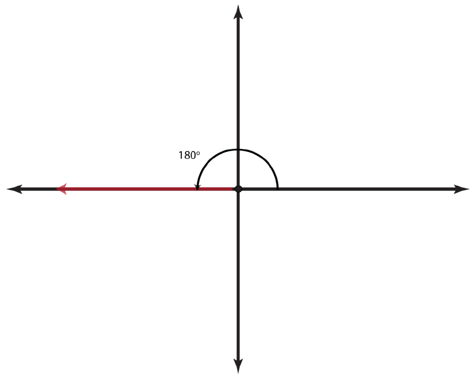
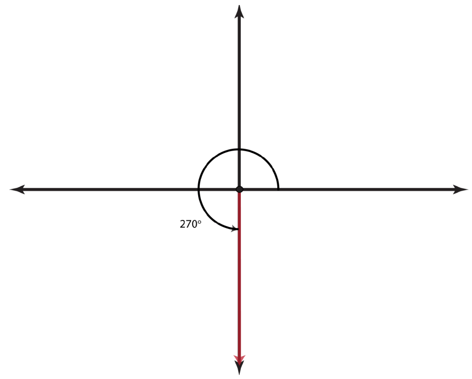
Quadrantal Angles

The 90-degree angle is one of four quadrantal angles. A **quadrantal angle** is one whose terminal side lies on an axis. Quadrantal angles are 0°, 90°, 180°, and 270°.







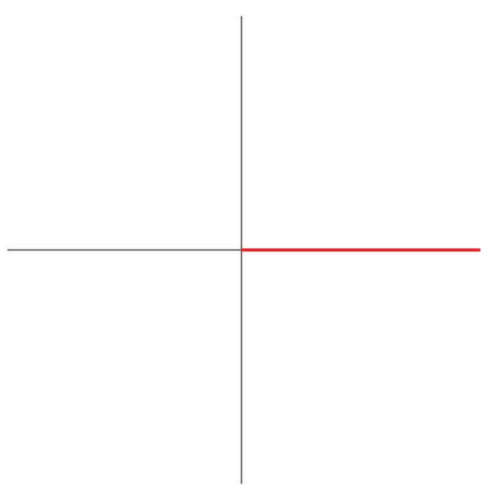


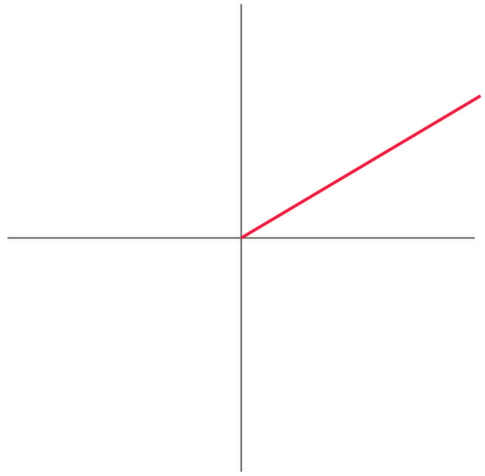
These angles are referred to as quadrantal because each angle defines a quadrant. Notice that, without the arrow indicating the rotation, 270° looks as if it is a $-90°$, defining the 4th quadrant. Notice also that 360° would look just like 0°.

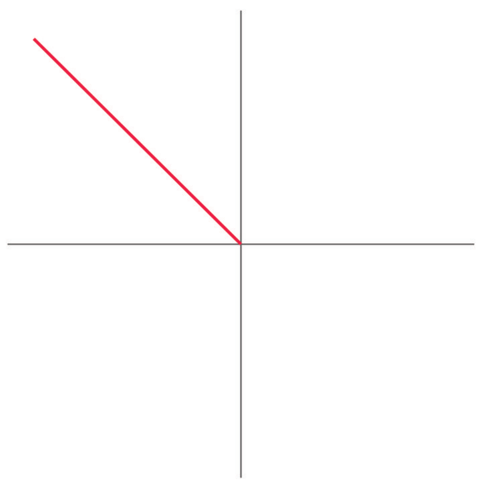
Examples

Example 1

What are the angles in standard position represented by these graphs?





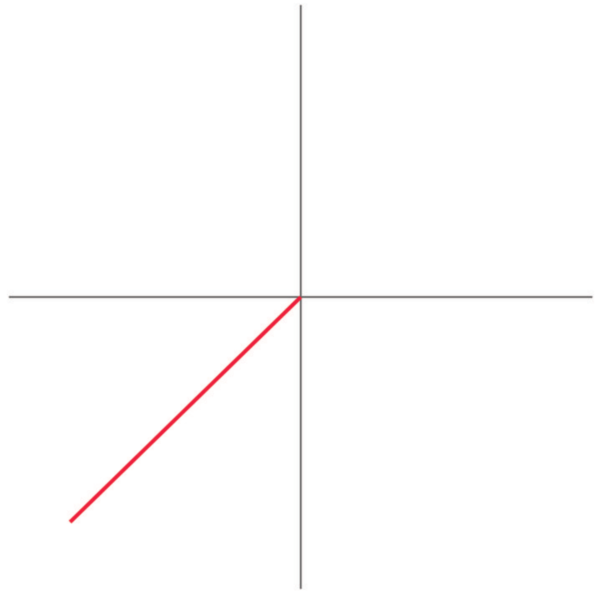


Solution:

The angles represented are 0°, 30° and 135°.

Example 2

Identify the measure of the angle represented below. Use a negative measure.

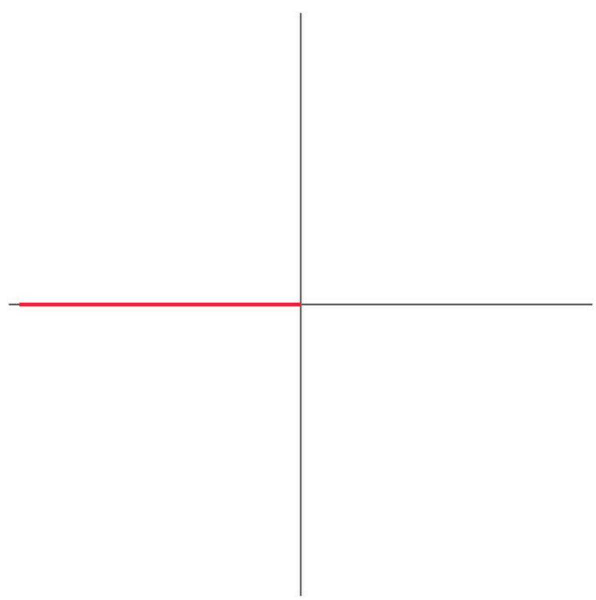


Solution:

The angle's measure is -135°.

Example 3

Identify the measure of the angle represented below. Use a positive and a negative measure.

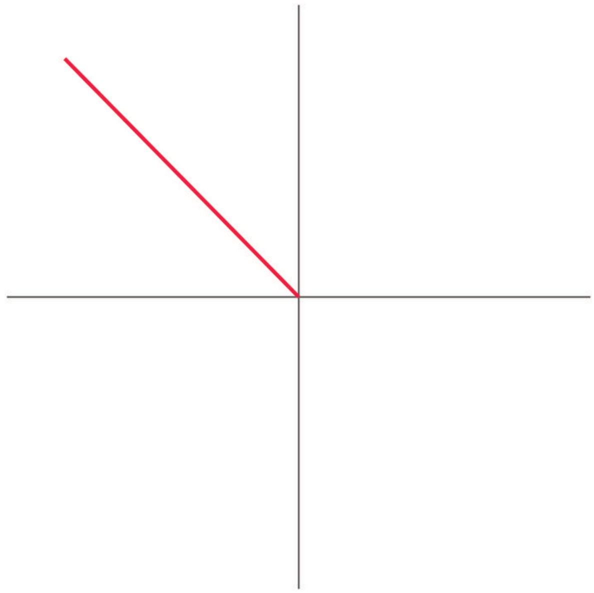


Solution:

The angle is 180° or -180° .

Example 4

Identify what the angle is in this graph, using negative angles:



Solution:

The angle's measure is $-225°$.

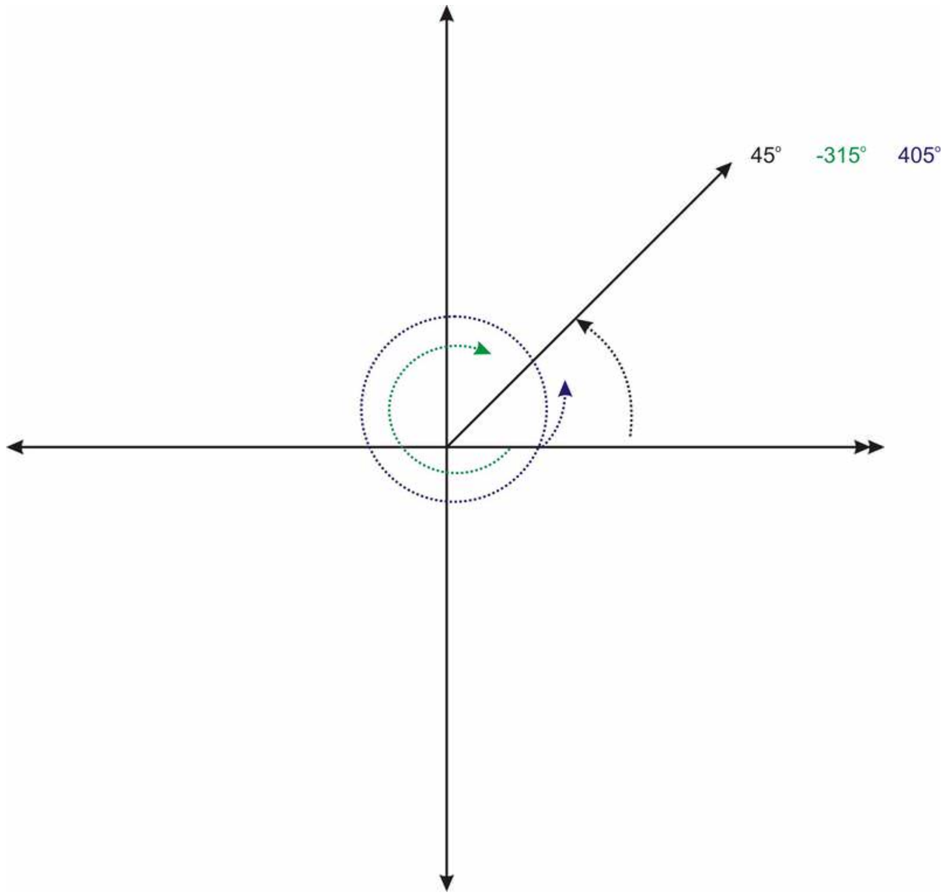
Example 5

Which of the following angles are coterminal with $45°$?

-45°, 405°, -315°, 135°

Solution:

405° and -315° are coterminal with 45°.



Summary

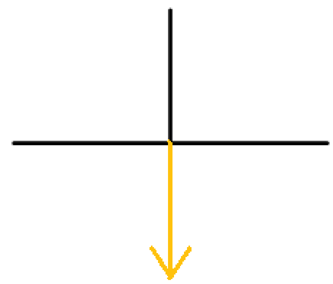
- Angles in standard position have their initial side on the positive $x-$axis. The positive direction for the angle is counterclockwise. The angle ends at its terminal side.
- Coterminal angles are angles that can be represented in many ways but are equivalent angles, because they have the same initial side and terminal side.
- Quadrantal angles have their terminal side on one of the axes.

Review

1. Draw an angle with measure $90°$.
2. Draw an angle with measure $45°$.
3. Draw an angle with measure $-135°$.
4. Draw an angle with measure $-45°$.
5. Draw an angle with measure $-270°$.
6. Draw an angle with measure $315°$.

For each diagram, identify the angle's positive measure:

7.



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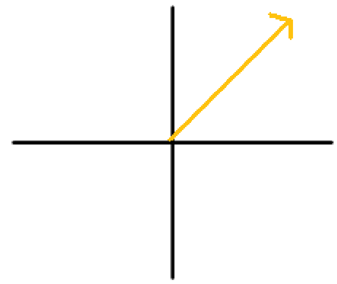

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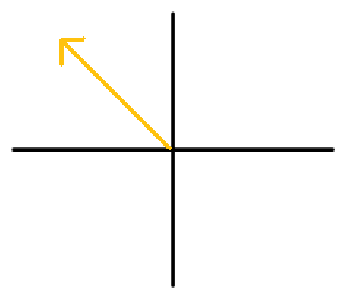
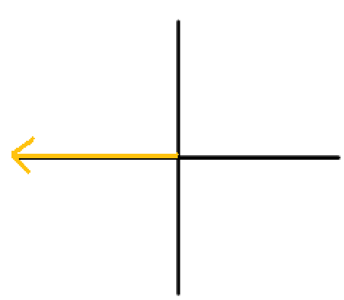


For each diagram below, identify the angle. Write the angle using negative degrees.

10.



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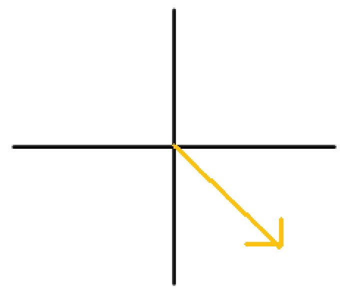

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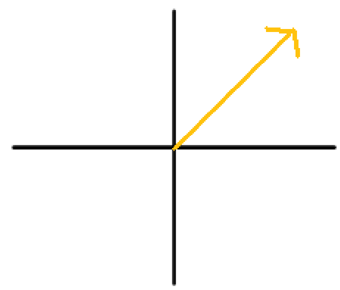


13. Which standard positive angle has a terminal angle that includes 7 on a standard 12-hour clock? Use positive degrees.

14. Which standard positive angle has a terminal angle that includes 2 on a standard 12-hour clock? Use positive degrees.

Review (Answers)

Please see the Appendix. 30°

5.3 Angles in Radians and Degrees

Learning Objectives

Learn to convert between degrees and radians.

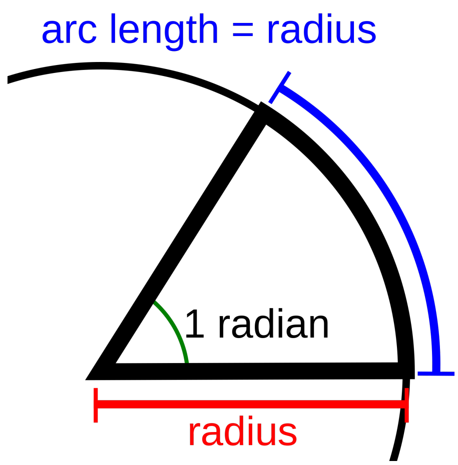
Introduction

A California vegetable grower is considering new options to irrigate crops. He needs to find a method that is both sustainable and economical. One possible choice is the center pivot irrigation system, a method widely used in the Great Plains region of the United States. The grower needs to learn how to convert the current drip irrigation system to a center pivot system. He needs critical information to answer his question about how to calculate the area of the fields to be irrigated with a center pivot irrigation system.

When setting the mechanism that turns the 400-foot irrigation pipe, the grower can set the rotation angle and speed. The system is set to start in a direction that is due east and rotate in a counterclockwise fashion, so the angle of rotation is in standard position. The grower found a formula on the internet to calculate the irrigated area, but the formula did not make sense. The formula requires that when he enters the rotation angle, he must first multiply it by 0.0175. Now the grower has more questions: "Where did 0.0175 come from? How is this formula created? Why must I enter the angle in this way?"

Definition of Radian Measure

A radian is a unit of measuring angles that is based on the properties of circles. One **radian angle** is defined to be the central angle where the subtended arc length, which is the part of the circle in between the two rays that make the central angle, is the same length as the radius.



Another way to think about radians is through the circumference of a circle. The circumference of a circle with radius r is $2\pi r$. Just over 6 radii (exactly 2π radii) would stretch around any circle.

Conversions between Degrees and Radians

Note that $360^\circ = 2\pi$ radians, so $\frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi}$ is the proportion used to convert from radians to degrees.

Conversions from Radians to Degrees

$$\theta \text{ radians} \cdot \frac{180^\circ}{\pi \text{ radians}} = \theta^\circ$$

Similarly, $360^\circ = 2\pi$ radians, so $\frac{2\pi}{360^\circ} = \frac{\pi}{180^\circ}$ is the proportion used to convert from degrees to radians.

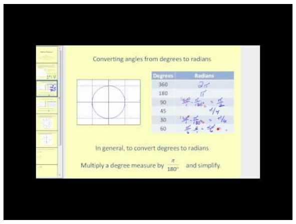
Conversion from Degrees to Radians

$$\theta^\circ \cdot \frac{\pi \text{ radians}}{180^\circ} = \theta \text{ radians}$$

Thus, the conversion factor to convert from degrees to radians is $\frac{\pi}{180^\circ}$ and from radians to degrees is $\frac{180^\circ}{\pi}$.

Learn, Play, and Explore with Radians: Radian Measure

Note that if an angle has no units listed, it is assumed to be in radians.



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Learn, Play, and Explore with Hours in the Day: Hours in theDay

Examples

Example 1

Convert 150° into radians.

Solution:

$$150^\circ \cdot \frac{\pi}{180^\circ} = \frac{15\pi}{18} = \frac{5\pi}{6} \textbf{ radians}$$

Note: Make sure the degree units cancel.

Example 2

Convert $\frac{\pi}{6}$ radians into degrees.

Solution:

$$\frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{180^\circ}{6} = 30^\circ$$

Note: Often the πs will cancel.

Example 3

Estimate the angle measure $(6\pi)^\circ$ by substituting 3.14 for π.

Solution:

$$6(3.14) = 19^\circ$$

Example 4

Convert $(6\pi)^\circ$ into radians.

Solution:

$$(6\pi)^\circ \cdot \frac{\pi}{180^\circ} = \frac{6\pi^2}{180} = \frac{\pi^2}{30} \approx 0.329 \text{ radians}$$

Example 5

Return to the problem from the Introduction of this section. The grower had a few questions: "Where did 0.0175 come from? How is this formula created? Why must I enter the angle in this way?"

Solution:

If the angle of rotation is in degrees, the conversion to radians would be done by multiplying by $\frac{\pi}{180} \approx 0.0175$. The formula found on the internet must have required that the angle be converted to radians.

Summary

- A ***radian*** is defined as the central angle where the subtended arc length is the same length as the radius.
- The conversion factor to convert from degrees to radians is $\frac{\pi}{180^\circ}$.
- The conversion factor to convert from radians to degrees is $\frac{180^\circ}{\pi}$.

Review

Find the radian measure of each angle:

1. 120°

2. 300°

3. 90°

4. 330°

5. 270°

6. 45°

7. $(5\pi)^\circ$

Find the degree measure of each angle:

8. $\frac{7\pi}{6}$

9. $\frac{5\pi}{4}$

10. $\frac{3\pi}{2}$

11. $\frac{5\pi}{3}$

12. π

13. $\frac{\pi}{6}$

14. 3

15. Explain why, if you are given an angle in degrees and you multiply it by $\frac{\pi}{180}$, you will get the same angle in radians.

16. Suppose you are working on a science lab in which your professor asks you to turn a knob 75° on the detector you are using. Unfortunately, you have been working in radians for a while, so you are having trouble remembering how far to turn the knob. How would you translate the instructions in degrees to radians?

Review (Answers)

Please see the Appendix.

5.4 Length of an Arc

Learning Objectives

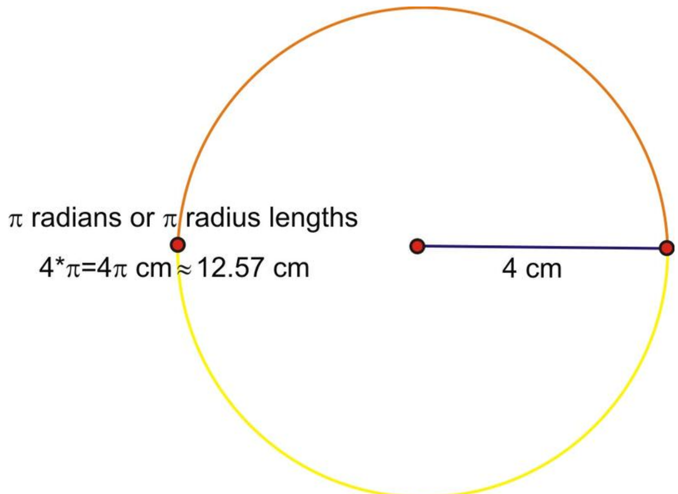
Learn how to find the length of a portion of the circumference of a circle using an angle in radians and the radius of the circle.

Introduction

The central pivot irrigation system provides moisture for crops to a portion of a field by slowly rotating a long pipe that sprays water onto the land. The shape of the area irrigated is a portion of a circle. Its radius is the length of the main pipe that is slowly rotated. The grower needs to know the length of the circular arc that is drawn by the end of the irrigation pipe. For example, some days the pipe is only turned through an angle of 54°. If the pipe is 400 ft long, what is the length of the arc traced by the end of the pipe?

Arc Length

The length of an arc on a circle depends on both the angle of rotation and the radius length of the circle. The circumference of a circle $C = 2\pi r$ is related to arc length. This is the arc length for one rotation of $360°$ or 2π radians. If the radius is 4 cm, then the length of the half-circle arc would be 4π cm.

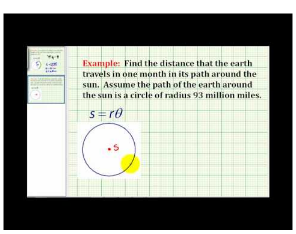


Extending this idea to an arbitrary angle θ inside the circle results in a formula that can be used to calculate the length of any arc.

Arc Length

$$s = r\theta,$$

where s is the length of the arc, r is the radius, and θ is the measure of the angle in radians.



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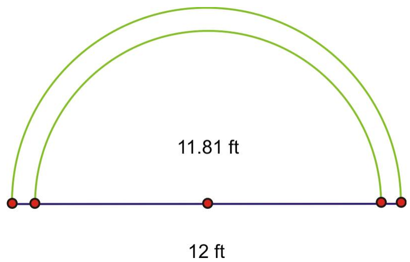
Examples

Example 1

The free-throw line on an NCAA basketball court is 12 ft wide. In international competition, it is only about 11.81 ft. How much longer is the half circle above the free-throw line on the NCAA court?



Solution:

Find both arc lengths.

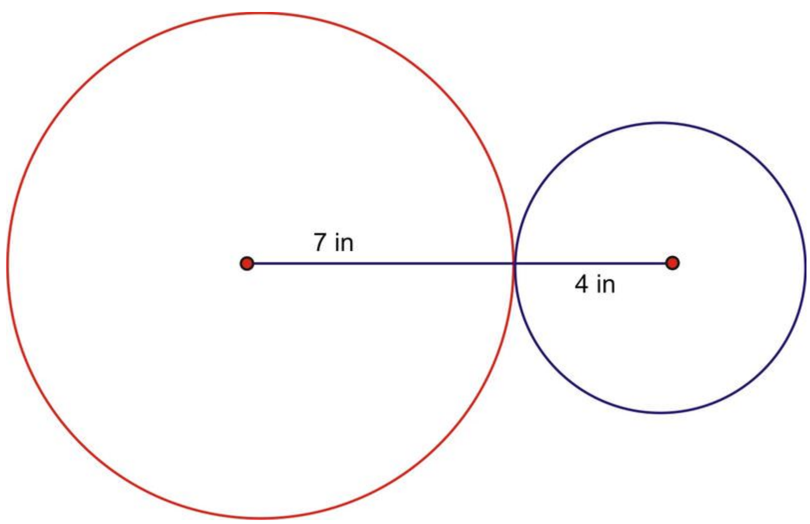
NCAA	INTERNATIONAL
$s_1 = r\theta$	$s_2 = r\theta$
$s_1 = \frac{12}{2}(\pi)$	$s_2 \approx \frac{11.81}{2}(\pi)$
$s_1 = 6\pi$	$s_2 \approx 5.905\pi$

So the answer is approximately $6\pi - 5.905\pi \approx 0.095\pi$.

This is approximately 0.3 ft, or about 3.6 in longer.

Example 2

Two connected gears are rotating. The smaller gear has a radius of 4 in and the larger gear's radius is 7 in. What is the angle through which the larger gear has rotated when the smaller gear has made one complete rotation?



Solution:

Step 1: Because the blue gear performs one complete rotation, the length of the arc traveled is calculated using the blue gear radius and the circumference of a whole circle:

$$s = r\theta$$
$$s = 4 \times 2\pi$$
$$s = 8\pi$$

Step 2: Use the 8π arc length and the radius of the larger gear to determine the angle formed on the larger circle:

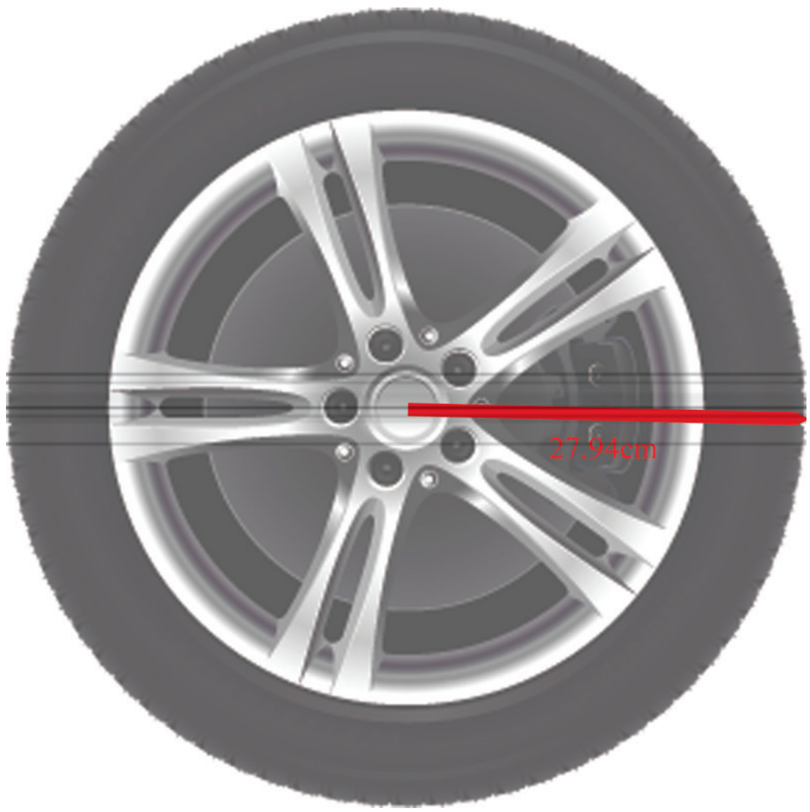
$$\theta = \frac{s}{r}$$
$$\theta = \frac{8\pi}{7}$$
$$\theta \approx 3.6 \text{ radians}$$

Step 3: The angle is approximately 3.6 radians. Convert this answer to degrees using the fact that π is equal to 180°.

$$3.6 \times \frac{180^\circ}{\pi} \approx 206^\circ$$

Example 3

The radius of a standard car tire is 27.94 cm. How far does a car go in one revolution of the tire?



Solution:

Since the distance traveled by the tire is equal to the distance around the tire, we can use the circumference of the tire to answer the question:

$$s = r\theta$$
$$s = (27.94 \text{ cm})(2\pi)$$
$$s = 175.55 \text{ cm}$$

Example 4

To return to the problem in the Introduction, the pipe is slowly rotated through an angle of 54°. If the pipe is 400 ft long, calculate the length of the arc traced by the end of the irrigation pipe.

Solution:

Step 1: Convert the angle from degrees to radians.

$$\theta = 54^\circ \cdot \frac{\pi}{180^\circ} = 0.3\pi = 0.942 \text{ radians}$$

Step 2: Calculate the arc length.

$$s = 400 \cdot 0.3\pi \text{ radians} = 120\pi \approx 377 \text{ ft}$$

Example 5

A stranded motorist is trying to push a car after it has broken down. If the car simply rocks back and forth rather than move forward, how far did the tire move each time the car rocked? If the radius of the car's tire is 14 in, and the change in the tire's angle is $\frac{\pi}{2}$ radians, how far did the tire move?

Solution:

Since the distance the tire moved is equal to the length of the arc the tire rolled, use $s = r\theta$ to determine how far the tire went:

$$\begin{aligned}
s &= r\theta \\
s &= (14)(\frac{\pi}{2}) \\
s &= 7\pi \\
s &\approx 21.99 in
\end{aligned}$$

Example 6

If an object with a radius of 10 cm spins so that its arc covers 54 cm, what is the change in angle of the object?

Solution:

Use the equation $s = r\theta$ to solve this problem:

$$\theta = \frac{s}{r} = \frac{54}{10} = 5.4$$

The disk moves 5.4 radians, which is a little less than a complete rotation, since a complete rotation is approximately 6.28 radians.

Example 7

A DVD has a radius of 2.25 in, how far does a point on the disk spin if the player turns it $\frac{\pi}{2}$ radians?

Solution:

Using $s = r\theta$,

$$s = (2.25)(\frac{\pi}{2})$$
$$s = 1.125\pi \approx 3.534$$

A point on the disk turns 3.534 in.

Summary

- The length of any arc is $s = r\theta$, where s is the length of the arc, r is the radius, and θ is the measure of the angle in radians.
- Use the fact that π is equal to 180° to convert between degrees and radians.
- A ***subtended arc*** is the part of the circle in between the two rays that make the central angle.

Review

The radius of a carousel is 8 meters. Assume you are on the outside edge. Use this information to answer questions 1-3.

1. You move half way around the carousel. How far did you travel?
2. You move all the way around the carousel. How far did you travel?
3. You have now traveled all the way around the carousel twice. How far did you travel?

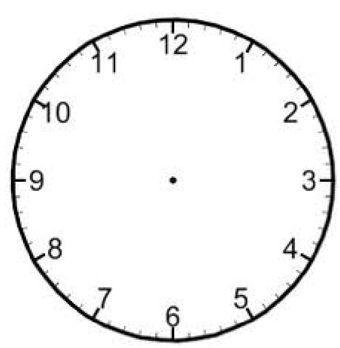
A pizza has a radius of 10 in. Use this information to answer questions 4-5.

4. A slice is removed. The length of the crust of the missing slice is 3 in. What is the central angle of the missing slice?
5. You eat three pieces with a total central angle of $\frac{4\pi}{5}$. What is the length of the crust you ate?
6. A large pizza has a radius of 12 in. What is the length of the crust of half the large pizza?

The diameter of a tire is 35 in. Use this information to answer questions 7-10.

7. What is the length around the whole tire?
8. The tire travels one mile (5280 ft). How many revolutions did the tire make?
9. You roll the tire so it rotates 7π radians. How far did it move?
10. The tire travels half a mile. How many radians did the tire rotate?

Consider a standard 12-hour clock like the one below with a radius of 5 in. Use this to answer questions 11-15.



11. What is the length of the arc between the 3 and the 7?
12. What is the length of the arc between the 3 and the 2?
13. It is 12:30. What is the length of the arc between the minute and hour hands?
14. It is 7:20. What is the length of the arc between the minute and hour hands?
15. It is 1:25. What is the length of the arc between the minute and hour hands?

Review (Answers)

Please see the Appendix.

5.5 Area of a Sector

Learning Objectives

Learn to find the area of a portion of a disk using the radius of the disk and an angle in radians.

Introduction

Very often, a grower waters their property at varying rates. For example, some days the irrigation pipe is only turned through an angle of 54°. The grower needs to know the area watered that day. The grower found a formula on the internet to calculate the irrigated area, but it did not make sense. The formula is $A = 0.0087r^2\theta$. The questions from the grower are: "How is this formula created? Why does the value .0087 appear in this formula?"

Area of Sector Formula

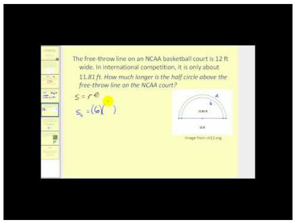
One of the most common geometric formulas is the area of a disk:

$$A = \pi r^2.$$

From this, we can derive the area of the sector with central angle, θ radians. Comparing the measure of the central angle with the full circle's angle of 2π, the sector of the area is proportional to the area of the disk:

$$A_s = \frac{\theta}{2\pi}\pi r^2$$
$$A_s = \frac{1}{2}r^2\theta,$$

where r is the radius.



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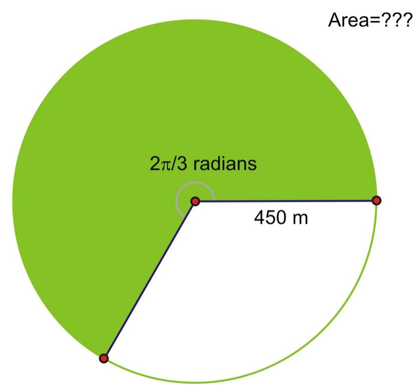
Examples

Example 1

Crops are often grown using a technique called center pivot irrigation, which results in circular- shaped fields.

Here is a satellite image taken over fields in Kansas that use this type of irrigation system:

If the irrigation pipe is 450 m in length, what is the area that can be irrigated after a rotation of $\frac{2\pi}{3}$ radians?



Solution:

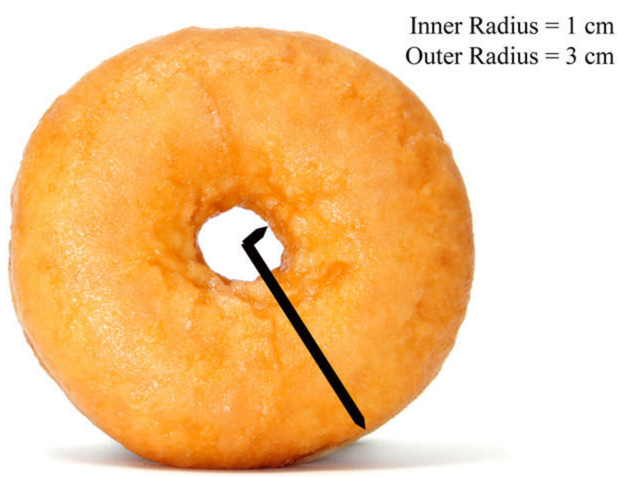
Use the formula

$$A = \frac{1}{2} r^2 \theta$$
$$A = \frac{1}{2}(450 \text{ m})^2 \left(\frac{2\pi}{3}\right).$$

The area is approximately 212,058 square meters.

Example 2

A drawing of a doughnut has a hole in the middle with a radius of 1 cm, and the distance from the center of the hole to the outer edge of the doughnut is 3 cm. What is the area of a sector of $\frac{1}{4}$ of the doughnut?



Solution:

The formula for the area of a sector is

$$A = \frac{1}{2} r^2 \theta.$$

Use this formula to find the area of the sector from the center outward:

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}3^2\frac{\pi}{2}$$
$$A = \frac{9\pi}{4}$$

Now subtract the area of the sector that is part of the hole, and therefore not part of the doughnut:

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}(1)^2\frac{\pi}{2}$$
$$A = \frac{\pi}{4}$$

Area of the sector of doughnut:

$$A = \frac{9\pi}{4} - \frac{\pi}{4} = \frac{8\pi}{4} = 2\pi \text{ cm}^2$$

Example 3

A driver is traveling around a circular track that has radius of 70 meters. If the angle from the starting line to her current position is $\frac{\pi}{3}$ radians, what is the area of the sector traced out by her car?

Solution:

The area of a sector is:

$$A = \frac{1}{2}r^2\theta$$

This leads us to:

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}(70 \text{ m})^2\frac{\pi}{3}$$
$$A \approx 2565.6 \text{ m}^2$$

Example 4

If the radius of a sector is 5 feet and the sector has a central angle of 43°, find the area of the sector.

Solution:

First, convert the angle from degrees to radians:

$$43^\circ \cdot \frac{\pi}{180^\circ} \approx 0.75 \text{ radians.}$$

Use $A = \frac{1}{2}r^2\theta$ to solve for the area:

$$A = \frac{1}{2}(5 \text{ ft})^2(.75)$$
$$A = 9.38 \text{ ft}^2.$$

Example 5

If a pie wedge has an area of 15 square inches and the pie has a radius of 9 inches, find the central angle.

Solution:

Since you know that $A = \frac{1}{2}r^2\theta$, you can solve for the angle swept out by the sector:

$$A = \frac{1}{2}r^2\theta$$
$$15 = \frac{1}{2}(81)(\theta)$$
$$\theta = \frac{(2)(15)}{81}$$
$$\theta \approx .37 \text{ radians}$$

Example 6

Returning to the grower's question, if the angle of rotation is in degrees, it must be converted to radians. That conversion factor is $\pi/180 \approx 0.0175$. The area formula is

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2\theta_d\frac{\pi}{180} \approx 0.0087r^2\theta_d,$$

where θ is the angle in radians, and θ_d is the angle in degrees.

Since the angle of rotation is 54° and the length of the irrigation pipe is 400 feet:

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}(400 \text{ ft})^2(54)\frac{\pi}{180}$$
$$A \approx 75{,}398 \text{ ft}^2$$

The area irrigated is approximately 75,398 ft^2.

Summary

- The area of a sector is $A = \frac{1}{2}r^2\theta$, where θ is in radians.
- Use the fact that π is equal to 180° to convert between degrees and radians.

Review

1. If the radius of a sector is 8 inches and the central angle of the sector is 40°, find the area of the sector.
2. If the radius of a sector is 12 inches and the central angle of the sector is $\frac{\pi}{6}$ radians, find the area of the sector.
3. If the radius of a sector is 6 inches and the central angle of the sector is 140°, find the area of the sector.
4. If the radius of a sector is 5 inches and the central angle of the sector is $\frac{5\pi}{3}$ radians, find the area of the sector.
5. If the radius of a sector is 10 inches and the central angle of the sector is 100°, find the area of the sector.

6. If a pie wedge has an area of 10 square inches and the pie has a radius of 6 inches, find the angle swept out by the sector.
7. If a pie wedge has an area of 15 square inches and the pie has a radius of 4 inches, find the angle swept out by the sector.
8. If a pie wedge has an area of 12 square inches and the pie has a radius of 3 inches, find the angle swept out by the sector.
9. If you have a piece of round cake that has an area of 20 square inches and you know the piece sweeps out an angle of $\frac{\pi}{3}$ radians, find the radius of the cake.
10. If you have a piece of round cake that has an area of 100 square inches and you know the piece sweeps out an angle of 50°, find the radius of the cake.
11. If you have a piece of round cake that has an area of 35 square inches and you know the piece sweeps out an angle of $\frac{2\pi}{5}$ radians, find the radius of the cake.

12. If you have a piece of round cake that has an area of 20 square inches and you know the piece sweeps out an angle of 30°, find the radius of the cake.

A pizza has a radius of 10 inches. Use this information to answer questions 13-14.

13. A slice is removed. The length of the crust of the missing slice is 3 inches. What is the area of the missing slice?
14. You eat 3 pieces with a combined central angle of $\frac{4\pi}{5}$. What is the area of the pizza you ate?
15. A large pizza has a radius of 12 inches. What is the area of half of the large pizza?

Review (Answers)

Please see the Appendix.

5.6 Angular Speed

Learning Objectives

Learn how to calculate linear and angular velocities for an object moving in a circle.

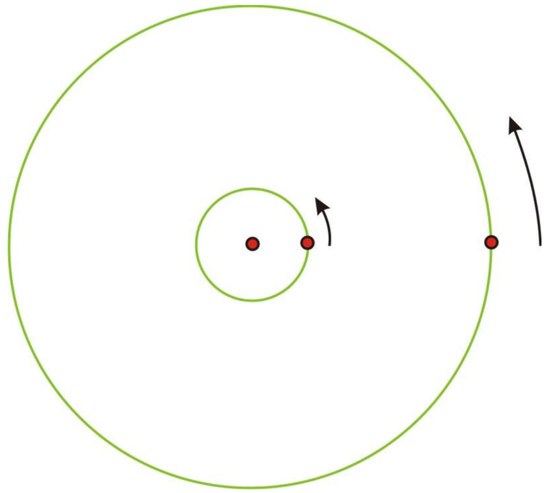
Introduction

Do you remember riding on a merry-go-round when you were younger? If two people are riding on the outer edge, the speeds at which they're moving should be the same. But what if one person is close to the center, and the other is on the edge? They are on the same object, but their speed is not the same.

Suppose you are standing 2.5 feet from the center, and a friend is on the outside edge 7 feet from the center. If it takes 6 seconds to complete a rotation, what is the speed that each of you is moving?

Angular Speed

Imagine the point on the larger circle is the person on the edge of the merry-go-round, and the point on the smaller circle is the person towards the middle. If the merry-go-round spins exactly once, then both individuals will also make one complete revolution in the same amount of time.

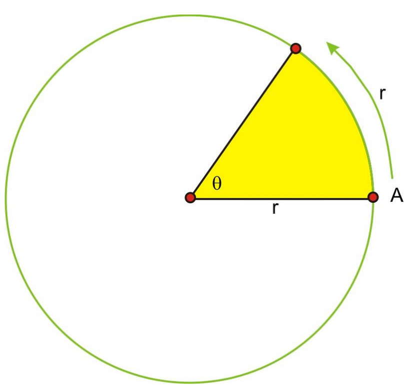


However, it is obvious that the person in the center did not travel as far as the person on the edge. The circumference (and of course the radius) of that circle is much smaller, and therefore the person who traveled a greater distance in the same amount of time is actually traveling faster, even though they are on the same rotating object. The person on the edge has a greater linear speed. But there is something about the two individuals traveling around that is the same: They both rotate at the same speed. This type of speed, measuring the angle of rotation over a given amount of time, is called the angular speed**.**

Angular Speed

$$\omega = \frac{\theta}{t},$$

where omega ω is the symbol for angular speed, θ is the angle of rotation expressed in radians, and t is the time to complete the rotation.



In this drawing, θ is exactly one radian, or the length of the radius bent around the circle. If it took a particle at point A exactly 2 seconds to rotate through the angle, the angular speed of it would be:

$$\omega = \frac{\theta}{t}$$
$$\omega = \frac{1}{2} \text{ radians per second.}$$

Learn, Play, and Explore with Angular Speed: Angular Speed 200 meter Dash

Linear Speed

In order to know the linear speed of the particle, we need the actual distance—that is, the length of the radius. Assume the radius is 5 cm.

If linear speed is $v = \frac{d}{t}$, then $v = \frac{5}{2}$ or 2.5 cm per second.

If the angle were not exactly 1 radian, then the distance traveled by the point on the circle is the length of the arc $s = r\theta$, or the radius length times the measure of the angle in radians.

Substituting into the formula for linear speed gives $v = \frac{r\theta}{t}$ or $v = r \cdot \frac{\theta}{t}$.

Look back at the formula for angular speed. Substituting ω gives the following relationship between linear and angular speed: $v = r\omega$. So the linear speed is equal to the radius times the angular speed.

Remember in a unit circle the radius is 1 unit, so in this case the linear speed is the same as the angular speed.

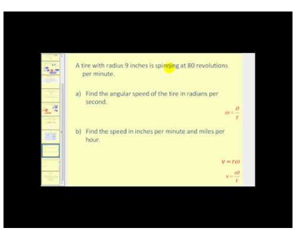
$$v = r\omega$$
$$v = 1 \times \omega$$
$$v = \omega$$

Here, the distance traveled around the circle is the same for a given unit of time as the angle of rotation, measured in radians.

Linear Speed

$$v = r\omega,$$

where v is the linear speed, r is the radius, and ω is the angular speed.



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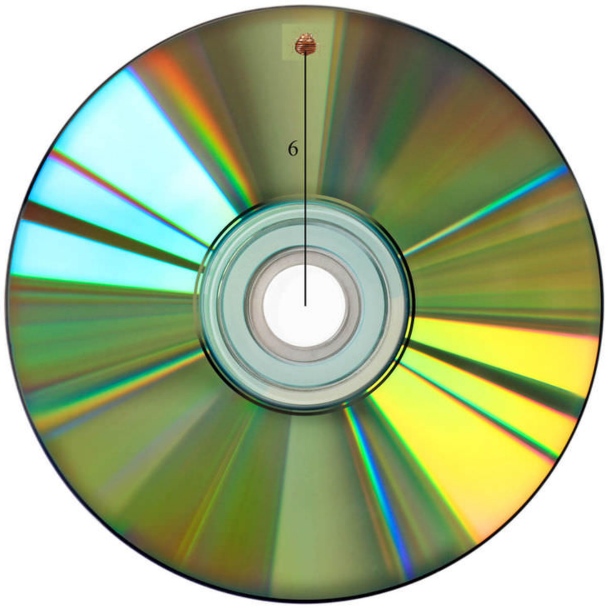
Examples

Example 1

A bug is standing near the outside edge of a compact disk that is rotating. His radius from the center of the disc is 6 cm. He notices that he has traveled π radians in 2 seconds. What is his angular speed? What is his linear speed?



Solution:

Step 1: The equation for angular speed is

$\omega = \frac{\theta}{t} = \frac{\pi}{2}$ radians per second.

Step 2: Use the given equation to find his linear speed:

$$v = r\omega = (6)\left(\frac{\pi}{2}\right) = 3\pi \approx 9.42 \text{ cm per sec.}$$

Example 2

How long does it take the bug in Example 1 to go through 2 complete turns?

Solution:

Step 1: Since the angular speed of the bug is $\frac{\pi}{2}$ radians per second, use the equation for angular speed and solve for time:

$$\omega = \frac{\theta}{t}$$

$$t = \frac{\theta}{\omega}$$

Step 2: Since there are 4π radians in 2 complete turns of the disc, use this for the value of θ:

$$t = \frac{4\pi}{\frac{\pi}{2}} = 4\pi \times \frac{2}{\pi} = 8 \text{ seconds}$$

Example 3

Doris and Lois go for a ride on a carousel. Doris rides on one of the outside horses, and Lois rides on a smaller one near the center. Lois's horse is 3 m from the center of the carousel, and Doris' horse is 7 m farther away from the center than Lois's. When the carousel starts, it takes them 12 seconds to complete a rotation.

Calculate the linear speed of each girl. Calculate the angular speed of the horses on the carousel.

Solution:

Step 1: Calculate the angular speed 1st. $\omega = \frac{2\pi}{12} = \frac{\pi}{6}$, so the angular speed is $\frac{\pi}{6}$ radians, or 0.524. Because the linear speed depends on the radius, each girl has her own.

Step 2: Lois: $v = r\omega = 3 \cdot \frac{\pi}{6} = \frac{\pi}{2} \approx 1.57$ m/sec

Step 3: Doris: $v = r\omega = 10 \cdot \frac{\pi}{6} = \frac{5\pi}{3} \approx 5.24$ m/sec

Example 4

Looking back to the Introduction, what is the angular speed of you and your friend?

Solution:

It takes 6 seconds to complete a rotation. A complete rotation is the same as 2π radians. So the angular speed is

$$\omega = \frac{\theta}{t} = \frac{2\pi}{6} = \frac{\pi}{3}$$

radians per second, which is slightly more than 1 (about 1.05) radian per second. Because both of you cover the same angle of rotation in the same amount of time, your angular speed is the same. In this case you rotate through approximately 60 degrees of the circle every second.

Example 5

Return again to the problem in the Introduction. You are standing 2.5 feet from the center, and your friend is riding on the outside edge, 7 feet from the center. If it takes 6 seconds to complete a rotation, what is the speed of each person?

Solution:

As discussed previously, the linear speeds of you and your friend are different. Using the formula, your linear speed is

$$v = r\omega = (2.5)\left(\frac{\pi}{3}\right) \approx 2.6 \text{ ft per sec.}$$

Your friend's linear speed is

$$v = r\omega = (7)\left(\frac{\pi}{3}\right) \approx 7.3 \text{ ft per sec.}$$

Example 6

The Large Hadron Collider (LHC) near Geneva, Switzerland began operation in 2008 and is designed to perform experiments that physicists hope will provide important information about the underlying structure of the universe. The LHC is circular with a circumference of approximately 27,000 m. Protons will be accelerated to a speed that is very close to the speed of light ($\approx 3 \times 10^8$ meters per second).

How long does it take a proton to make a complete rotation around the collider? What is the approximate (to the nearest meter per second) angular speed of a proton traveling around the collider? Approximately how many times would a proton travel around the collider in one full second?

Solution:

Part 1:

$$v = \frac{d}{t}$$
$$3 \times 10^8 = \frac{27{,}000}{t}$$
$$t = \frac{2.7 \times 10^4}{3 \times 10^8} = 0.9 \times 10^{-4} = 9 \times 10^{-5} \text{ or } 0.00009 \text{ seconds.}$$

The proton rotates around once in 0.00009 seconds.

Part 2:

$$\omega = \frac{\theta}{t} = \frac{2\pi}{0.00009} \approx 69,813 \text{ rad/sec}$$

Part 3:

In one second it will rotate around the LHC,

$$1 \div 0.00009 \approx 11,111.11$$

times, or just over 11,111 rotations.

Example 7

Ted is standing 2 meters from the center of a merry-go-round. If his linear speed is 6 m/s, what is his angular speed?

Solution:

Since the equation relating linear and angular speed is given by $v = r\omega$, we can solve for omega:

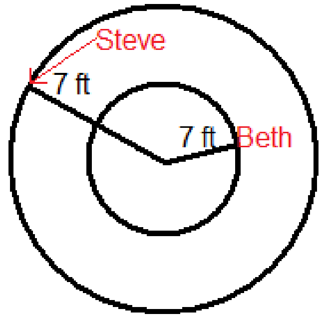
$$\omega = \frac{v}{r} = \frac{6 \text{ m/s}}{2 \text{ m}} = 3 \text{ radians per second.}$$

Summary

- Angular speed: $\omega = \frac{\theta}{t}$ where omega ω is the symbol for angular speed, θ is the angle of rotation expressed in radian measure, and t is the time to complete the rotation.
- Linear speed: $v = r\omega$ where v is the linear speed, r is the radius, and ω is the angular speed.

Review

Beth and Steve are on a carousel. Beth is 7 feet from the center and Steve is right on the edge, 7 feet further from the center than Beth. Use this information and the following picture to answer questions 1-6.



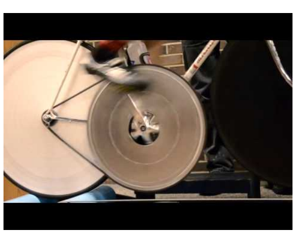
1. The carousel makes a complete revolution in 12 seconds. How far did Beth go in one revolution? How far did Steve go in one revolution?
2. If the carousel continues making revolutions every 12 seconds, what is the angular speed of the carousel?
3. What are Beth and Steve's linear speeds?
4. How far away from the center would Beth have to be in order to have a linear speed of π feet per second?
5. The carousel changes to a new angular speed of $\frac{\pi}{3}$ radians per second. Now how long does it take to make a complete revolution?
6. With the carousel's new speed, what are Beth and Steve's new linear speeds?
7. Beth and Steve go on another carousel that has an angular speed of $\frac{\pi}{8}$ radians per second. Beth's linear speed is 2π feet per second. How far is she standing from the center of the carousel?
8. Steve's linear speed is only $\frac{\pi}{3}$ feet per second. How far is he standing from the center of the carousel?
9. What is the angular speed of the minute hand on a clock (in radians per minute)?
10. What is the angular speed of the hour hand on a clock (in radians per minute)?
11. A certain clock has a radius of 1 foot. What is the linear speed of the tip of the minute hand? (Assume the tip of the minute hand is 1 foot away from the center.)
12. What is the linear speed of the tip of the hour hand if it is 1 foot from the center of another clock?
13. The tip of the minute hand on a clock has a linear speed of 2 inches per minute. What is the radius of the clock?
14. What is the angular speed of the second hand on a clock (in radians per minute)?
15. The tip of the second hand on a clock has a linear speed of 2 feet per minute. What is the radius of the clock? (Assume the second hand has the same length as the radius.)

Review (Answers)

Please see the Appendix.

5.7 Project: Angles

The fixed-gear bicycle has had new popularity as the interest in bicycles has increased. The simple construction of this vehicle makes it a predecessor of the more typical freewheel bicycles used today. The bicycle is constructed with a rear sprocket (or gear) and a sprocket in front of it that is fixed to the rear wheel. A fixed gear means there is no freewheel, so the rear sprocket is attached directly to the rear wheel. When the wheel turns, then the rear sprocket (and therefore the front sprocket and pedals) turns. It is not possible for the rider to coast on such a bike. In early bicycle racing, the fixed-gear bicycle was used to set speed information for a specific track at a specific event. These bicycles also have no brakes, making it difficult to make sudden speed changes, which is intended to improve their safety. In the following video, a person is attempting to break the speed record on a fixed-gear bicycle:



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Here is the situation:

- The sprocket assembly of a bike consists of a front gear and a back gear connected by a chain.
- The back gear is attached to the back wheel.
- The front gear is 9 inches in diameter.
- The back gear is 6 inches in diameter.
- The back wheel is 26 inches in diameter.

In cycling, if we know how fast the wheels turn, then we'll know how fast the rear sprocket turns. If we know how fast the rear sprocket turns, then we'll know how fast the chain moves. If we know how fast the chain moves, then we'll know how fast the front sprocket turns. If we know how fast the front sprocket turns, then we'll know how fast the pedals turn. The rate at which the pedals turn is known as the cadence. Since the system turns, it has an angular speed and a linear speed. The **angular speed** is the rate at which the pedals turn, described in units such as revolutions per second. The linear speed is the speed at which some point on the edge of the sprocket travels, in units such as meters per second.

For this project, you will determine how fast the bike is traveling if the cyclist is peddling at 1.2 revolutions per second.

1. Find the linear speed of a point on the sprocket attached to the pedal. (Use the diameter of the front gear.)

The chain moves at this calculated linear speed. Also, a point on the gear sprocket attached to the wheel travels at this linear speed.

2. Calculate the angular speed of the gear attached to the wheel.

The wheel travels at this calculated angular speed.

3. Calculate the linear speed of a point on the back wheel to determine how fast the bicycle is traveling.

Resources:

"Bike Wheel Diagram Angular and Linear Velocity." GeoGebra. N.p., 13 Dec. 2013. Web. 02 Apr. 2016.

"Fixed Gear Conversions." Fixed Gear Bicycles for the Road. N.p., n.d. Web. 02 Apr. 2016.

"Roller Speed Record Attemp." YouTube. N.p., n.d. Web. 02 Apr. 2016.

"Video: Rider Attempts to Hit 100mph by Bicycle." The Telegraph. Telegraph Media Group, n.d. Web. 02 Apr. 2016.

Weyland, Jocko. "Unstoppable." The New York Times. The New York Times, 28 Apr. 2007. Web. 02 Apr. 2016.

5.8 Summary: Angles

Chapter Summary

The study of trigonometry can be summarized by analyzing the roots of the very word:

"tri" is the prefix meaning *three,*

"gonia" is a root meaning *angle,* and

"metry" is from the Greek word *metron,* meaning *to measure.*

Trigonometry is the measuring of angles in three-sided figures. In this chapter we have learned many tools to understand how to measure angles.

In this chapter, we learned:

- Angles in standard position have their initial side on the positive $x-$axis.
- Coterminal angles are angles that can be represented in many ways, but are equivalent angles because they have the same initial side and terminal side.
- The conversion factor to convert from degrees to radians is $\frac{\pi}{180^\circ}$.
- The length of any arc is $s = r\theta$, where s is the length of the arc, r is the radius, and θ is the measure of the angle in radians.
- The area of a sector is $A = \frac{1}{2}r^2\theta$, where θ is in radians.
- Angular speed is $\omega = \frac{\theta}{t}$, where omega ω is the symbol for angular speed, θ is the angle of rotation expressed in radians, and t is the time to complete the rotation.
- Linear speed is $v = r\omega$, where v is the linear speed, r is the radius, and ω is the angular speed.

Chapter Review

Try the following cumulative review problems to practice the concepts studied in this chapter:

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URL: http://www.ck12.org/flx/render/embeddedobject/205299

5.9 References

1. David Goehring. https://flic.kr/p/42r64 .
2. University of Delaware Carvel REC. https://www.flickr.com/photos/carvel/14159224397/in/photolist-nzcJ7z-nzbVX1-aRWEo4-59pLRD-TB9wYH-bnNhVA-aRWA82-aRWE4c-nRB9Pf-aRWzWa-nTsNax-yL7zb-nPCLXu-7fX6Zc-nzbZ4y-nRB9No-yL7se-aRWyGv-aRWAx2-aRWAS4-aRWAH6-nzbYQY-nzbTXa-aRWDPH-raKvR2-aRWB8R-bzo3ru-aRWBqi-aRWHEV-9vwkXa-5rodJk-yL7tC-yL7GW-qptUiS-q1Dz9D-yL7Cm-yL7AK-yL6td-3qKKr-yL6uG-hd2Pvo-yL6iC-aRWHWt-aRWCrR-aRWH46-aRWDBz-yL7FK-ccFXam-8TahhB-aRWHov .
3. . Angle Standard Position.
4. . Coterminal Angles.
5. CK-12. Example 1b .
6. CK-12. Example 1c .
7. . Example 2.
8. . Example 3.
9. . Example 4.
10. CK-12. Example 5 .
11. U.S. Department of Agriculture. https://www.flickr.com/photos/usdagov/16018285828/in/photolist-qptUiS-yL7zb-yL7se-mc6oRR-fK3sY-yL7tC-aRWE4c-yL7Cm-yL7FK-yL6td-yL6uG-yL7AK-yL6iC-aRWEo4-aRWA82-aRWzWa-nzbVX1-yL7xP-q1Dz9D-8ZVJZJ-azAqrp-8ZVK4u-td56fP-ccFXam-nRxm9f-aRWDPH-8L9QYp-aRWzmk-aRWEXX-aRWxBc-aRWEEP-aRWz3F-aRWBJZ-bVjGrV-aRWyGv-aRWCcc-8TahhB-8Tai7a-6abBtS-8TdqPj-8TdpVm-8TdsHj-8Tdw93-bzo3ru-9vwkXa-9AeSdA-8ZSE7M-bzo2HA-dgq1JQ-8MXmhd .
12. CK-12. Definition Radian Measure .
13. U.S. Department of Agriculture. https://www.flickr.com/photos/usdagov/17215649691 .
14. CK-12. Arc Length .
15. CK-12. Example 1 .
16. CK-12. Example 2 .
17. CK-12. Example 3 .
18. Michael Pereckas. https://www.flickr.com/photos/beigephotos/1143287144/in/photolist-2K2Drj-9tL5oj-8oFu2B-9peb7g-5QXFyF-Djjy-9uj4ds-8skuEC-JKxEZw-nzbVX1-5hK5in-nE3NTi-cPVd9L-7uLqtr-4ETqSS-9ug3it-9HYGar-95TMvv-pnNoBP-J4XZbx-raKvR2-eXYnWJ-e9GFhd-pDYhkZ-gsb9gP-bGZTZa-hyMKpv-6wukYv-v1FUf-nTt2mx-hKywge-6HAC5B-mc6oRR-73NUMz-JWGzqD-T56cjY-9tL1ph-hKyt8F-hKyaVW-hKxrGi-hKxNxN-hKxn2r-hKyyiR-7hKNae-5aYnmq-nJGYZU-J4Y2ja-8oJGMy-hKxTZj-JSb9W6/ .
19. Courtesy of the USDA. http://commons.wikimedia.org/wiki/File:PivotIrrigationOnCotton.jpg .
20. Courtesy of NASA. http://commons.wikimedia.org/wiki/File:Crops_Kansas_AST_20010624.jpg .
21. CK-12. Example 1 .
22. CK-12. Example 2 .
23. machu.. https://www.flickr.com/photos/machu/62788591/in/photolist-gW4he-KjfSj-gdyRub-gVYon-6xNKu-5bPgXY-6xNRV-bkqHYz-4J3NNj-gVVhE-gWdM4-dJZa2e-61MVRo-9YPSGw-2HmZQ-gVQT4-gdzcED-6xNC1-gW27s-5fAXxY-4r9uK5-92CfMD-5fAXGw-541JBq-aK5p16-5bK2ya-6xNNR-5fwACx-4r9ohN-2HmZP-7YBNeZ-5fwAPK-4HYnBv-6xNGh-oW3LQq-oW9wQS-pdpBA8-iKrUr5-dUzZ2v-7ix9ab-bvGM3z-9A7UrJ-gVRLL-6xNLz-ds5gTw-cTCWFW-541JUd-amaXCQ-4r9qRY-541LmA/ .
24. Casey Marshall. https://www.flickr.com/photos/rsdio/2103025616/ .
25. mira66. https://www.flickr.com/photos/21804434@N02/4240342583/in/photolist-conQ7o-7sGS7T-7W14f9-dBew79-rjwbZW-4H6kUG-SQ9Qu7-7r6Gvm-6MzT84-pWBYAW-7wYXQF-aGQHs8-7uQxfM-d3DJru-7ngx1x-4gjLJC-4rEze-6m9csQ-5RrEah-rrcXrG-c1Tox-aa5Mv5-n2tR7-dj7ErP-a5P5cn-4iX5QF-c88F-4zNscM-6rijP6-BtdUn1-51ybwj-e27ZQf-51ybyY-q4sUGE-3eWTWp-8YhJCM-g1xyh-9bERuH-dChMWY-eeAUWA-51y

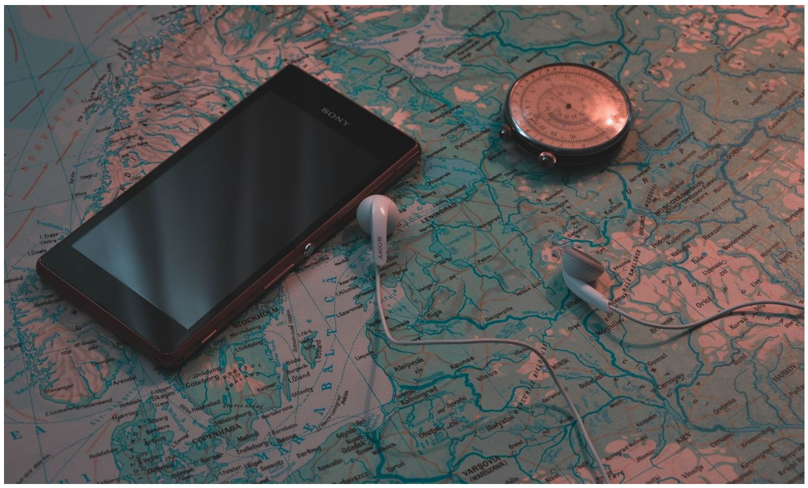
bL3-51tY2v-28aKXz-51ybkh-5iae29-9hqyUc-51ybtY-6QLXH3-HpEEk-arsrXn .
26. . Angular Velocity.
27. CK-12. Example 2 .
28. nooxnet. https://commons.wikimedia.org/wiki/File:SRAM_X0_Blackbox_bike_gears.jpg .

CHAPTER 6 Basic Triangle Trigonometry

Chapter Outline

6.1 Introduction: Basic Triangle Trigonometry

Today, anyone with a cell phone knows that with the GPS functionality of the phone, they can reasonably be located at any point on Earth. Part of this technology is the mathematical process of triangulation. One way to understand triangulation is as a method of surveying. Surveyors mark points on the ground at the vertices of several triangles.



The angles of the triangles are measured instrumentally, and the sides are derived by computation. The lengths of the sides of those triangles are distances between geographic points that cannot be measured directly. Even distances in space can be measured using these techniques. Cell phones are easily located by calculating their distances from nearby cell phone towers. In this chapter, we will explore the mathematical definitions and tools used to calculate these distances.

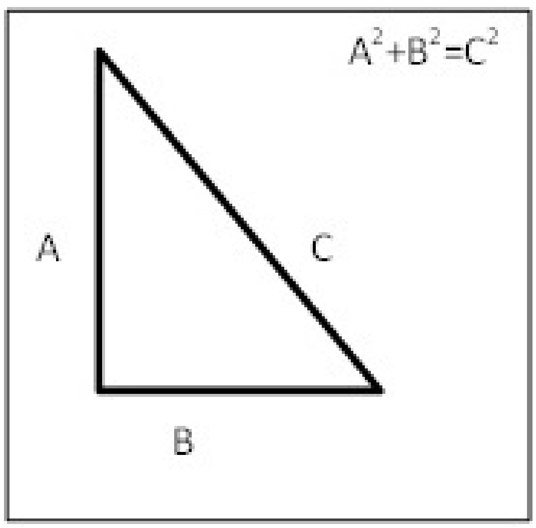
6.2 Special Right Triangles

Learning Objectives

Learn about properties of 30-60-90 and 45-45-90 right triangles.

Introduction

The Pythagorean Theorem is a 1st step to start the exploration of trigonometric relationships. There are some special triangles like 45-45-90 and 30-60-90 triangles that are so common that it is useful to know the side ratios without using the Pythagorean Theorem each time. These patterns are used throughout trigonometry.



Given a 45-45-90 right triangle with sides 6 inches, 6 inches, and x inches, what is the value of x?

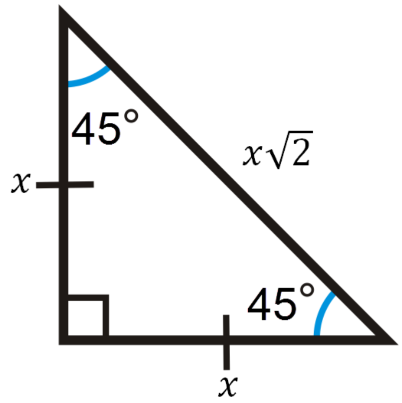
Special Triangles

45-45-90 Triangles

A 45-45-90 right triangle is an isosceles triangle with two sides having the same side length. Let these side lengths be equal to x. Use the Pythagorean Theorem to determine the length of the 3rd side.

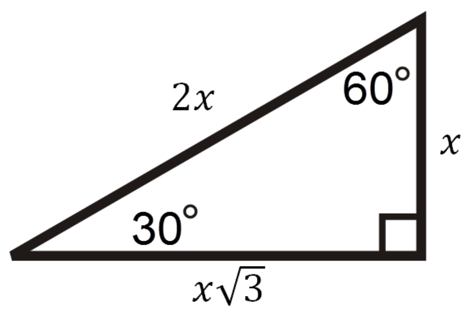
$$\begin{aligned}x^2+x^2&=c^2\\2x^2&=c^2\\x\sqrt{2}&=c\end{aligned}$$

Thus, a 45-45-90 right triangle has side ratios $x, x,$ and $x\sqrt{2}$.



30-60-90 Triangles

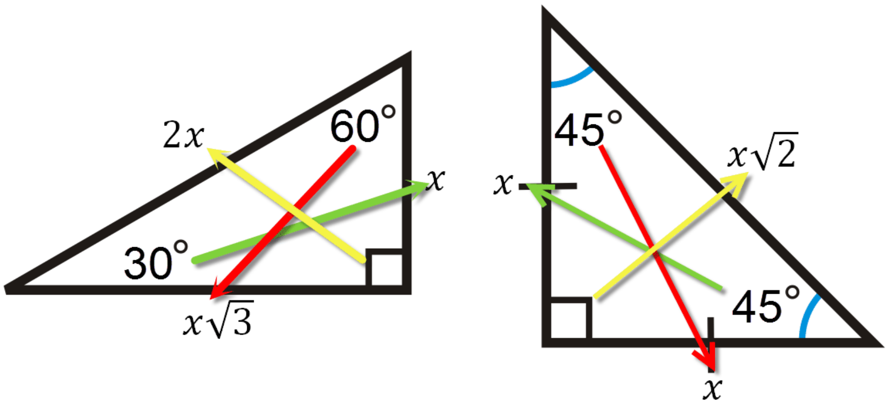
A 30-60-90 right triangle has side ratios $x, x\sqrt{3}$, and $2x$.



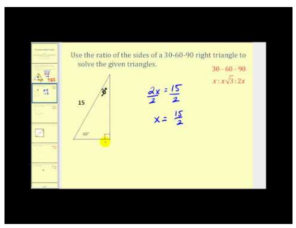
Confirm with the Pythagorean Theorem:

$$
\begin{aligned}
x^2+\left(x\sqrt{3}\right)^2 &= (2x)^2\\
x^2+3x^2 &= 4x^2\\
4x^2 &= 4x^2
\end{aligned}
$$

Note that the order of the side ratios—x, x, and $x\sqrt{2}$ and $x, x\sqrt{3}$, and $2x$—is important because each side ratio has a corresponding angle. In all triangles, the smallest sides correspond to the smallest angles, and the largest sides always correspond to the largest angles.



The side ratios for the 45-45-90 right triangle and the 30-60-90 right triangle with examples are also explained in the following video:



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Pythagorean Triples

Pythagorean triples are special right triangles with integral side lengths. While the angle measures are not integers, the side ratios are very useful to know. Here are some examples of Pythagorean triples:

- 3, 4, 5
- 5, 12, 13
- 7, 24, 25
- 8, 15, 17
- 9, 40, 41

Additional Pythagorean triples can be found by scaling any other Pythagorean triple. For example,

$3,4,5 \rightarrow 6,8,10$ (scaled by a factor of 2).

Examples

Example 1

A right triangle has two sides that are 3 inches in length. What is the length of the 3rd side?

Solution:

Since it is a right triangle with two sides of equal length, then it must be a 45-45-90 right triangle. Thus, the 3rd side is $3\sqrt{2}$ inches.

Example 2

A 30-60-90 right triangle has a hypotenuse of length 10. What are the lengths of the other two sides?

Solution:

The hypotenuse is the side opposite the 90°. Sometimes it is helpful to draw a picture or make a table:

TABLE 6.1:

30°	60°	90°
x	$x\sqrt{3}$	$2x$
		10

From the table, solve the subsequent equations for the missing sides:

$$18 = x\sqrt{3}$$
$$\frac{18}{\sqrt{3}} = x$$
$$x = \frac{18}{\sqrt{3}} = \frac{18}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{18\sqrt{3}}{3} = 6\sqrt{3}$$
$$2x = 2 \cdot 6\sqrt{3} = 12\sqrt{3}$$

Example 3

A 30-60-90 right triangle has a side length of 18 inches corresponding to 60 degrees. What are the lengths of the other two sides?

Solution:

Make a table with the side ratios and the information given, then write equations and solve for the missing side lengths:

TABLE 6.2:

30	60	90
x	$x\sqrt{3}$	$2x$
	18	

$$18 = x\sqrt{3}$$
$$\frac{18}{\sqrt{3}} = x$$
$$x = \frac{18}{\sqrt{3}} = \frac{18}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{18\sqrt{3}}{3} = 6\sqrt{3}$$
$$2x = 2 \cdot 6\sqrt{3} = 12\sqrt{3}$$

Note that the denominator is rationalized so that there is no a square root in the denominator.

Example 4

Return to the question in the Introduction. Given a 45-45-90 right triangle with sides 6 inches, 6 inches, and x inches, what is the value of x?

Solution:

Using the pattern for 45-45-90 right triangles, a right triangle with legs 6 inches and 6 inches has a hypotenuse that is $6\sqrt{2}$ inches, so $x = 6\sqrt{2}$.

Summary

- **Corresponding angles and sides** are angles and sides that are on opposite sides of each other in a triangle. Capital letters like $A, B,$ and C are often used for the angles in a triangle, and the lowercase letters $a, b,$ and c are used for their corresponding sides (angle A corresponds to side a, etc.).
- A 30-60-90 right triangle has side ratios $x, x\sqrt{3},$ and $2x$.
- A 45-45-90 right triangle has side ratios $x, x,$ and $x\sqrt{2}$.
- **Pythagorean triples** are special right triangles with integer sides.

Review

For 1-4, find the missing sides of the 45-45-90 triangle based on the information given in each row:

TABLE 6.3:

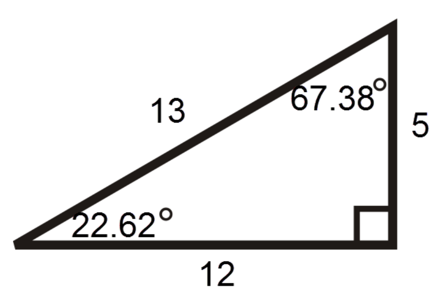
Problem Number	Side Opposite 45°	Side Opposite 45°	Side Opposite 90°
1.	3		
2.		7.2	
3.			16
4.	$5\sqrt{2}$		

For 5-8, find the missing sides of the 30-60-90 triangle based on the information given in each row:

TABLE 6.4:

Problem Number	Side Opposite 30°	Side Opposite 60°	Side Opposite 90°
5.	$3\sqrt{2}$		
6.		4	
7.			15
8.			$12\sqrt{3}$

Use the picture below for 9-11.



9. Which angle corresponds to the side that is 12 units?

10. Which side corresponds to the right angle?

11. Which angle corresponds to the side that is 5 units?

For 12-17, verify the Pythagorean triple using the Pythagorean Theorem:

12. 3, 4, 5

13. 5, 12, 13

14. 7, 24, 25

15. 8, 15, 17

16. 9, 40, 41

17. 6, 8, 10

18. Find another Pythagorean triple by using the scaling method for 11, 60, 61.

Review (Answers)

Please see the Appendix.

6.3 Right Triangle Trigonometry

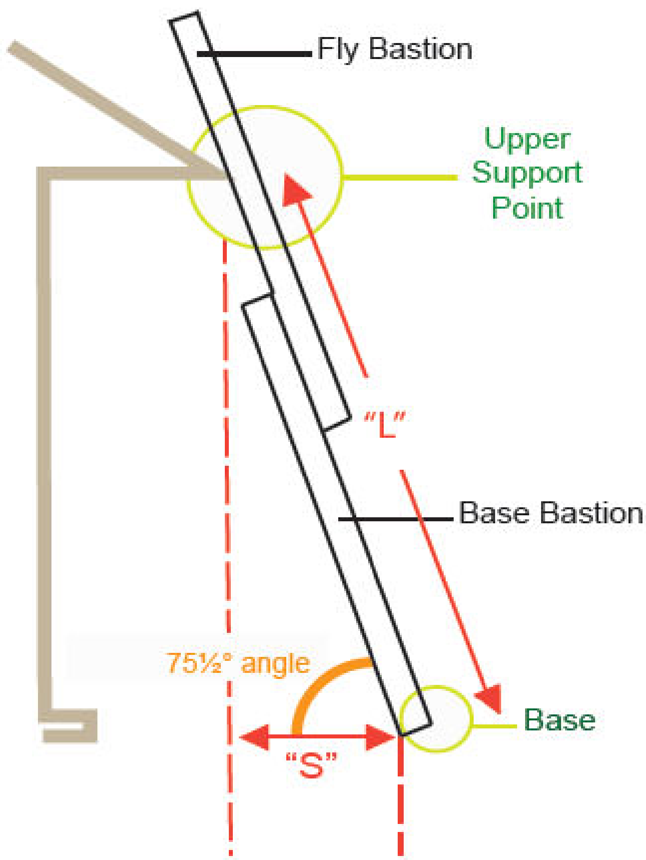
Learning Objectives

Learn the six right triangle ratios and how to use them to completely solve for the missing sides and angles of any right triangle.

Introduction

Laner Property Management Company is very proactive concerning exterior maintenance. The company is also very careful to follow established workplace safety guidelines. When a worker prepared to wash all the windows on a three-story apartment building, the manager consulted workplace guidelines and learned that, for safety, the ladder needed to be long enough so that:

- The ladder extends 3 feet above the highest point.
- The bottom of the ladder forms a 75.5° angle with the ground.

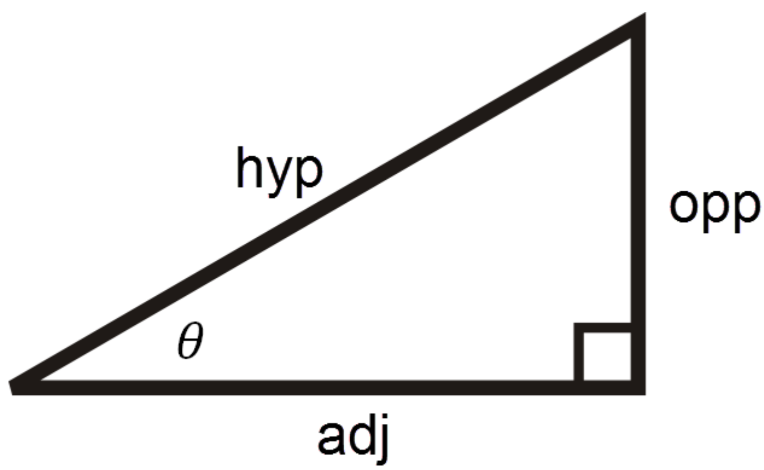


The manager knows that the tallest building on the property is 38 feet high. What is the length of the ladder required to do this job safely?

Trigonometry—in part, the study of triangles, will be needed to answer this question. Using trigonometry, if we know some angles and side lengths of a triangle, we can determine all other dimensions of the triangle.

Introduction to Right Triangle Trigonometry

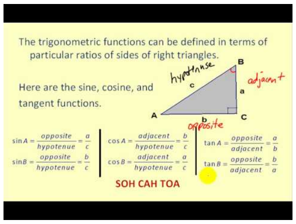
The six trigonometric functions are sine, cosine, tangent, cotangent, secant, and cosecant, where the functions' inputs are angles. For a right triangle, these functions can be determined as ratios of sides of the triangle. *Opp* stands for the leg opposite of the angle θ, *hyp* stands for hypotenuse, and *adj* stands for the leg adjacent to the angle θ.



The sine of an angle, $\sin\theta$, is defined as the leg opposite divided by the hypotenuse. The cosine of an angle, $\cos\theta$, is defined as the leg adjacent divided by the hypotenuse. The tangent of an angle, $\tan\theta$, is defined as the leg opposite divided by the leg adjacent. The cosecant of an angle, $\csc\theta$, is defined as the hypotenuse divided by the leg opposite. The secant of an angle, $\sec\theta$, is defined as the hypotenuse divided by the leg adjacent. The cotangent of an angle, $\cot\theta$, is defined as the leg adjacent divided by the leg opposite.

Trigonometric Functions

$$\sin\theta = \frac{opp}{hyp} \qquad \csc\theta = \frac{hyp}{opp}$$
$$\cos\theta = \frac{adj}{hyp} \qquad \sec\theta = \frac{hyp}{adj}$$
$$\tan\theta = \frac{opp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$



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Basic Trigonometric Identities

Notice that the sine function and cosecant function are reciprocals of each other.

$$\sin\theta = \frac{opp}{hyp} = \frac{1}{\csc\theta} = \frac{1}{\frac{hyp}{opp}}$$
$$\csc\theta = \frac{hyp}{opp} = \frac{1}{\sin\theta} = \frac{1}{\frac{opp}{hyp}}$$

Likewise, cosine and secant functions and tangent and cotangent functions are reciprocals of each other.

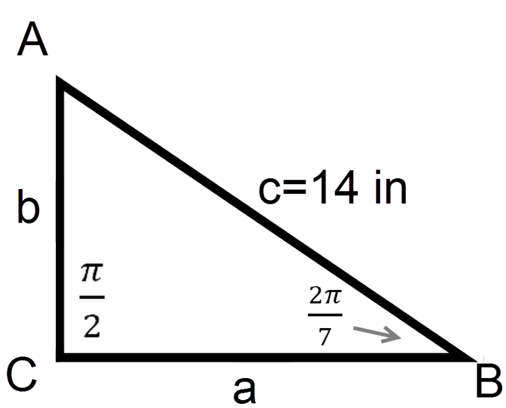
$$\cos\theta = \frac{adj}{hyp} = \frac{1}{\sec\theta}$$
$$\sec\theta = \frac{hyp}{adj} = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{opp}{adj} = \frac{1}{\cot\theta}$$
$$\cot\theta = \frac{adj}{opp} = \frac{1}{\tan\theta}$$

Examples

Example 1

Solve for side *b*.



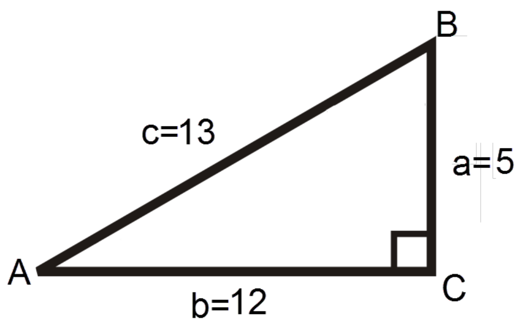
Solution:

Since the hypotenuse is given, and b is the leg opposite to the given angle $\frac{2\pi}{7}$, the sine function is the most appropriate to use in this problem.

$$\sin\left(\frac{2\pi}{7}\right) = \frac{b}{14}$$
$$b = 14 \cdot \sin\left(\frac{2\pi}{7}\right) \approx 10.9 \text{ in}$$

Example 2

Solve for angle A.



Solution:

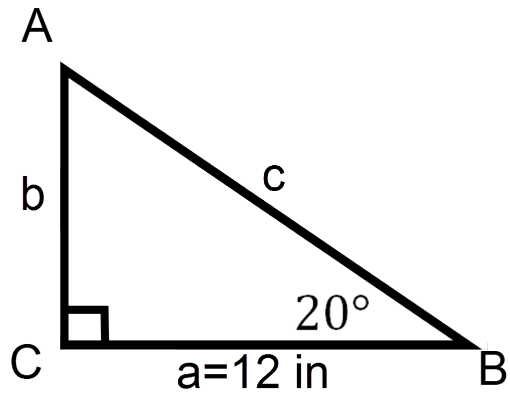
This problem can be solved using sine, cosine, or tangent relationships because the opposite, adjacent, and hypotenuse lengths are all given.

The argument of a sine function is always an angle. The inverse function of a sine function is arcsin or $\sin^{-1}$ function. The argument of the inverse function of a sine function is always a ratio of sides of the triangle. Inverse trigonometric functions are used to find an angle that corresponds to a trigonometric ratio.

$$\sin A = \frac{5}{13}$$
$$A = \sin^{-1} \text{[U+0081]} \left(\frac{5}{13}\right) \approx 0.39 \text{ radian} \approx 22.6^\circ$$

Example 3

Given a right triangle with $a = 12$ in, $m\angle B = 20^\circ$, and $m\angle C = 90^\circ$, find the length of the hypotenuse.



Solution:

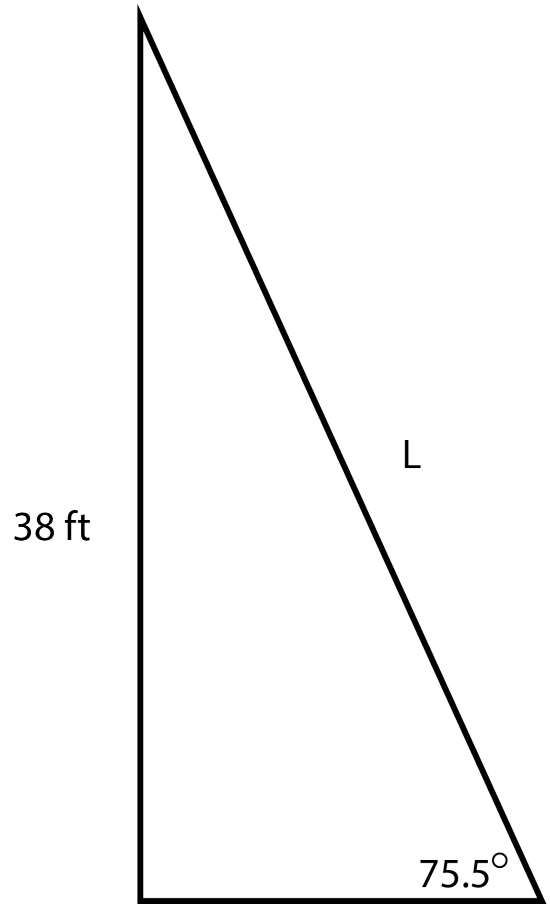
$$\cos 20^\circ = \frac{12}{c}$$
$$c = \frac{12}{\cos 20^\circ} \approx 12.77 \text{ in}$$

Example 4

To return to the problem in the Introduction, the bottom of the ladder forms a 75.5° angle with the ground, and the manager knows that the tallest building on the property is 38 feet high. What is the length of the ladder required to safely do this job?

Solution:

Step 1: Create a diagram from the given information.



Step 2: Solve for L.

$$\sin 75.5^\circ = \frac{38}{L}$$
$$L = \frac{38}{\sin 75.5^\circ} \approx 39.25 \text{ ft}$$

Recall that the ladder extends 3 feet above the highest point, so add 3 to the value of L.

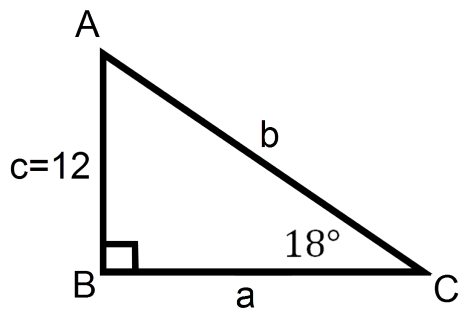
$$39.25 + 3 = 42.25 \text{ ft}$$

Example 5

Given $\triangle ABC$, where B is a right angle, $m\angle C = 18^\circ$, and $c = 12$. What is a?

Solution:

Step 1: Create a diagram from the given information.



Step 2: Solve for a.

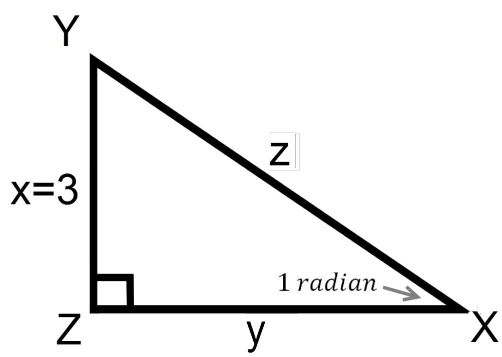
$$\tan 18^\circ = \frac{12}{a}$$
$$a = \frac{12}{\tan 18^\circ} \approx 36.9$$

Example 6

Given $\triangle XYZ$, where Z is a right angle, $m\angle X = 1$ radian, and $x = 3$, what is z?

Solution:

Step 1: Create a diagram from the given information.



Step 2: Solve for z.

$$\sin 1 = \frac{3}{z}$$
$$z = \frac{3}{\sin 1} \approx 3.6$$

Summary

- The Basic Trigonometric Ratios:
  - $\sin\theta = \frac{opp}{hyp}$
  - $\cos\theta = \frac{adj}{hyp}$

- $\tan\theta = \frac{opp}{adj}$
- $\cot\theta = \frac{adj}{opp}$
- $\sec\theta = \frac{hyp}{adj}$
- $\csc\theta = \frac{hyp}{opp}$

Review

For 1-15, information about the sides and/or angles of right triangle *ABC* is given. Completely solve the triangle (find all missing sides and angles) to one decimal place. Use your knowledge of trigonometric functions, the Pythagorean Theorem, and special right triangles.

TABLE 6.5:

Problem Number	A	B	C	a	b	c
1.	$90°$	$29.745°$				7
2.	$90°$		$37°$	18		
3.		$90°$	$15°$		32	
4.	$33.06°$		$90°$			11
5.	$90°$	$12°$		19		
6.	$54°$	$90°$				10
7.	$90°$	$10°$			2	
8.	$4°$	$90°$		0.3		
9.	$\frac{\pi}{2}$ radians		1 radian			15
10.	0.93 radian	$\frac{\pi}{2}$ radians		12		
11.		0.70 radian	$\frac{\pi}{2}$ radians			14
12.	$\frac{\pi}{4}$ radian	$\frac{\pi}{4}$ radian			5	
13.	$\frac{\pi}{2}$ radians			26	13	
14.		$\frac{\pi}{2}$ radians	1 radian		19	
15.			$\frac{\pi}{2}$ radians	10		$10\sqrt{2}$

Review (Answers)

Please see the Appendix.

6.4 Inverse Trigonometric Functions

Learning Objectives

Learn how to use the inverses of trig functions to solve for unknown angles.

Introduction

While running on your school's track, you notice a flagpole at the very end of the football field. The pole is 50 feet tall. Standing at the end of the track, you know that the distance between you and the flagpole is 350 feet. Since you are curious about such things, you want to determine the angle between the ground and the top of the flagpole from where you are standing. You have a firm grasp of trigonometric functions, but you're still unsure how to solve for the angle of a trigonometric function.

Inverse Trigonometric Functions

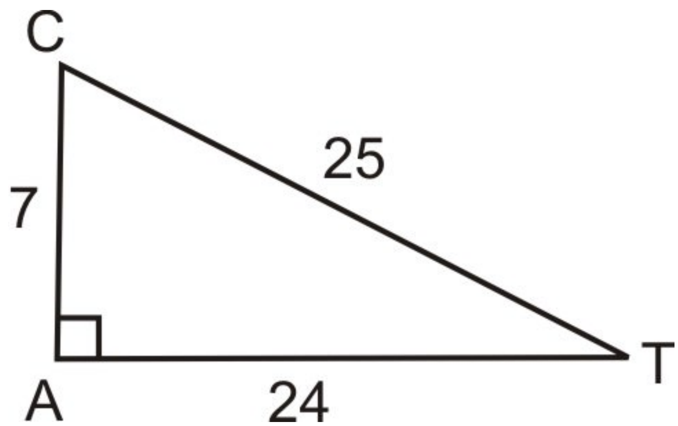
The word *inverse* is probably familiar to you. In mathematics, once you learn how to do an operation, you also learn how to "undo" it. For example, you may remember that addition and subtraction are considered inverse operations. Multiplication and division are also inverse operations. In learning about functions, you studied how to algebraically solve for the inverse of a given function. You also learned to graph a function and its inverse. When we apply the word inverse to the trigonometric ratios, we can find the acute angle measures within a right triangle. Normally, if you are given an angle and a side of a right triangle, you can find the other two sides using sine, cosine, or tangent. With the inverse trigonometric function, you can find the angle measure for the given two sides.

Inverse Tangent: If you know the opposite side and adjacent side of an angle in a right triangle, you can use inverse tangent to find the measure of the angle. Inverse tangent is also called *arctangent* and is labeled $\tan^{-1}$ or *arctan*. The "-1" indicates inverse.

Inverse Sine: If you know the opposite side of an angle and the hypotenuse in a right triangle, you can use inverse sine to find the measure of the angle. Inverse sine is also called *arcsine* and is labeled $\sin^{-1}$ or *arcsin*.

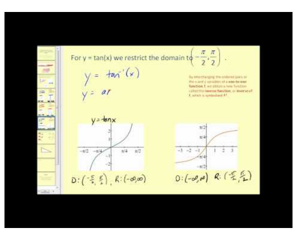
Inverse Cosine: If you know the adjacent side of an angle and the hypotenuse in a right triangle, you can use inverse cosine to find the measure of the angle. Inverse cosine is also called *arccosine* and is labeled $\cos^{-1}$ or *arccos*.

Consider the right triangle below.



From this triangle, we know how to determine all six trigonometric functions for both $\angle C$ and $\angle T$. However, none of the angles are known. The inverse of the trigonometric function must be used to determine the measure of the angle.

From any of these functions, we can also find the value of the angle using a calculator. You might recall that $\sin 30^\circ = \frac{1}{2}$. The inverse function for this example would then be $\sin^{-1}\left(\frac{1}{2}\right) = 30^\circ$. On some calculators you would press 2^{nd} and sin to get $\sin^{-1}($ and then type in $\frac{1}{2}$, close the parentheses, and press ENTER.



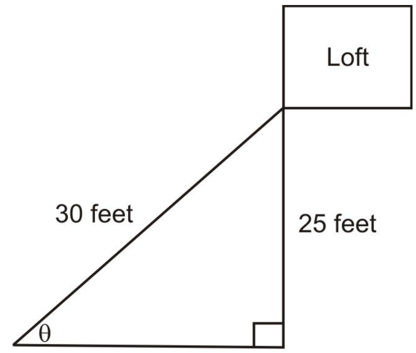
MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/176168

Real-World Application: Conveyor Belt

You live on a farm and your chore is to move hay from the loft of the barn down to the stalls for the horses. The hay is very heavy, and to move it manually down a ladder would take too much time and effort. You decide to devise a makeshift conveyor belt made of bedsheets, which you'll attach to the door of the loft and anchor securely in the ground. If the door of the loft is 25 feet above the ground, and you have 30 feet of sheeting, at what angle do you need to anchor the sheets to the ground?



Using the picture, we need to use the inverse sine function:

$$\begin{aligned}
\sin\theta &= \frac{25\ feet}{30\ feet} \\
\sin\theta &= 0.8333 \\
\sin^{-1}(\sin\theta) &= \sin^{-1}0.8333 \\
\theta &\approx 56.4^\circ.
\end{aligned}$$

The sheets should be anchored at an angle of 56.4°.

Learn, Play, and Explore with Inverse Trig Functions: Inverse Trig Ladder

Examples

Example 1

Determine the angle measure for the trig function below.

$$\sin x = 0.823$$

Solution:

$$\sin^{-1}0.823 \approx 55.39^\circ$$

Example 2

Determine the angle measure for the trig function below.

$$\cos x = -0.112$$

Solution:

$$\cos^{-1}-0.112 \approx 96.43^\circ$$

Example 3

Determine the angle measure for the trig function below.

$$\tan x = 0.2$$

Solution:

$$\tan^{-1} 0.2 \approx 11.31^\circ$$

Example 4

Recall from the Introduction that you noticed a 50-foot-tall flagpole at the very end of the football field. While standing at the end of the track, you are a distance of 350 feet from the flagpole. Determine the angle between the ground and the top of the flagpole from where you are standing.

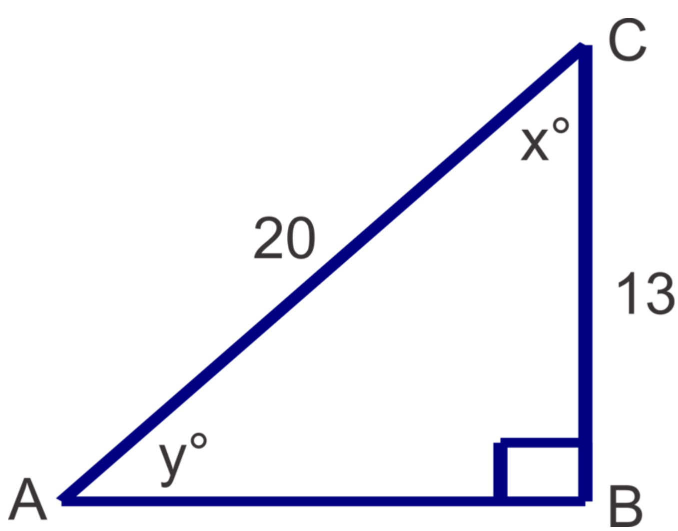
Solution:

Using your knowledge of inverse trigonometric functions, you can set up a tangent relationship to solve for the angle:

$$\tan \theta = \frac{50}{350}$$
$$\theta = \tan^{-1} \frac{50}{350}$$
$$\theta \approx 8.13^\circ$$

Example 5

Determine the measures of the unknown angles in the triangle shown. Round answers to the nearest degree.

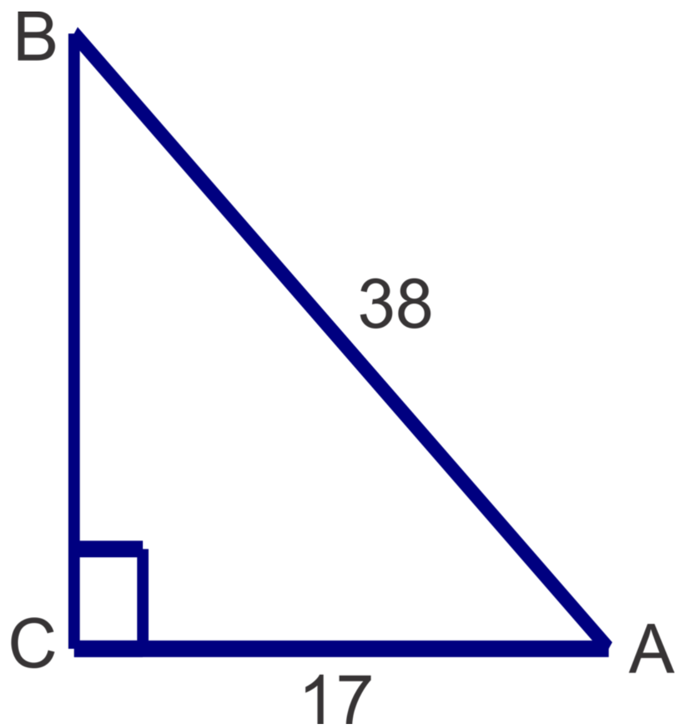


Solution:

$$x = \cos^{-1}\left(\frac{13}{20}\right) \approx 49^\circ$$
$$y = \sin^{-1}\left(\frac{13}{20}\right) \approx 41^\circ$$

Example 6

Solve the triangle for all missing angles and side lengths. Round side lengths to the nearest tenth and angles to the nearest degree.



Solution:

$$m\angle A = \cos^{-1}\left(\frac{17}{38}\right) \approx 63^\circ$$
$$m\angle B = \sin^{-1}\left(\frac{17}{38}\right) \approx 27^\circ$$
$$a = \sqrt{38^2 - 17^2} \approx 34.0$$

Review

Use inverse trigonometry to find the angle measure of angle A for each angle below.

1. $\sin A = 0.839$

2. $\cos A = 0.19$

3. $\tan A = 0.213$

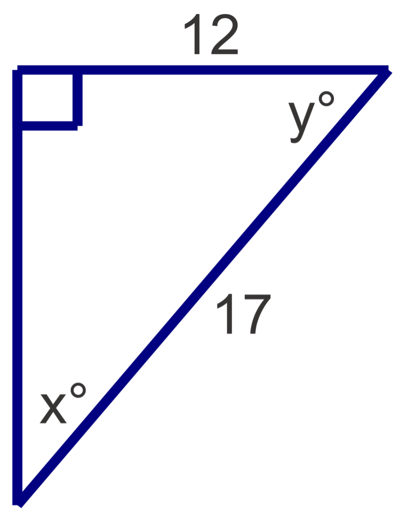
4. $\csc A = 1.556$

5. $\sec A = 2.063$

6. $\cot A = 2.356$

Find the measures of the unknown acute angles. Round measures to the nearest degree.

7.



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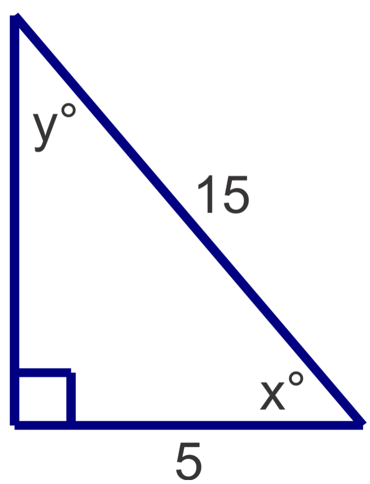

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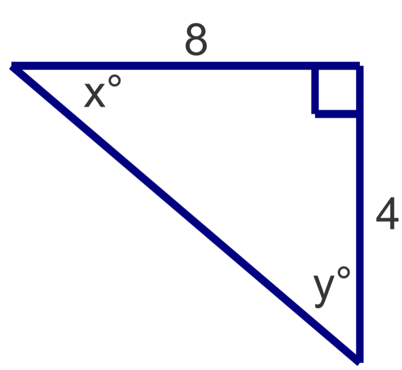

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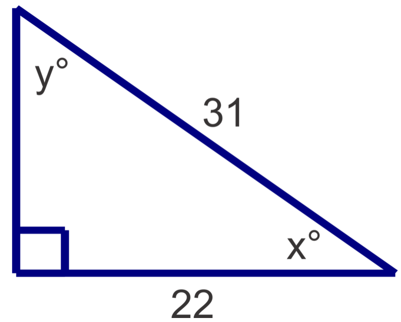

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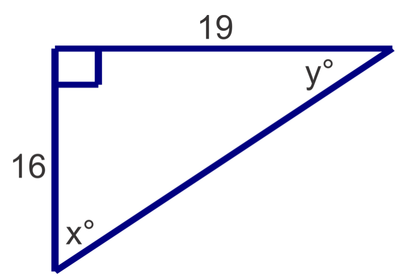
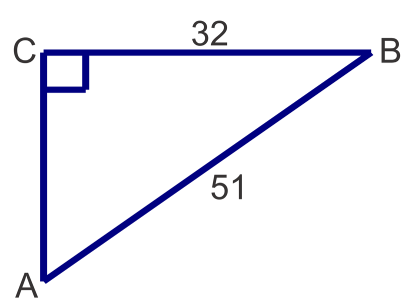


Solve the following right triangles for all missing angles and side lengths. Round angle measures to the nearest degree and side lengths to the nearest tenth.

12.



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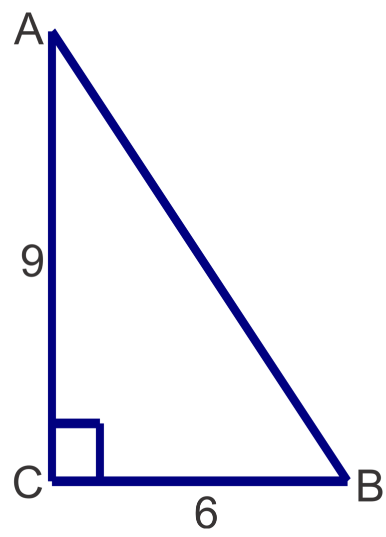
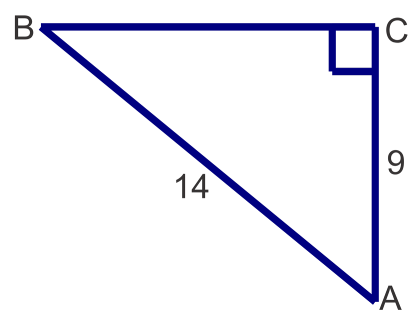


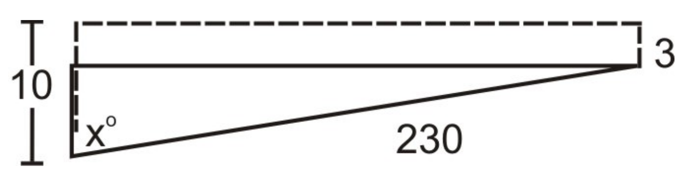
14.



15. A 70-foot building casts an 100-foot shadow. At what angle does the sun hit the building?

16. Whitney is sailing and spots a shipwreck 100 feet below the water. She jumps from the boat and swims 250 feet to reach the wreck. What is the angle of depression from the boat to the shipwreck? (Note this is the angle from the horizontal line of the water down.)

17. A boat is docked at the end of a 10-foot pier. The boat leaves the pier and drops anchor 230 feet away, 3 feet straight out from shore (which is perpendicular to the pier). What was the angle (x) of the boat from a line drawn from the end of the pier through the foot of the pier, as shown below?



Review (Answers)

Please see the Appendix.

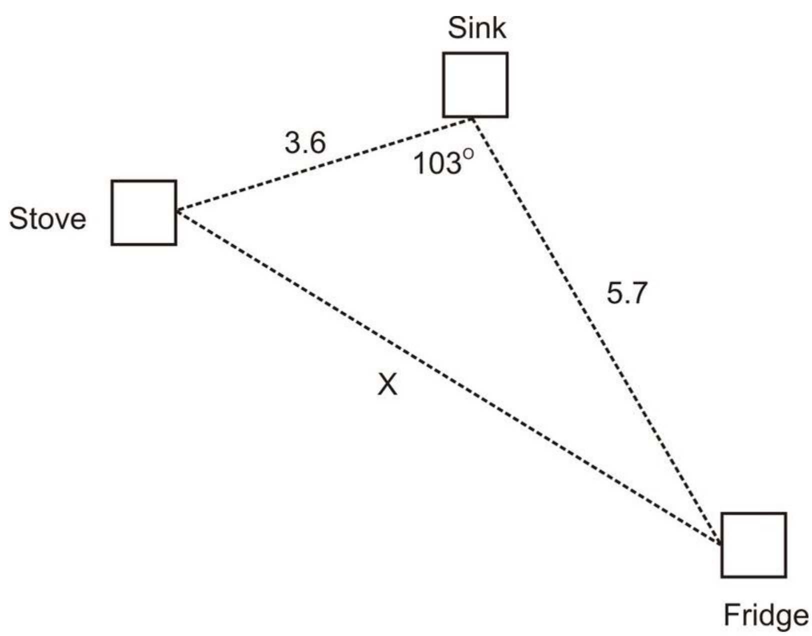
6.5 Law of Cosines

Learning Objectives

Learn to use the Law of Cosines to solve for missing parts of a triangle.

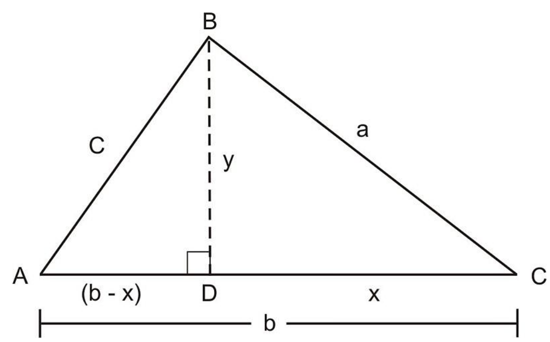
Introduction

An architect is designing a kitchen for a client. When designing a kitchen, the architect must pay special attention to the placement of the stove, sink, and refrigerator. In order for a kitchen to be utilized effectively, these three amenities must form a triangle with one another, known as the "work triangle." By design, the three parts of the work triangle must be *no less than 3 feet apart and no more than 7 feet apart*. Based on the dimensions of the current kitchen, the architect has determined that the sink will be 3.6 feet away from the stove and 5.7 feet away from the refrigerator. If the sink forms a 103° angle with the stove and refrigerator, will the distance between the stove and refrigerator remain within the confines of the work triangle?



This question cannot be answered using the right triangle trigonometric relationships because the triangle is not a right triangle. Fortunately, two mathematical laws provide a way to solve the problem. The trigonometric ratios can be extended so they can be used for any type of triangle.

Derivation of the Law of Cosines



$\triangle ABC$ contains an altitude BD that extends from B and intersects AC. Refer to the length of BD as y. The sides of $\triangle ABC$ measure a units, b units, and c units. If DC is x units long, then AD measures $(b-x)$ units.

Using the Pythagorean Theorem:

$c^2 = y^2 + (b-x)^2$	Pythagorean Theorem
$c^2 = y^2 + b^2 - 2bx + x^2$	Expand $(b-x)^2$
$c^2 = a^2 + b^2 - 2bx$	$a^2 = y^2 + x^2$ by Pythagorean Theorem
$c^2 = a^2 + b^2 - 2b(a\cos C)$	$\cos C = \frac{x}{a}$, so $a\cos C = x$ (cross multiply)
$c^2 = a^2 + b^2 - 2ab\cos C$	Simplify

Use a similar process to derive all three forms of the **Law of Cosines:**

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Note that if $\angle A, \angle B$, or $\angle C$ is 90° then $\cos 90^\circ = 0$, and the Law of Cosines is identical to the Pythagorean Theorem.

The Law of Cosines is one tool to use in certain situations involving all triangles: right, obtuse, and acute. (Recall that a right triangle is a triangle with one right angle, an obtuse triangle is a triangle with one obtuse angle, and an acute triangle is a triangle with all acute angles.) It is a general statement relating the lengths of the sides of any general triangle to the cosine of one of its angles. There are two situations in which to use the Law of Cosines:

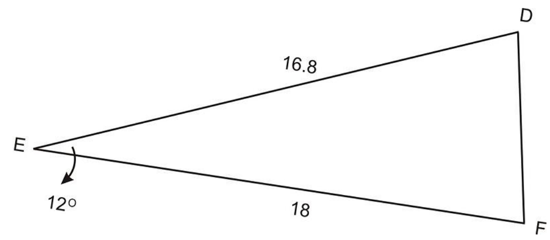
1. When we know two sides and the included angle in an oblique triangle (any triangle that is not a right triangle), and we need the 3rd side (SAS).
2. When we know all three sides in an oblique triangle, and we need one of the angles (SSS).

Case 1: Finding the Side of an Oblique Triangle

In this situation, two sides and the included angle in a triangle (SAS) are known and the length of the 3rd side is needed.

Example 1

For $\triangle DEF$, let $\angle E = 12^\circ$, $d = 18$, and $f = 16.8$. Find e.



Solution:

Since $\triangle DEF$ isn't a right triangle, the Pythagorean Theorem does not apply. This is an SAS problem, so the Law of Cosines does apply.

$$\begin{aligned}
e^2 &= 18^2 + 16.8^2 - 2(18)(16.8)\cos 12^\circ && \text{Use Law of Cosines.}\\
e^2 &= 324 + 282.24 - 2(18)(16.8)\cos 12^\circ && \text{Simplify squares.}\\
e^2 &\approx 324 + 282.24 - 591.5836689 && \text{Multiply.}\\
e^2 &\approx 14.6563311 && \text{Add and subtract from left to right.}\\
e &\approx 3.8 && \text{Find square root.}
\end{aligned}$$

* Note that the negative answer is an extraneous solution, which means it it not a valid solution.

Example 2

We can now solve the architect's problem in the Introduction.

Solution:

To find the distance from the sink to the refrigerator, find side x. To find side x, use the Law of Cosines because this is an obtuse triangle. We know the lengths of two sides: the sink to the stove and the sink to the refrigerator. We also know that the included angle (the angle between the two known lengths) is 103°. This is an SAS case, so the Law of Cosines applies.

$$\begin{aligned}
x^2 &= 3.6^2 + 5.7^2 - 2(3.6)(5.7)\cos 103^\circ && \text{Use Law of Cosines.}\\
x^2 &= 12.96 + 32.49 - 2(3.6)(5.7)\cos 103^\circ && \text{Simplify squares.}\\
x^2 &\approx 12.96 + 32.49 + 9.23199127 && \text{Multiply.}\\
x^2 &\approx 54.68199127 && \text{Evaluate.}\\
x &\approx 7.4 && \text{Find square root.}
\end{aligned}$$

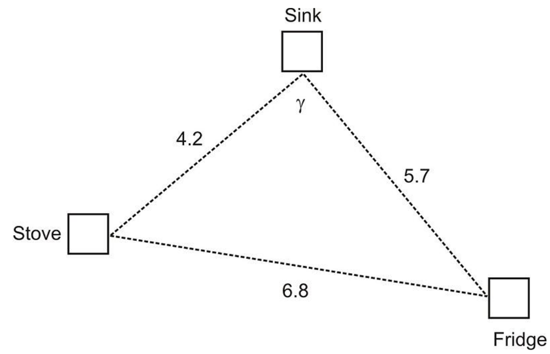
No, this triangle does not conform to the definition of a work triangle. The sink and the refrigerator are too far apart by 0.4 feet.

Case 2: Finding any Angle of a Triangle

Another situation in which the Law of Cosines applies is when all three sides of a triangle are known (SSS), and the measure of one of the angles is needed.

Example 3

Continuing from Example 2, if the architect moves the stove so it is 4.2 feet from the sink and 6.8 feet from the fridge, how does this affect the angle that the sink forms with the stove and the refrigerator?



Solution:

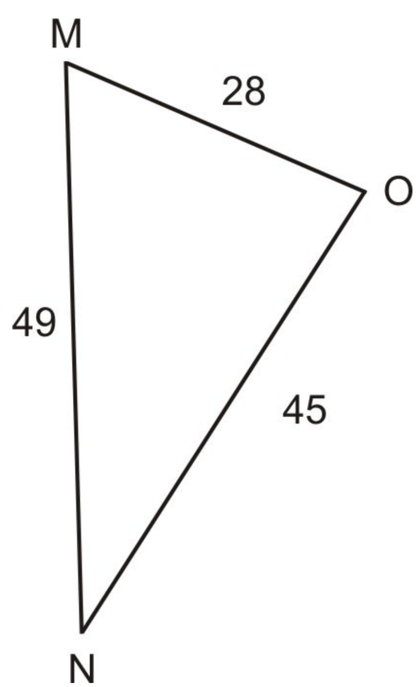
To find how the angle is affected, use the Law of Cosines to solve for γ.

$6.8^2 = 4.2^2 + 5.7^2 - 2(4.2)(5.7)\cos\gamma$	Use Law of Cosines.
$46.24 = 17.64 + 32.49 - 2(4.2)(5.7)\cos\gamma$	Simplify squares.
$46.24 = 17.64 + 32.49 - 47.88\cos\gamma$	Multiply.
$46.24 = 50.13 - 47.88\cos\gamma$	Add.
$-3.89 = -47.88\cos\gamma$	Subtract.
$0.0812447786 = \cos\gamma$	Divide.
$\cos^{-1}(0.081244786) = \cos^{-1}(\cos\gamma)$	Take the inverse.
$85.3^\circ \approx \gamma$	Simplify.

The new angle would be 85.3°, which means it would be 17.7° less than the original angle.

Example 4

In the oblique $\triangle MNO$, $m = 45$, $n = 28$, and $o = 49$. Find $\angle M$.



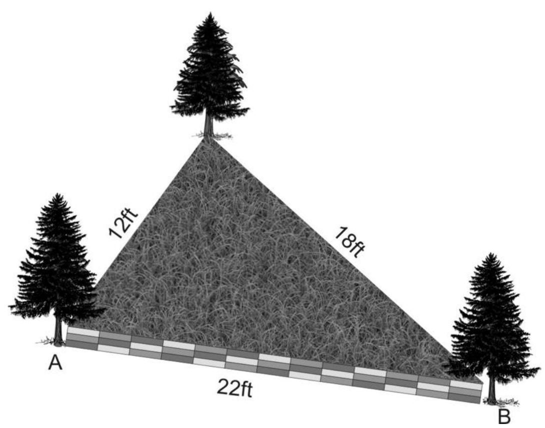
Solution:

Since all three sides of the triangle are known, use the Law of Cosines to find $\angle M$.

$45^2 = 28^2 + 49^2 - 2(28)(49)\cos M$	Use Law of Cosines.
$2{,}025 = 784 + 2{,}401 - 2(28)(49)\cos M$	Simplify squares.
$2{,}025 = 784 + 2{,}401 - 2{,}744\cos M$	Multiply.
$2{,}025 = 3{,}185 - 2{,}744\cos M$	Add.
$-1160 = -2{,}744\cos M$	Subtract 3,185.
$0.422740525 = \cos M$	Divide by $-2{,}744$.
$\cos^{-1}(0.422740525) = \cos^{-1}(\cos M)$	Take the inverse.
$65^\circ \approx M$	Simplify.

Example 5

Sam is building a retaining wall for a garden he plans to put in the back corner of his yard. Due to the placement of some trees, the dimensions of his wall need to be as follows: side 1 is 12 feet, side 2 is 18 feet, and side 3 is 22 feet. Determine the angle measure formed by side 1 and side 2, the angle measure formed by side 2 and side 3, and the angle measure formed by side 1 and side 3.



Solution:

Since the measures of all three sides of the retaining wall are known, use the Law of Cosines to find the measures of the angles formed by adjacent walls. Refer to the angle formed by side 1 and side 2 as $\angle C$, the angle formed by side 2 and side 3 as $\angle B$, and the angle formed by side 1 and side 3 as $\angle A$. First, find $\angle C$.

$22^2 = 12^2 + 18^2 - 2(12)(18)\cos C$	Use Law of Cosines.
$484 = 144 + 324 - 2(12)(18)\cos C$	Simplify squares.
$484 = 144 + 324 - 432\cos C$	Multiply.
$484 = 468 - 432\cos C$	Add.
$16 = -432\cos C$	Subtract 468.
$-0.037037037 \approx \cos C$	Divide by -432.
$\cos^{-1}(-0.037037037) \approx \cos^{-1}(\cos C)$	Take the inverse.
$92.12^\circ \approx C$	Simplify.

Next, find the measure of $\angle A$ also by using the Law of Cosines.

$$
\begin{aligned}
18^2 &= 12^2+22^2-2(12)(22)\cos A && \text{Use Law of Cosines.}\\
324 &= 144+484-2(12)(22)\cos A && \text{Simplify squares.}\\
324 &= 144+484-528\cos A && \text{Multiply.}\\
324 &= 628-528\cos A && \text{Add.}\\
-304 &= -528\cos A && \text{Subtract 628.}\\
0.575757576 &= \cos A && \text{Divide by } -528.\\
\cos^{-1}(0.575757576) &= \cos^{-1}(\cos A) && \text{Take the inverse.}\\
54.85^\circ &\approx A && \text{Simplify.}
\end{aligned}
$$

Now that we know two of the angles, we can find the 3rd angle using the Triangle Sum Theorem, which states that the three angles of a triangle equal 180°.

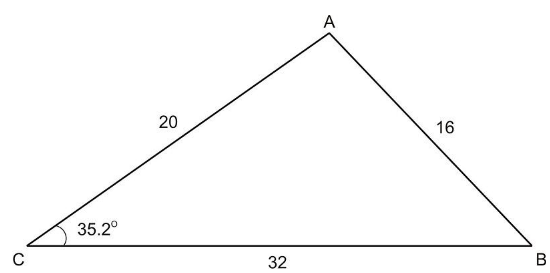
$$\angle B = 180-(92.12+54.85) = 33.03^\circ$$

Identify Accurate Drawings of General Triangles

The Law of Cosines can also be used to verify that drawings of oblique triangles are accurate.

Example 6

In $\triangle ABC$ at the right, $a = 32$, $b = 20$, and $c = 16$. Is the drawing accurate if it labels $\angle C$ as 35.2°? If not, what should $\angle C$ measure?



Solution:

Use the Law of Cosines to check whether or not $\angle C$ is 35.2°.

$$
\begin{aligned}
16^2 &= 20^2+32^2-2(20)(32)\cos 35.2^\circ && \text{Use Law of Cosines.}\\
256 &= 400+1{,}024-2(20)(32)\cos 35.2^\circ && \text{Simply squares.}\\
256 &\approx 400+1{,}024-1{,}045.94547 && \text{Multiply.}\\
256 &\neq 378.05453 && \text{Add and subtract.}
\end{aligned}
$$

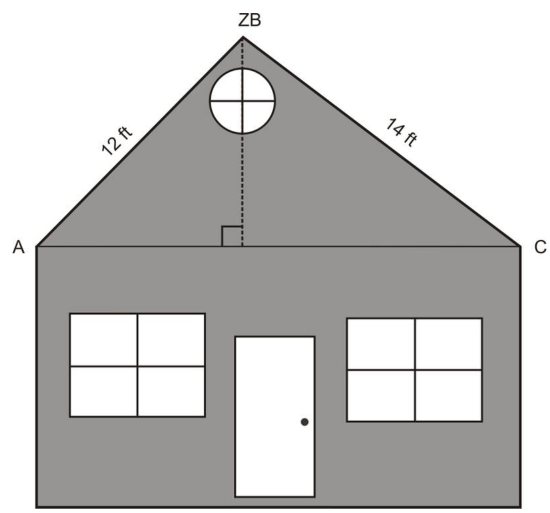
Since $256 \neq 378.05453$, $\angle C$ is not 35.2°. Using the Law of Cosines, calculate the correct measurement of $\angle C$.

$16^2 = 20^2 + 32^2 - 2(20)(32)\cos C$	Law of Cosines
$256 = 400 + 1024 - 2(20)(32)\cos C$	Simplify Squares
$256 = 400 + 1024 - 1280\cos C$	Multiply
$256 = 1424 - 1280\cos C$	Add
$-1168 = -1280\cos C$	Subtract 1424
$0.9125 = \cos C$	Divide
$24.1^\circ \approx \angle C$	$\cos^{-1}(0.9125)$

For some situations, it is necessary to utilize not only the Law of Cosines, but also the Pythagorean Theorem and trigonometric ratios to verify that a triangle or quadrilateral has been drawn accurately.

Example 7

A builder received plans for the construction of a second-story addition on a house. The diagram shows how the architect wants the roof framed, while the length of the house is 20 feet. The builder decides to add a perpendicular support beam from the peak of the roof to the base. He estimates that new beam should be 8.3 feet high, but he wants to doublecheck before he begins construction. Is the builder's estimate of 8.3 feet for the new beam correct? If not, what is his error?



Solution:

If either $\angle A$ or $\angle C$ were known, right triangle trigonometric ratios could be used to find the height of the support beam. However, neither of these angle measures is given. Since all three sides of $\triangle ABC$ are known, use the Law of Cosines to find one of these angles. Find $\angle A$:

$14^2 = 12^2 + 20^2 - 2(12)(20)\cos A$	Use Law of Cosines.
$196 = 144 + 400 - 480\cos A$	Simplify.
$196 = 544 - 480\cos A$	Add.
$-348 = -480\cos A$	Subtract.
$0.725 = \cos A$	Divide.
$43.5^\circ \approx \angle A$	$\cos^{-1}(0.725)$

Use$\angle A$ to find the length of BD.

$$\sin 43.5 = \frac{x}{12}$$
$$12 \sin 43.5 = x$$
$$8.3 \approx x$$

Yes, the builder's estimate of 8.3 feet for the support beam is accurate.

Summary

- Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

- There are two cases when the Law of Cosines can be applied:
  1) When two sides and the included angle in an oblique triangle are known and the 3rd side is needed (SAS).

  2) When all three sides in a triangle (SSS) are known and the measure of one of the angles is needed.

Review

For $\triangle ABC$, let $a = 12$, $b = 15$, and $c = 20$.

1. Find $m\angle A$.
2. Find $m\angle B$.
3. Find $m\angle C$.

For $\triangle DEF$, let $d = 25$, $e = 13$, and $f = 16$.

4. Find $m\angle D$.
5. Find $m\angle E$.
6. Find $m\angle F$.

For $\triangle KBP$, let $k = 19$, $\angle B = 61^\circ$, and $p = 12$.

7. Find the length of b.
8. Find $m\angle K$.
9. Find $m\angle P$.
10. While hiking one day you walk for 5 miles due east, then turn to the left and walk 3 more miles 30° west of north. At this point you want to return home. How far are you from home if you were to walk in a straight line?
11. A parallelogram has sides of 20 and 31 feet, and an angle of 46°. Find the length of the longer diagonal of the parallelogram.
12. Dirk wants to find the length of a long building from one side (point A) to the other (point B). He stands outside of the building (at point C), where he is 500 feet from point A and 220 feet from point B. The angle at C is 94°. Find the length of the building.

Determine whether or not each triangle is possible.

13. $a = 12$, $b = 15$, and $c = 10$
14. $a = 1$, $b = 5$, and $c = 4$
15. $a = 8$, $b = 10$, and $\angle A = 32^\circ$

Review (Answers)

Please see the Appendix.

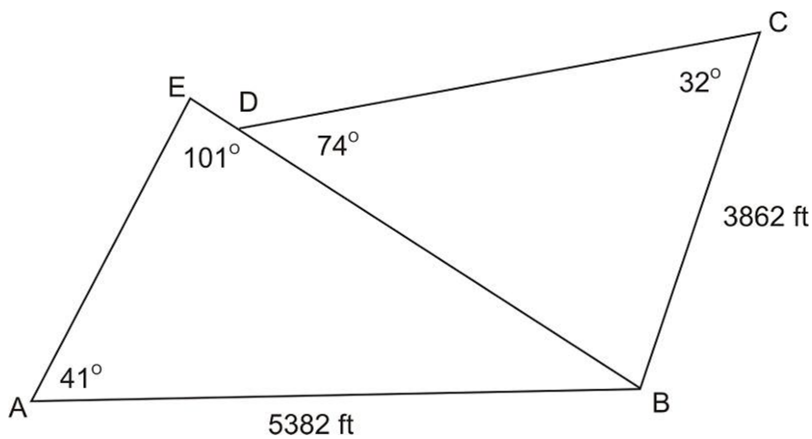
6.6 Law of Sines

Learning Objectives

Learn to use the Law of Sines to solve for missing parts of a triangle.

Introduction

A business group wants to build a golf course on a plot of land that was once a farm. The deed to the land is old, and information about the land is incomplete. If AB is 5,382 feet, BC is 3,862 feet, $\angle AEB$ is 101°, $\angle BDC$ is 74°, $\angle EAB$ is 41°, and $\angle DCB$ is 32°, what are the lengths of the sides of each triangular piece of land?



The Law of Cosines, which is a generalization of the Pythagorean Theorem for any triangle, can be used when:

1. Two sides of a triangle and the included angle are known (SAS), or
2. All three sides of the triangle are known (SSS).

If the triangle doesn't fit either of those scenarios, the Law of Sines may apply.

The Law of Sines is a statement about the relationship between the sides and the angles in any triangle. While the Law of Sines will yield one correct answer in many situations, there are times when it is ambiguous, meaning it can produce more than one answer, or no answer.

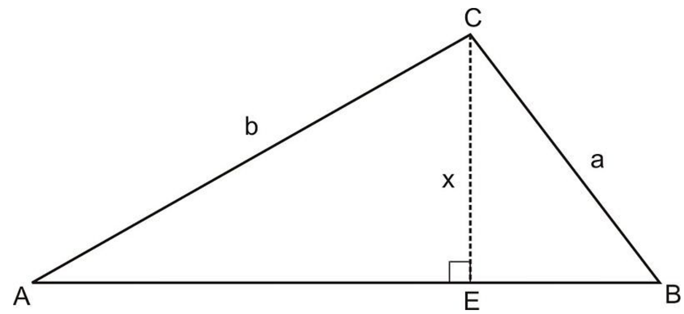
Use the Law of Sines when:

1. Two angles and a non-included side are known (AAS), or
2. Two angles and the included side (ASA) are known.

When solving a triangle where two sides and the non-included side are known (SSA), this case is known as the Ambiguous Case of the Law of Sines.

Derivation of the Law of Sines

$\triangle ABC$ contains altitude CE, which extends from C and intersects AB. Refer to the length of altitude CE as x.



First, $\sin A = \frac{x}{b}$ and $\sin B = \frac{x}{a}$, by the definition of sine. Equivalently,

$$b(\sin A) = x \qquad \text{and} \qquad a(\sin B) = x$$
$$\searrow \qquad \swarrow$$
$$b(\sin A) = a(\sin B)$$
$$\frac{\sin A}{a} = \frac{\sin B}{b} \text{ or } \frac{a}{\sin A} = \frac{b}{\sin B}$$

Extending these ratios to angle C and side c, the Law of Sines is developed.

Law of Sines

Form 1 : $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

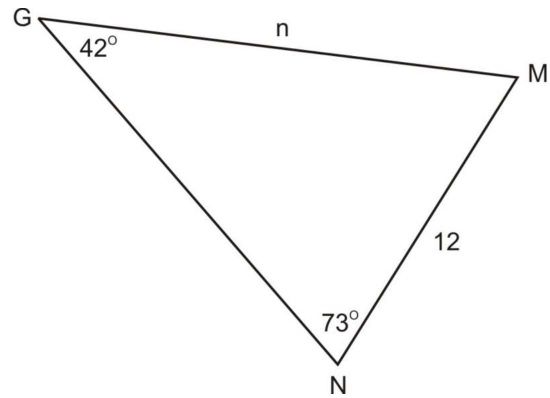
Form 2 : $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Note: Use any two of the three ratios from either form when solving.

AAS (Angle-Angle-Side) Case

Example 1

For $\triangle GMN, \angle G = 42^\circ, \angle N = 73^\circ$, and $g = 12$. Find n.



Solution:

Two angles and one non-included side (g) are known. The other non-included side (n) can be found.

$$\begin{aligned}\frac{\sin 73^\circ}{n} &= \frac{\sin 42^\circ}{12}\\ n\sin 42^\circ &= 12\sin 73^\circ\\ n &= \frac{12\sin 73^\circ}{\sin 42^\circ}\\ n &\approx 17.15\end{aligned}$$

Example 2

From the diagram in Example 1, find $\angle M$ and m.

Solution:

$\angle M$ is simply $180^\circ - 42^\circ - 73^\circ = 65^\circ$. To find side m, use either the Law of Sines or Law of Cosines. Using the Law of Sines:

Option 1: $\angle G$ and g

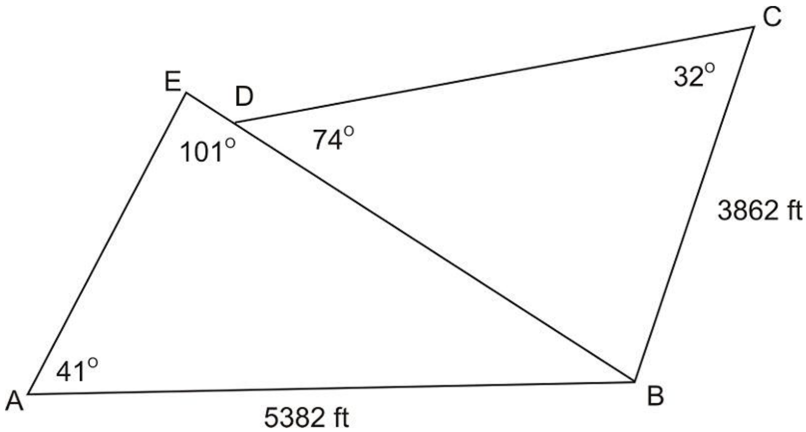
$$\begin{aligned}\frac{\sin 65^\circ}{m} &= \frac{\sin 42^\circ}{12}\\ m\sin 42^\circ &= 12\sin 65^\circ\\ m &= \frac{12\sin 65^\circ}{\sin 42^\circ}\\ m &\approx 16.25\end{aligned}$$

Option 2: $\angle N$ and n

$$\begin{aligned}\frac{\sin 65^\circ}{m} &= \frac{\sin 73^\circ}{17.15}\\ m\sin 73^\circ &= 17.15\sin 65^\circ\\ m &= \frac{17.15\sin 65^\circ}{\sin 73^\circ}\\ m &\approx 16.25\end{aligned}$$

Example 3

Recall the problem from the Introduction. A business group wants to build a golf course on a plot of land that was once a farm. If AB is 5,382 feet, BC is 3,862 feet, $\angle AEB$ is 101°, $\angle BDC$ is 74°, $\angle EAB$ is 41°, and $\angle DCB$ is 32°, what are the lengths of the sides of each triangular piece of land?



Solution:

Use the Law of Sines to determine the length of side EB.

$$\begin{aligned} \frac{\sin 41^\circ}{EB} &= \frac{\sin 101^\circ}{5{,}382} \\ EB \sin 101^\circ &= 5{,}382 \sin 41^\circ \\ EB &= \frac{5{,}382 \sin 41^\circ}{\sin 101^\circ} \\ EB &\approx 3{,}597 \text{ feet} \end{aligned}$$

Use the Law of Sines to determine the length of side DB.

$$\begin{aligned} \frac{\sin 32^\circ}{DB} &= \frac{\sin 74^\circ}{3{,}862} \\ DB \sin 74^\circ &= 3{,}862 \sin 32^\circ \\ DB &= \frac{3{,}862 \sin 32^\circ}{\sin 74^\circ} \\ DB &\approx 2{,}129 \text{ feet} \end{aligned}$$

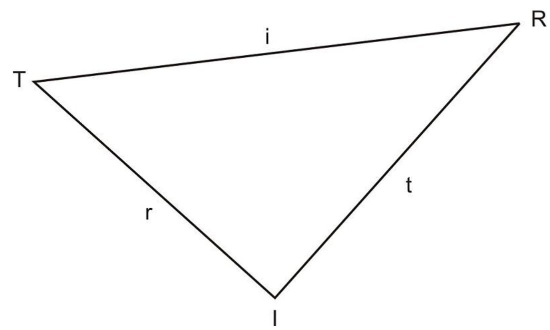
To find the length of AE, start by finding $\angle ABE = 180 - (41 + 101) = 38^\circ$. Then use the Law of Sines.

$$\begin{aligned} \frac{5382}{\sin 101^\circ} &= \frac{AE}{\sin 38^\circ} \\ \sin 38^\circ \cdot \frac{5382}{\sin 101^\circ} &= AE \\ AE &= 3{,}375.5 \text{ feet} \end{aligned}$$

To solve for the last side, find the angle: $7 < 9, 9 \sin 51.06^\circ = 7.00024$. Since there are two equal angles, there will be two equal sides and $DC = BC = 3{,}862$ feet.

ASA (Angle-Side-Angle) Case

The 2nd case to use the Law of Sines is when two angles in a triangle and the included side are known (ASA). For instance, in $\triangle TRI$:



$\angle T, \angle R$, and i are known.

$\angle T, \angle I$,and r are known.

$\angle R, \angle I$, and t are known.

Example 4

In the picture above, $\triangle TRI, \angle T = 83^\circ, \angle R = 24^\circ$, and $i = 18.5$. Find the measure of t.

Solution:

Since two angles and the included side are known, you can find either of the non-included sides using the Law of Sines. Start by finding the 3rd angle using the fact that the sum of all of the angles in a triangle must equal 180°.

$$\angle I = 180 - (83 + 24)$$
$$\angle I = 180 - 107$$
$$\angle I = 73^\circ$$

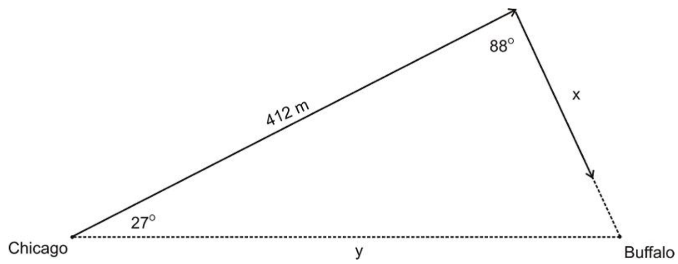
Now use the Law of Sines to find t:

$$\frac{\sin 73^\circ}{18.5} = \frac{\sin 83^\circ}{t}$$
$$t(\sin 73^\circ) = 18.5(\sin 83^\circ)$$
$$t = \frac{18.5(\sin 83^\circ)}{\sin 73^\circ}$$
$$t \approx 19.2$$

Example 5

To avoid a snowstorm on a flight from Chicago to Buffalo, the pilot moved 27° off the normal flight path. After flying 412 miles in this direction, the pilot turned the plane back toward Buffalo. The angle formed by the 1st flight course and the 2nd flight course is 88°. For the pilot, two issues are pressing:

1. What is the total distance of the modified flight path?
2. How much farther did the plane travel than if he had chosen the original course?



Solution:

Part 1: In order to find the total distance of the modified flight path, find side x. To find side x, use the Law of Sines. Since two angles and the included side are known, this is an ASA case.

$$
\begin{aligned}
\text{Missing Angle} &= 180^\circ - (27^\circ + 88^\circ) = 65^\circ && \text{The sum of angles in a triangle is } 180^\circ.\\
\frac{\sin 65^\circ}{412} &= \frac{\sin 27^\circ}{x} && \text{Use Law of Sines.}\\
x(\sin 65^\circ) &= 412(\sin 27^\circ) && \text{Cross multiply.}\\
x &= \frac{412(\sin 27^\circ)}{\sin 65^\circ} && \text{Divide by sin } 65^\circ.\\
x &\approx 206.4 \text{ mi}
\end{aligned}
$$

The total distance of the modified flight path is $412 + 206.4 = 618.4$ mi.

Part 2: To find how much farther the pilot had to travel, find y, the original flight path. Use the Law of Sines again to find y.

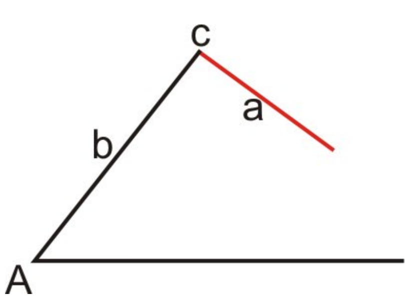
$$
\begin{aligned}
\frac{\sin 65^\circ}{412} &= \frac{\sin 88^\circ}{y} && \text{Use Law of Sines.}\\
y(\sin 65^\circ) &= 412(\sin 88) && \text{Cross multiply.}\\
y &= \frac{412(\sin 88^\circ)}{\sin 65^\circ} && \text{Divide by } \sin 65^\circ.\\
y &\approx 454.3 \text{ mi}
\end{aligned}
$$

John had to travel $618.4 - 454.3 = 164.1$ mi farther.

SSA (Side-Side-Angle) Case

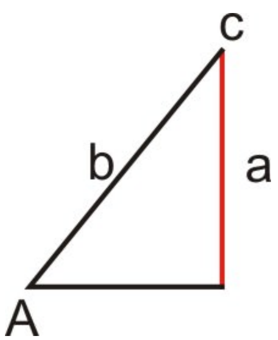
In geometry, two sides and a non-included angle do not necessarily define a unique triangle. Consider the following cases given a, b, and $\angle A$:

Case 1: No triangle exists $(a < b)$



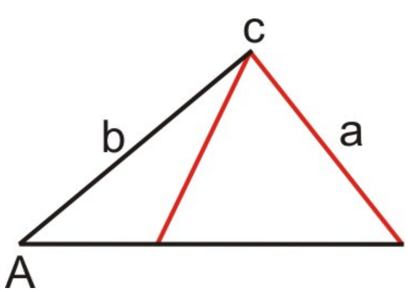
In this case, $a < b$ and side a is too short to reach the base of the triangle. Since no triangle exists, there is no solution.

Case 2: One triangle exists $(a < b)$



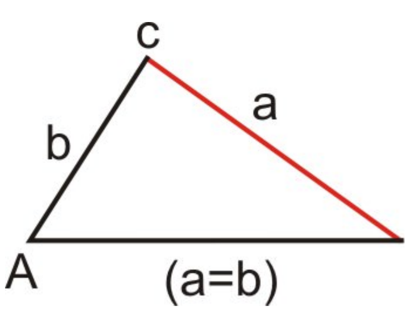
In this case, $a < b$ and side a is perpendicular to the base of the triangle. Since this situation yields exactly one triangle, there is exactly one solution.

Case 3: Two triangles exist $(a < b)$



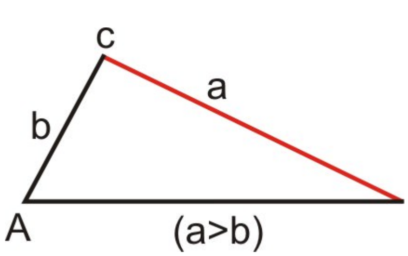
In this case, $a < b$ and side a meets the base at exactly two points. Since two triangles exist, there are two solutions. Note that only the angle opposite the *longest side* of a triangle may be obtuse. To eliminate the need for considering two possible outcomes, avoid using the Law of Sines to determine the largest angle of a triangle.

Case 4: One triangle exists $(a = b)$



In this case, $a = b$ and side a meets the base at exactly one point. Since there is exactly one triangle, there is one solution.

Case 5: One triangle exists $(a > b)$



In this case, $a > b$ and side a meets the base at exactly one point. Since there is exactly one triangle, there is one solution.

Case 3 is referred to as the Ambiguous Case because there are two possible triangles and two possible solutions. One way to check to see how many possible solutions (if any) a triangle will have is to compare sides a and b. In the situation, where $a < b$, determine how many solutions there will be by analyzing a and $b\sin A$.

TABLE 6.6:

	If:	**Then:**
i.	$a < b$	No solution, one solution, two solutions
a.	$a < b\sin A$	No solution
b.	$a = b\sin A$	One solution
c.	$a > b\sin A$	Two solutions
ii.	$a = b$	One solution
iii.	$a > b$	One solution

Example 6

Determine if the sides and angle given determine zero, one, or two triangles. All sets contain an angle, its opposite side, and the side between them.

a. $a = 5, b = 8, A = 62.19^\circ$

Solution:

Even though a, b, and $\angle A$ are not used in every example, follow the same pattern from the table by multiplying the non-opposite side (of the angle) by the sine of the angle.

$5 < 8, 8\sin 62.19^\circ = 7.076$. Since $5 < 7.076$, there is no solution.

b. $c = 14, b = 10, B = 15.45^\circ$

Solution:

$10 < 14, 14\sin 15.45^\circ = 3.73$. Since $10 > 3.73$, there are two solutions.

c. $d = 16, g = 11, D = 44.94^\circ$

Solution:

$16 > 11$, so there is one solution.

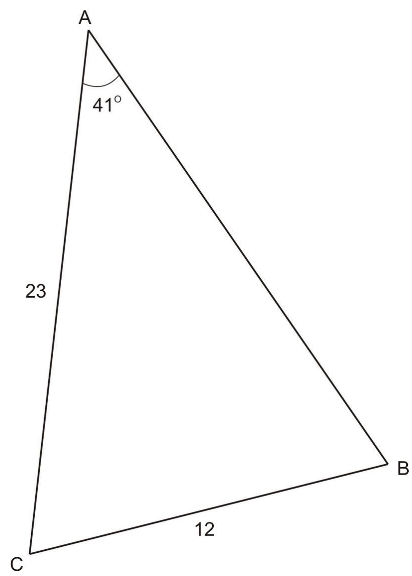
d. $a = 9, b = 7, B = 51.06^\circ$

Solution:

$7 < 9, 9\sin 51.06^\circ = 7.00024$. Since $7 < 7.00024$, there is no solution.

Example 7

Find $\angle B$.



Solution:

Use the Law of Sines to determine the angle.

$$\frac{\sin 41^\circ}{12} = \frac{\sin B}{23}$$
$$23 \sin 41^\circ = 12 \sin B$$
$$\frac{23 \sin 41^\circ}{12} = \sin B$$
$$1.257446472 \approx \sin B$$

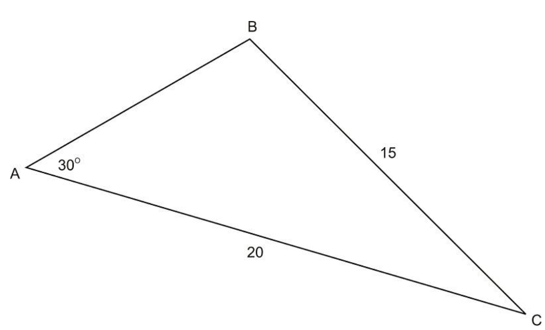
Since no angle exists with a sine greater than 1, there is no solution to this problem.

Similarly, the 1st step could have compared a and $b \sin A$ before proceeding to see how many solutions were possible.

$a = 12, b \sin A = 15.1$: since $12 < 15.1$ and $a < b \sin A$, there are no solutions.

Example 8

In triangle $ABC, a = 15, b = 20$, and $\angle A = 30^\circ$. Find $\angle B$.



Solution:

Again in this case, $a < b$ and two sides and a non-included angle are known. By comparing a and $b \sin A$, $a = 15, b \sin A = 10$. Since $15 > 10$ there will be two solutions to this problem.

$$\frac{\sin 30^\circ}{15} = \frac{\sin B}{20}$$
$$20\sin 30^\circ = 15\sin B$$
$$\frac{20\sin 30^\circ}{15} = \sin B$$
$$0.6666667 \approx \sin B$$
$$\angle B = 41.8^\circ$$

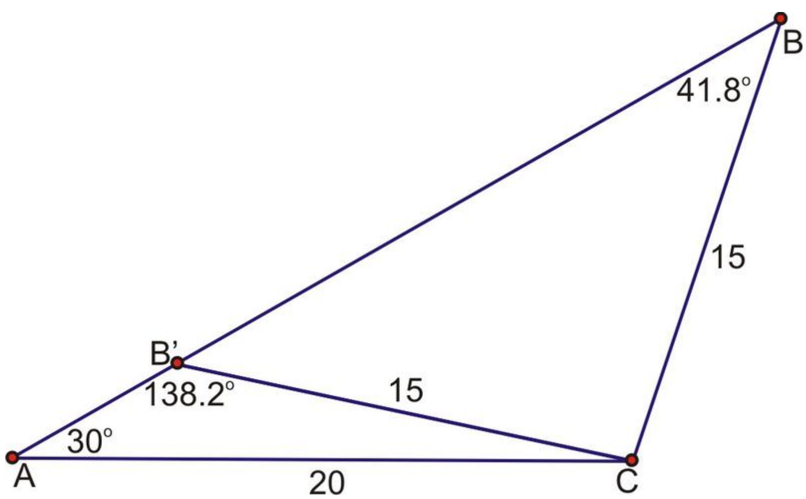
However, there are two angles less than 180° with a sine of 0.6666667. The 1st one is 41.8°, which was found by using the inverse sine function. To find the 2nd one, subtract this 1st solution from 180°.

$$180^\circ - 41.8^\circ = 138.2^\circ$$

To check to make sure 138.2° is a solution, use the Triangle Sum Theorem to find the 3rd angle. All three angles must add up to 180°.

$$180^\circ - (30^\circ + 41.8^\circ) = 108.2^\circ \qquad \text{or} \qquad 180^\circ - (30^\circ + 138.2^\circ) = 11.8^\circ$$

Since both solutions yield a valid 3rd angle with measure between $0^\circ < \theta < 180^\circ$, this problem yields two solutions. Either $\angle B = 41.8^\circ$ or 138.2°.



Summary

- Law of Sines:

$$\text{Form 1}: \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\text{Form 2}: \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

- The Ambiguous Case (SSA): Either zero, one, or two triangles result.

TABLE 6.7:

	If:	Then:
i.	$a < b$	No solution, one solution, two solutions
a.	$a < b\sin A$	No solution
b.	$a = b\sin A$	One solution
c.	$a > b\sin A$	Two solutions
ii.	$a = b$	One solution
iii.	$a > b$	One solution

Review

Find all possible measures of angle B if any exist for each of the following triangle values.

1. $A = 30^\circ, a = 13, b = 15$
2. $A = 42^\circ, a = 21, b = 12$
3. $A = 22^\circ, a = 36, b = 37$
4. $A = 87^\circ, a = 14, b = 12$
5. $A = 31^\circ, a = 25, b = 44$
6. $A = 59^\circ, a = 37, b = 41$
7. $A = 81^\circ, a = 22, b = 20$
8. $A = 95^\circ, a = 31, b = 34$
9. $A = 112^\circ, a = 12, b = 15$
10. $A = 78^\circ, a = 20, b = 16$
11. In $\triangle ABC$, a=10 and $m\angle B = 39^\circ$. What's a possible value for b that would produce two triangles?
12. In $\triangle ABC$, a=15 and $m\angle B = 67^\circ$. What's a possible value for b that would produce no triangles?
13. In $\triangle ABC$, a=21 and $m\angle B = 99^\circ$. What's a possible value for b that would produce one triangle?
14. Bill and Connie are each leaving for school. Connie's house is 4 miles due east of Bill's house. Bill can see the school in the direction 40° east of north. Connie can see the school on a line 51° west of north. What is the straight line distance of each person from the school?
15. Rochelle and Rose are each looking at a hot air balloon. They are standing 2 miles apart. The angle of elevation (meaning the angle from the ground up to the balloon in this case) for Rochelle is 30° and the angle of elevation for Rose is 34°. How high off the ground is the balloon?

Answers for Review Problems

Please see the Appendix.

6.7 Area of a Triangle

Learning Objectives

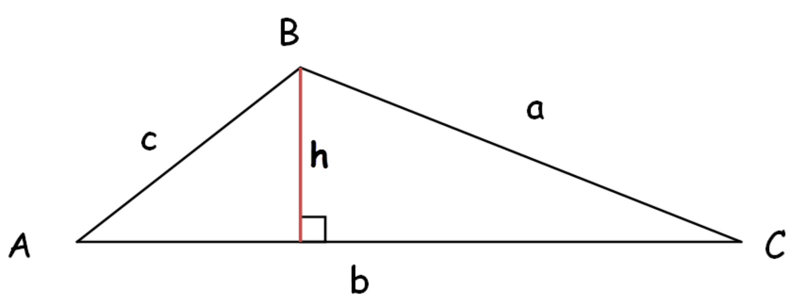
Learn to use the sine ratio to find the area of non-right triangles in which two sides and the included angle measure are known.

Introduction

Trigonometry to this point has allowed us flexibility in analyzing geometric relationships that could be sketched as a combination of triangles. As we continue to explore the triangle, let's assume that an area needs to be calculated for a triangular region where only two sides and the angle between them are able to be measured. The given sides of the triangle are 5 and 6, and the angle between the sides is $\frac{\pi}{3}$ radians. The height of the triangle is not given. Can the area of the triangle be calculated?

Calculating an Area Using Sine

When we know two sides of a triangle and its included angle, we can use the sine function to calculate the area of the triangle. More specifically, the sine function allows the height of any triangle to be calculated, and that value can be used in the familiar triangle area formula.



Using the sine function, isolate the height h:

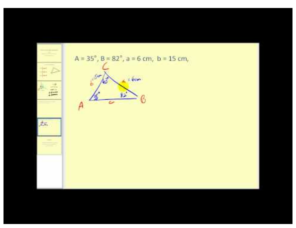
$$\sin C = \frac{h}{a}$$
$$a \sin C = h$$

Substituting into the area formula:

$$\text{Area} = \frac{1}{2} b \cdot h$$
$$\text{Area} = \frac{1}{2} b \cdot a \cdot \sin C$$
$$\text{Area} = \frac{1}{2} \cdot a \cdot b \cdot \sin C$$

Area Formula Given Two Sides & an Included Angle

$$\text{Area} = \frac{1}{2} \cdot a \cdot b \cdot \sin C$$



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Calculating an Area Using Heron's Formula

The area formula for a triangle requires the height of the triangle to be known. However, sometimes we know only the side lengths of a triangle. When we know the three side lengths, Heron's Formula can be used to calculate the area of the triangle.

Since the height of the triangle is unknown, we must do several different calculations to derive Heron's Formula.

Using the Pythagorean Theorem, solve for x in terms of the other two legs of the triangle ADB.

$$x^2 + h^2 = c^2$$
$$x^2 = c^2 - h^2$$
$$x = \sqrt{c^2 - h^2}$$

Using the Pythagorean Theorem, solve for $(b-x)^2$ in terms of the other two legs of the triangle CDB, and then expand $(b-x)^2$.

$$(b-x)^2 + h^2 = a^2$$
$$(b-x)^2 = a^2 - h^2$$
$$b^2 - 2bx + x^2 = a^2 - h^2$$

Since $a,\ b,$ and c are the only known variables, eliminate the variable x using substitution.

$$\begin{aligned}
x &= \sqrt{c^2-h^2} \\
x^2 &= c^2-h^2 \\
b^2-2b\sqrt{c^2-h^2}+(c^2-h^2) &= a^2-h^2 \\
b^2+c^2-a^2 &= 2b\sqrt{c^2-h^2}
\end{aligned}$$

Solve for the height of the triangle, h, using algebra.

$$\begin{aligned}
\left(b^2+c^2-a^2\right)^2 &= \left(2b\sqrt{c^2-h^2}\right)^2 \\
\left(b^2+c^2-a^2\right)^2 &= 4b^2(c^2-h^2) \\
\frac{\left(b^2+c^2-a^2\right)^2}{4b^2} &= c^2-h^2
\end{aligned}$$

$$\begin{aligned}
h^2 &= c^2-\frac{\left(b^2+c^2-a^2\right)^2}{4b^2} \\
h^2 &= \frac{4b^2c^2-\left(b^2+c^2-a^2\right)^2}{4b^2} \\
h^2 &= \frac{(2bc)^2-\left(b^2+c^2-a^2\right)^2}{4b^2} \\
h^2 &= \frac{\left[2bc+\left(b^2+c^2-a^2\right)\right]\left[2bc-\left(b^2+c^2-a^2\right)\right]}{4b^2} \\
h^2 &= \frac{\left[\left(b^2+2bc+c^2\right)-a^2\right]\left[a^2-\left(b^2-2bc+c^2\right)\right]}{4b^2} \\
h^2 &= \frac{\left[(b+c)^2-a^2\right]\left[a^2-(b-c)^2\right]}{4b^2} \\
h^2 &= \frac{[(b+c)+a]\,[(b+c)-a]\,[a+(b-c)]\,[a-(b-c)]}{4b^2} \\
h^2 &= \frac{(a+b+c)\,(b+c-a)\,(a+b-c)\,(a-b+c)}{4b^2} \\
h^2 &= \frac{(a+b+c)\,(a+b+c-2a)\,(a+b+c-2c)\,(a+b+c-2b)}{4b^2} \\
h &= \sqrt{\frac{(a+b+c)\,(a+b+c-2a)\,(a+b+c-2b)\,(a+b+c-2c)}{4b^2}} \\
h &= \frac{\sqrt{(a+b+c)\,(a+b+c-2a)\,(a+b+c-2b)\,(a+b+c-2c)}}{2b}
\end{aligned}$$

Note that the perimeter of the triangle P is equal to $P=a+b+c$.

$$h = \frac{\sqrt{(a+b+c)(a+b+c-2a)(a+b+c-2b)(a+b+c-2c)}}{2b}$$
$$h = \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{2b}$$

The area of the triangle ABC can found be found using the formula $A = \frac{1}{2}bh$.

$$A = \frac{1}{2}b\frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{2b}$$
$$A = \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{4}$$
$$A = \sqrt{\frac{P(P-2a)(P-2b)(P-2c)}{16}}$$
$$A = \sqrt{\frac{P}{2}\left(\frac{P-2a}{2}\right)\left(\frac{P-2b}{2}\right)\left(\frac{P-2c}{2}\right)}$$
$$A = \sqrt{\frac{P}{2}\left(\frac{P}{2}-a\right)\left(\frac{P}{2}-b\right)\left(\frac{P}{2}-c\right)}$$

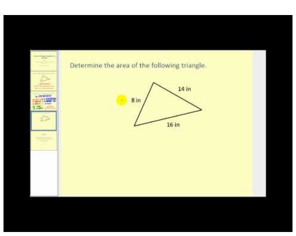
The semiperimeter of the triangle is half the perimeter: $s = \frac{P}{2} = \frac{a+b+c}{2}$.

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Heron's Formula

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } a, b, \text{ and } c \text{ are the side lengths}$$

$$s = \frac{a+b+c}{2}, \text{ where } s \text{ is the semiperimeter of the triangle}$$



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Examples

Example 1

Given ΔABC with $A = 22^\circ$, $b = 6$, and $c = 7$, what is the area?

Solution:

Since two sides and an angle are given, use the sine function to calculate the area.

$$\begin{aligned} Area &= \frac{1}{2} \cdot 6 \cdot 7 \cdot \sin 22^\circ \\ &\approx 7.87 \text{ units}^2 \end{aligned}$$

Example 2

Given that ΔXYZ has an area of 28 in^2, what is the angle included between side length 8 and 9?

Solution:

Since the included angle needs to be determined for two given side lengths, use the sine function.

$$\begin{aligned} \text{Area} &= \frac{1}{2} \cdot a \cdot b \cdot \sin C \\ 28 &= \frac{1}{2} \cdot 8 \cdot 9 \cdot \sin C \\ \sin C &= \frac{28 \cdot 2}{8 \cdot 9} \\ \sin^{-1}(\sin C) &= \sin^{-1}\left(\frac{28 \cdot 2}{8 \cdot 9}\right) \\ C &= \sin^{-1}\left(\frac{28 \cdot 2}{8 \cdot 9}\right) \\ &\approx 51.06^\circ \text{ or } 128.94^\circ \end{aligned}$$

Example 3

What is the area of ΔABC with $A = 31^\circ$, $b = 12$, and $c = 14$?

Solution:

Since two sides and the included angle are given, use the area formula.

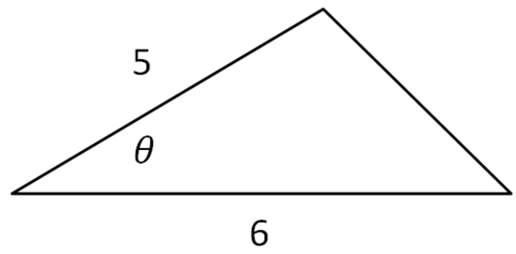
$$Area = \frac{1}{2} \cdot 12 \cdot 14 \cdot \sin 31^\circ \approx 43.26 \text{ units}^2$$

Example 4

In the problem in the Introduction, the sides of the given triangle are 5 and 6, and the angle between the sides is $\theta = \frac{\pi}{3}$. Calculate the area.

Solution:

Since two sides and an angle are given, use the sine function to calculate the area.



$$\text{Area} = \frac{1}{2} \cdot 5 \cdot 6 \cdot \sin\frac{\pi}{3} \approx 12.99 \text{ units}^2$$

Example 5

Calculate the area of ΔABC with side lengths $a = 11$, $b = 12$, and $z = 13$.

Solution:

Since three sides are given, use Heron's Formula to calculate the area.

$$\begin{aligned} s &= \frac{11+12+13}{2} = 18 \\ \text{Area} &= \sqrt{18 \cdot (18-11) \cdot (18-12) \cdot (18-13)} \\ \text{Area} &= \sqrt{18 \cdot (7) \cdot (6) \cdot (5)} \\ \text{Area} &= \sqrt{3{,}780} \\ \text{Area} &\approx 61.48 \text{ units}^2 \end{aligned}$$

Example 6

The area of a triangle is 3 units^2. Two sides of the triangle are 4 units and 5 units. What is the measure of their included angle?

Solution:

Since the included angle needs to be determined for two given side lengths, use the sine function.

$$\begin{aligned} 3 &= \frac{1}{2} \cdot 4 \cdot 5 \cdot \sin\theta \\ \theta &= \sin^{-1}\left(\frac{3 \cdot 2}{4 \cdot 5}\right) \\ &\approx 17.46^\circ \text{ and } 162.54^\circ \end{aligned}$$

Example 7

Given ΔABC with $A = 12^\circ$, $b = 4$, and Area $= 1.7$ units^2, what is the length of side c?

Solution:

Since a side length, an angle, and an area are given, use the sine function.

$$\begin{aligned} Area &= \frac{1}{2} \cdot c \cdot b \cdot \sin A \\ 1.7 &= \frac{1}{2} \cdot c \cdot 4 \cdot \sin 12^\circ \\ c &= \frac{1.7 \cdot 2}{4 \cdot \sin 12^\circ} \approx 4.09 \text{ units} \end{aligned}$$

Summary

- If two sides of a triangle and an included angle are known, the sine function can be used to calculate the area: $A = \frac{1}{2} ab \sin C$.
- If three sides of a triangle are known, Heron's Formula can be used to calculate the area: Area $= \sqrt{s(s-a)(s-b)(s-c)}$ wl the semiperimeter $s = \frac{a+b+c}{2}$.

Review

For 1-11, calculate the area of each triangle.

1. ΔABC if $a = 13, b = 15$, and $\angle C = 70^\circ$.
2. ΔABC if $b = 8, c = 4$, and $\angle A = 58^\circ$.
3. ΔABC if $b = 34, c = 29$, and $\angle A = 125^\circ$.
4. ΔABC if $a = 3, b = 7$, and $\angle C = 81^\circ$.
5. ΔABC if $a = 4.8, c = 3.7$, and $\angle B = 54^\circ$.
6. ΔABC if $a = 12, b = 5$, and $\angle C = 22^\circ$.
7. ΔABC if $a = 3, b = 10$, and $\angle C = 65^\circ$.
8. ΔABC if $a = 5, b = 9$, and $\angle C = 11^\circ$.
9. ΔABC if $a = 5, b = 7$, and $c = 8$.
10. ΔABC if $a = 7, b = 8$, and $c = 14$.
11. ΔABC if $a = 12, b = 14$, and $c = 13$.
12. The area of a triangle is 12 square units. Two sides of the triangle are 8 units and 4 units. What is the measure of their included angle?
13. The area of a triangle is 23 square units. Two sides of the triangle are 14 units and 5 units. What is the measure of their included angle?
14. Given ΔDEF has area 32 square inches, what is the angle included between side length 9 and 10?
15. Given ΔGHI has area 15 square inches, what is the angle included between side length 7 and 11?

Review (Answers)

Please see the Appendix.

6.8 Applications of Basic Triangle Trigonometry

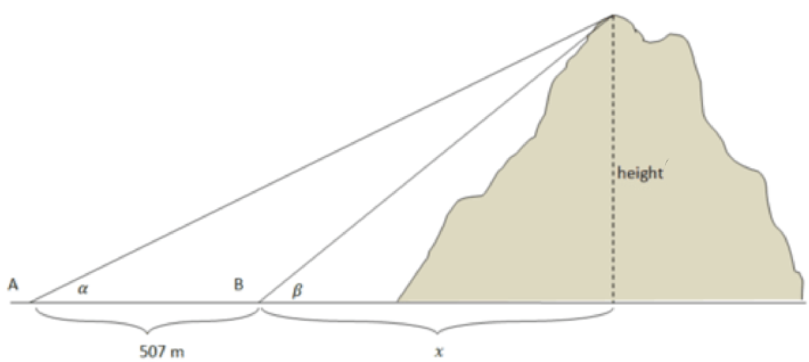
Learning Objectives

Learn to apply your knowledge of trigonometry and problem solving in context.

Introduction

A surveying crew is given the job of verifying the height of a cliff. From point A, they measure an angle of elevation to the top of the cliff to be $\alpha = 21.567^\circ$. They move 507 meters closer to the cliff and find that the angle to the top of the cliff is now $\beta = 25.683^\circ$. How tall is the cliff?

First, sketch the image and label what you know.

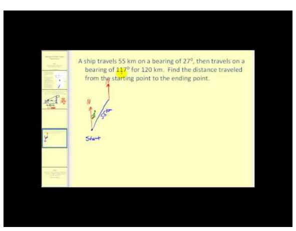


Deciding when to use the right triangle trigonometric functions, Law of Cosines or the Law of Sines, is not always obvious. Sometimes more than one approach will work, and sometimes even correct computations can lead to incorrect results. This is because correct interpretation is still essential.

Solving Problems Using Right Triangle Trigonometry

Right triangles can be used to calculate the angle of elevation or depression in real-world applications. The **angle of elevation** is the angle at which you view an object above the horizon. The **angle of depression** is the angle at which you view an object below the horizon. This can be thought of as a negative angle of elevation.

Bearings in navigation can also be used to determine distances. **Bearing** is how direction is measured in navigation. On a compass, North is 0°, East is 90°, South is 180°, and West is 270°. For example, a ship traveling at a bearing of 75° is traveling 75° from North towards East. This ship could also be said to be traveling $90^\circ - 75^\circ = 15^\circ$ from East on the compass towards North.



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How to Solve Triangles

When applying trigonometry, it is important to have a full toolbox of mathematical techniques to use:

- The Pythagorean Theorem states that for legs a, b and hypotenuse c in a right triangle, $a^2 + b^2 = c^2$.
- There are 360° in a circle. Negative angles may need to be re-expressed as positive angles using an angle in this range.
- The three angles in a triangle add up to 180°.
- 30-60-90 right triangles have side ratios $x, x\sqrt{3}, 2x$.
- 45-45-90 right triangles have side ratios $x, x, x\sqrt{2}$.
- Pythagorean number triples are exceedingly common and are helpful if recognized in right triangle problems. Examples of triples are 3, 4, 5 and 5, 12, 13.
- The Triangle Inequality Theorem states that for any triangle, the sum of any two of the sides must be greater than the 3rd side.
- SOH CAH TOA is a mnemonic device to help you remember the three right triangle trig functions:

$$\sin\theta = \frac{opp}{hyp} \quad \cos\text{[U+0081]}\theta = \frac{adj}{hyp} \quad \tan\text{[U+0081]}\theta = \frac{opp}{adj}$$

- The Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$
- The Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B}$ or $\frac{\sin A}{a} = \frac{\sin B}{b}$ (Consider the Ambiguous Case.)
- Area Formula: Area $= \frac{1}{2} \cdot a \cdot b \sin C$
- Heron's Formula: Area $= \sqrt{s(s-a)(s-b)(s-c)}$ with $s = \frac{a+b+c}{2}$.

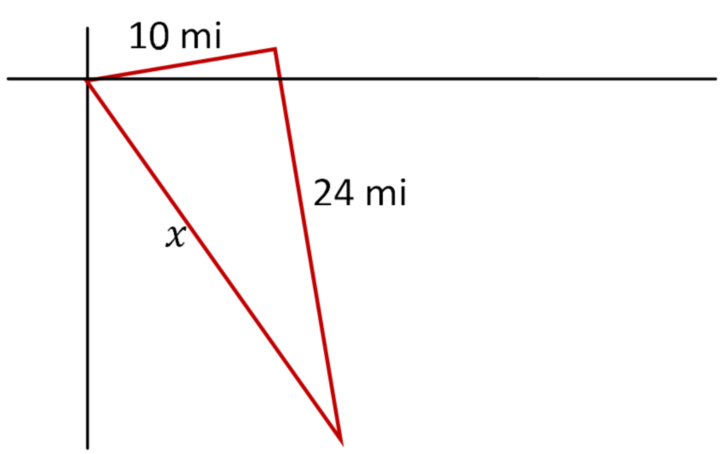
Examples

Example 1

Bearing is the term used to describe how direction is measured at sea. North is 0°, East is 90°, South is 180°, and West is 270°. A ship travels 10 miles at a bearing of 88° and then turns 90° to the right to avoid an iceberg for 24 miles. How far is the ship from its original position?

Solution:

First, draw a clear sketch.



Next, recognize the right triangle with legs 10 and 24. This is a multiple of the 5, 12, 13 Pythagorean triple, so the distance x must be 26 miles.

Example 2

Returning to the surveyor's question, how tall is the cliff?

Solution:

The height is measured at a right angle with the ground. Set up two equations. Remember that α and β are fixed values, not variables.

$$\tan\alpha = \frac{h}{507+x}$$
$$\tan\beta = \frac{h}{x}$$

Solve both of these equations for h, and then set them equal to each other to find x.

$$h = \tan \text{[U+0081]}\,\alpha(507+x) = x\tan\beta$$
$$507\tan\alpha + x\tan\alpha = x\tan \text{[U+0081]}\,\beta$$
$$507\tan\alpha = x\tan\beta - x\tan\alpha$$
$$507\tan\alpha = x(\tan \text{[U+0081]}\,\beta - \tan\alpha)$$
$$x = \frac{507\tan\alpha}{\tan \text{[U+0081]}\,\beta - \tan\alpha} = \frac{507\tan 21.567^\circ}{\tan 25.683^\circ - \tan 21.567^\circ} \approx 2{,}339.997 \text{ m}$$

Since the problem asked for the height, substitute x back and solve for h.

$h = x\tan\beta = 2339.997\tan 25.683^\circ \approx 1{,}125.31$ m

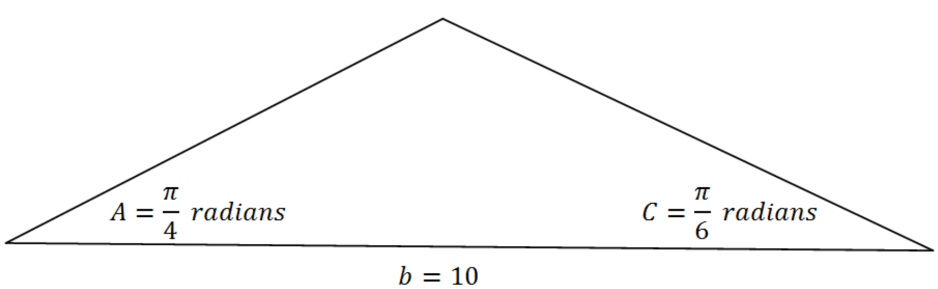
Example 3

Given a triangle with SSS or SAS, you know how to use the Law of Cosines. In triangles where there are corresponding angles and sides like AAS or SSA, it makes sense to use the Law of Sines. What about ASA?

Given ΔABC with $A = \frac{\pi}{4}$ radians, $C = \frac{\pi}{6}$ radians, and $b = 10$ in. What is a?

Solution:

First, draw a picture.



The sum of the angles in a triangle is 180°. Since this problem is in radians, convert each angle to degrees.

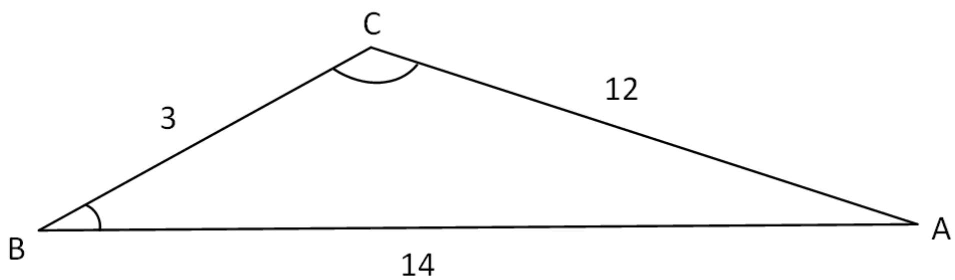
$$A = \frac{\pi}{4} \cdot \frac{180^\circ}{\pi} = 45^\circ$$
$$C = \frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = 30^\circ$$

The missing angle must be $\angle B = 105^\circ$. Now use the Law of Sines to solve for a.

$$\frac{\sin 105^\circ}{10} = \frac{\sin 45^\circ}{a}$$
$$a = \frac{10 \sin 45^\circ}{\sin 105^\circ} \approx 7.32 \text{ in}$$

Example 4

Sometimes when using the Law of Sines, you find the results are not consistent with the Law of Cosines. Both answers can be correct computationally, but the Law of Sines may require further analysis when the triangle is obtuse. The Law of Cosines does not require this 2nd step.



First, use the Law of Cosines to find $\angle B$.

$$12^2 = 3^2 + 14^2 - 2 \cdot 3 \cdot 14 \cdot \cos B$$
$$\angle B = \cos^{-1}\left(\frac{12^2 - 3^2 - 14^2}{-2 \cdot 3 \cdot 14}\right) \approx 43.43^\circ$$

Next, use the Law of Sines to find $\angle C$. Use the unrounded value for B, even though a rounded value is shown.

$$\frac{\sin 43.43^\circ}{12} = \frac{\sin C}{14}$$
$$\frac{14 \sin 43.43^\circ}{12} = \sin C$$
$$\angle C = \sin^{-1}\left(\frac{14 \sin 43.43^\circ}{12}\right) \approx 53.3^\circ$$

Use the Law of Cosines to doublecheck $\angle C$.

$$14^2 = 3^2 + 12^2 - 2 \cdot 3 \cdot 12 \cdot \cos C$$
$$C = \cos^{-1}\left(\frac{14^2 - 3^2 - 12^2}{-2 \cdot 3 \cdot 12}\right) \approx 126.7^\circ$$

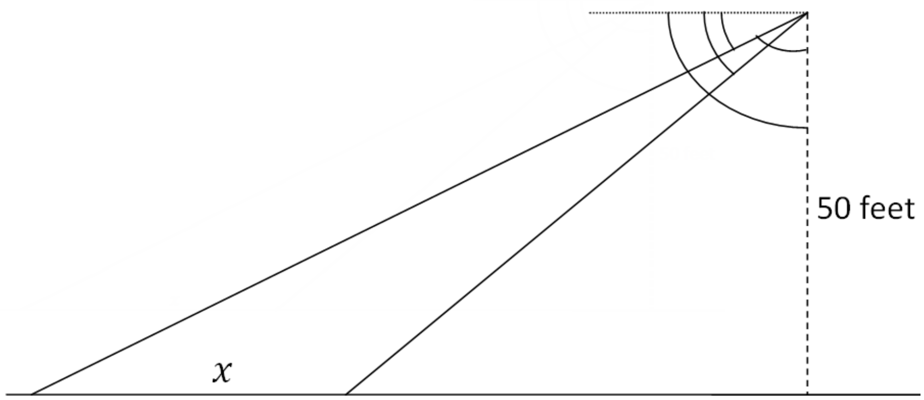
Notice that the last two answers do not match, but they are supplementary. This is because this triangle is obtuse, and the $\sin^{-1}\left(\frac{opp}{hyp}\right)$ function is restricted to only producing acute angles. Don't forget to check for the ambiguous case when using the Law of Sines.

Example 5

From the 3rd story of a 50-foot building, someone observes a car driving on the street below towards the building. If the angle of depression of the car changes from 21° to 45° while the person watches, how far did the car travel?

Solution:

Draw a very clear picture.



In the upper right corner of the picture, four important angles are marked. The measures of these angles from the outside in are $90^\circ, 45^\circ, 21^\circ, 69^\circ$. There is a 45-45-90 right triangle on the right, so the base must also be 50. Therefore, set up and solve an equation for x.

$$\tan 69^\circ = \frac{x+50}{50}$$
$$x = 50\tan 69^\circ - 50 \approx 80.25 \text{ ft}$$

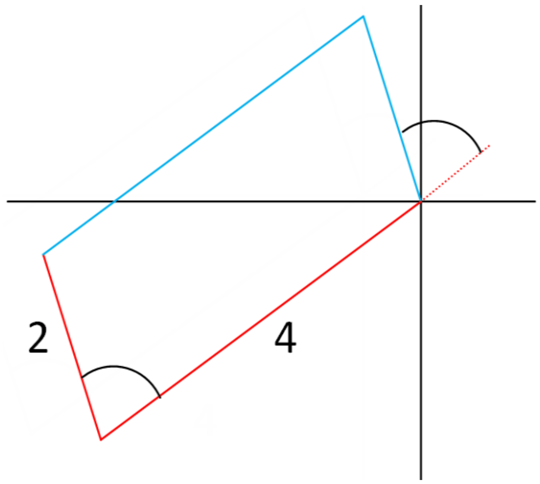
Often, the hardest parts of this problem are drawing a picture and working with the angles.

Example 6

If a boat travels 4 miles SW and then 2 miles NNW, how far away is it from its starting point?

Solution:

Translate SW and NNW into degrees-bearing. SW is a bearing of 225°, and NNW is a bearing of 337.5°. Draw a picture in two steps. Draw the original 4 miles traveled, and draw the 2nd 2 miles traveled from the origin. Then translate the 2nd leg of the trip so it follows the 1st leg. This way you end up with a parallelogram, which has interior angles that are easier to calculate.



The angle between the two red line segments is 67.5°, which can be seen if the red line is extended past the origin and you find the complement of the difference between the two bearings.

$$180^\circ - (337.5^\circ - 225^\circ) = 67.5^\circ$$

The shorter diagonal of the parallelogram is the required unknown information.

$$x^2 = 4^2 + 2^2 - 2 \cdot 4 \cdot 2 \cdot \cos 67.5^\circ$$
$$x \approx 3.7 \text{ m}$$

Summary

- When working with triangles, numerous mathematical techniques can be used, including trigonometry.
- **Angle of elevation** is the angle at which you view an object above the horizon.
- **Angle of depression** is the angle at which you view an object below the horizon. This can be thought of as a negative angle of elevation.
- **Bearing** is how direction is measured in navigation. North is 0°, East is 90°, South is 180°, and West is 270°.

Review

The angle of depression of a boat in the distance from the top of a lighthouse is $\frac{\pi}{6}$. The lighthouse is 150 feet tall. Find the distance from the base of the lighthouse to the boat.

1. Draw a picture of this situation.

2. What methods or techniques will you use?

3. Solve the problem.

From the 3rd story of a building (60 feet), Jeff observes a car moving towards the building driving on the streets below. The angle of depression of the car changes from 34° to 62° while he watches. How far has the car traveled?

4. Draw a picture of this situation.
5. What methods or techniques will you use?
6. Solve the problem.

A boat travels 6 miles NW and then 2 miles SW. You want to know how far away the boat is from its starting point.

7. Draw a picture of this situation.
8. What methods or techniques will you use?
9. Solve the problem.

You want to figure out the height of a building. From point A, you measure an angle of elevation to the top of the building to be $\alpha = 10^\circ$. You move 50 feet closer to the building to point B, and find that the angle to the top of the building is now $\beta = 60^\circ$.

10. Draw a picture of this situation.
11. What methods or techniques will you use?
12. Solve the problem.
13. Given ΔABC with $A = 40^\circ, C = 65^\circ$ and $b = 8$ in, what is a?
14. Given ΔABC with $A = \frac{\pi}{3}$ radians, $C = \frac{\pi}{8}$ radians and $b = 12$ inches what is a?
15. Given ΔABC with $A = \frac{\pi}{6}$ radians, $C = \frac{\pi}{4}$ radians and $b = 20$ inches what is a?

Review (Answers)

Please see the Appendix.

6.9 Project: Basic Triangle Trigonometry

Baseball has often been called America's national pastime. The game is played on a field that consists of four bases. These four bases create the corners of a large square that is called the baseball diamond. Each side of the diamond has a length of 94 feet. Right triangle trigonometry can be used to determine how far a ball must be thrown between bases. The Law of Cosines and Law of Sines can also be used to determine how far a ball must be thrown, and how far a baseball player needs to run from other locations on the field.

1. If the baseball player standing at 3rd base has the ball and wants to throw it to the 1st baseman, what is the shortest distance the ball can travel from 3rd base to 1st base?
2. If the catcher is attempting to throw out a runner trying to steal 2nd base, how far would the catcher need to throw the ball if he stands 3 feet behind home plate?
3. A standard pitcher's mound is 60 feet and 6 inches from home plate. Break the diamond into four triangles using the pitcher's mound as one of the vertices of each triangle. Determine if the pitcher's mound is precisely in the middle of the baseball diamond by calculating the distance from all the bases.
4. During a baseball game, an outfielder catches a ball hit to dead center field, 400 feet from home plate. How far does the outfielder have to throw the ball to get it to 1st base?
5. During a televised baseball game, the outfielder in center field is playing approximately 330 feet from the television camera that is behind home plate. A batter hits a fly ball that goes to the wall, 420 feet from the camera. How far does the center fielder need to run to make the catch if the camera turns 8° to follow the play?

6.10 Summary: Basic Triangle Trigonometry

When we apply trigonometry, it is important to have a complete toolbox of mathematical techniques to use. Trigonometric functions extend Euclidean geometry to solve problems in many areas, such as surveying and astrophysics.

Chapter Summary

In this chapter we learned about:

Special Right Triangles

- A 30-60-90 right triangle has side ratios $x, x\sqrt{3},$ and $2x$.
- A 45-45-90 right triangle has side ratios $x, x,$ and $x\sqrt{2}$.
- Pythagorean triples are special right triangles with integer sides.

Right Triangle Trigonometry

- $\sin\theta = \frac{opp}{hyp}$
- $\cos\theta = \frac{adj}{hyp}$
- $\tan\theta = \frac{opp}{adj}$
- $\cot\theta = \frac{adj}{opp}$
- $\sec\theta = \frac{hyp}{adj}$
- $\csc\theta = \frac{hyp}{opp}$

Law of Cosines

- $a^2 = b^2 + c^2 - 2bc\cos A$
  $b^2 = a^2 + c^2 - 2ac\cos B$
  $c^2 = a^2 + b^2 - 2ab\cos C$

Law of Sines

- Form 1 : $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

  Form 2 : $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Area Formulas

- $\text{Area} = \frac{1}{2} \cdot a \cdot b \sin C$
- Heron's Formula: $\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$ with $s = \frac{a+b+c}{2}$

Review

Try the following cumulative review problems to practice the concepts in this chapter:

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6.11 References

1. Pexels. https://pixabay.com/photo-1283012/ .
2. . 45-45-90 Triangle.
3. CK-12. 30-60-90 Triangle .
4. CK-12. Special Traingles .
5. Electronic Library of Construction Occupational Safety and Health. Safe Ladder Positioning .
6. CK-12 Foundation. .
7. Jim, the Photographer. https://www.flickr.com/photos/jcapaldi/8648949913/ .
8. CK-12. Kitchen Diagram .
9. CK-12. Proof of Law Cosine .
10. CK-12. Example 1 .
11. CK-12. Example 3 .
12. CK-12. Example 4 .
13. CK-12. Example 5 .
14. CK-12. Example 6 .
15. CK-12. Example 7 .
16. CK-12 Foundation. Law of Sines Motivation .
17. CK-12. Derivation Law of Sines .
18. CK-12. Example 1 .
19. . Law of Sines Motivation.
20. CK-12. Angle Side Angle Case .
21. CK-12. Example 5 .
22. . LOS - No Triangle.
23. . LOS - 1 Triangle.
24. . LOS - 2 Triangles.
25. . Example 2.
26. . Example 3.
27. CK-12 Foundation. Area Formula Diagram .
28. . Example 4.
29. . Motivating Application.
30. Margaret W. Carruthers. https://commons.wikimedia.org/wiki/File:Compass_Rose,_Fort_McHenry.jpg .
31. CK-12. Example 1 .
32. CK-12. Example 3 .
33. CK-12. Example 4 .
34. CK-12. Example 5 .
35. CK-12. Example 6 .
36. Emily Chris. https://www.flickr.com/photos/chrisandemily/2868452221/in/photolist-5ntyUM-bUwN5h-bUxmBb-bUyV21-bUx3NS-bUxGjy-bUwXiY-bUwK9o-bUwyw5-bUxkSS-bUxuRw-bUxmey-bUxXXU-bUxsgo-bUx6Sd-bUwAjb-bUyja5-bUy8Ef-bUxHaY-bUxXsN-bUwMas-bUxQFW-bUwVim-bUxqsY-bUyfHw-bUxMz7-bUyf1q-bUwPhy-bUwyVY-bUxqTb-bUyTvW-bUyRi7-bUxBDA-bUywCf-bUwE27-bUz8r9-bUziUd-bUxGJN-bUxvHU-bUxMZN-bUz8VW-bUxkry-bUxtLw-bUxf4m-bUxsEQ-bUwzmo-bUxzRj-bUxoVd-bUxur3-bUxdAY .

CHAPTER 7 The Unit Circle and Trigonometric Functions

Chapter Outline

7.1 Introduction: The Unit Circle and Trigonometric Functions

Many researchers model data that has a cyclical and repeating nature by using trigonometric functions. For instance, Stuart Kenny, an early scientist, coined the term "sinusoidal" while observing the growth and harvest of soybeans. He noticed that since the growth and harvest pattern was directly connected to the time of the year, the plant maturity reached the same level every few months. That made its maturity level periodic, with a period length of 60 days.

Other researchers have noticed the same repetitive nature with ocean tides and seasonal temperature. Even the height of a person on a Ferris wheel can be modeled through the use of sine and cosine functions. In this chapter, we will study characteristics of the six basic trigonometric functions, such as how these periodic functions all repeat their pattern over time. As with all function families, the characteristics will be established and applied to real-world examples.

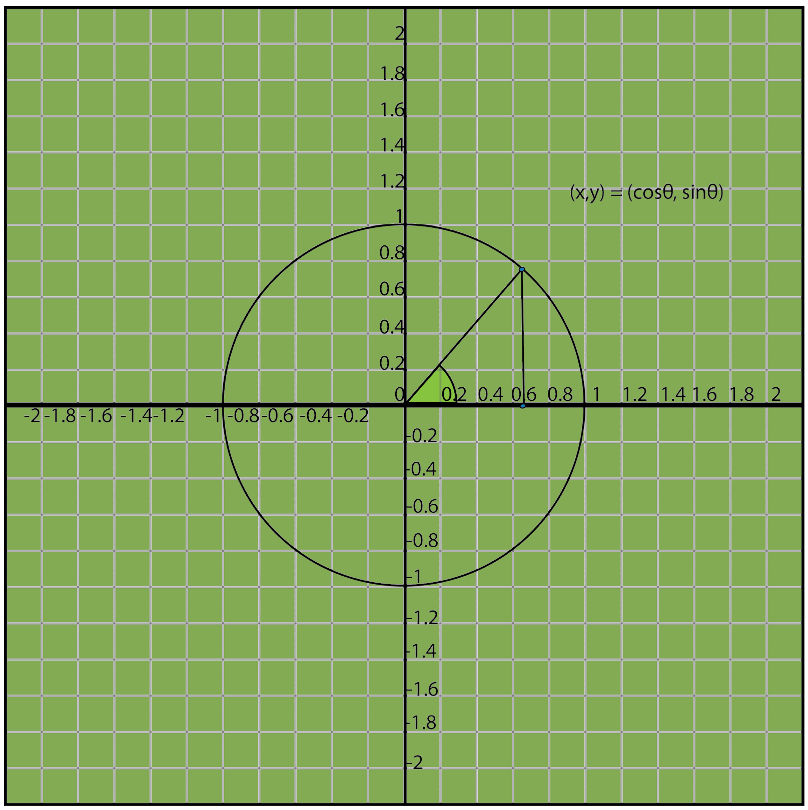
7.2 The Unit Circle

Learning Objectives

Learn to use your knowledge of basic triangle trigonometry to identify key points and angles around a circle of radius one centered at the origin.

Introduction

The unit circle is a circle of radius 1 that is centered at the origin.

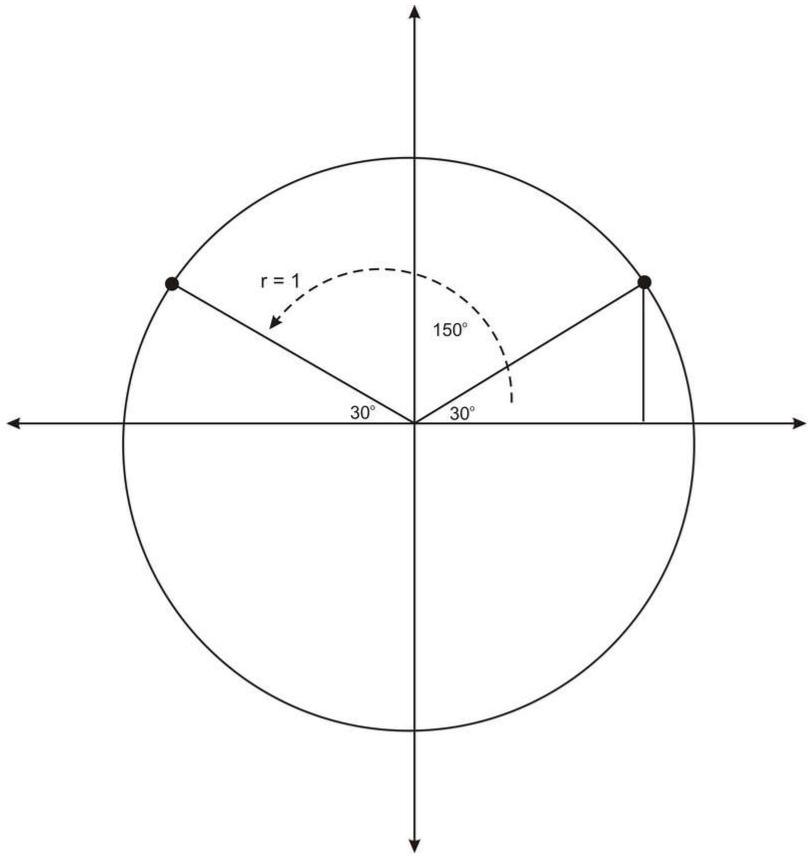


Sketch the central angle (θ) in standard position. This angle is formed by the x-axis, the origin, and the terminal side of the angle. Its terminal side intersects the unit circle at (x,y). Since the radius is one, $\cos\theta = \frac{x}{1} = x$ and $\sin\theta = \frac{y}{1} = y$. Thus, $(x,y) = (\cos\theta, \sin\theta)$.

This framework makes it easy to evaluate expressions like $\cos(135^\circ)$ or $\sin\left(-\frac{5\pi}{3}\right)$. It also helps to produce the parent graphs of sine and cosine.

Reference Angles

Consider the angle 150°. With this angle in standard position, the terminal side of this angle is a reflection (across the $y-$axis) of an angle with terminal side of 30°.

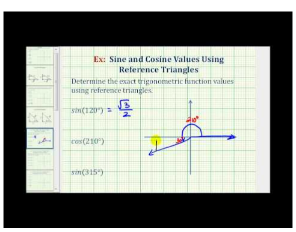


Notice that in the preceding figure, the terminal side of the 150° angle makes a 30° angle with the negative $x-$axis. Therefore, 30° is the nonnegative **reference angle** for 150°.

Formally, the **reference angle** of an angle in standard position is the angle formed with the terminal side of the angle, and the part of the $x-$axis closest to the terminal side.

Notice that 30° is the reference angle for many angles. For example, it is the reference angle for 210° and for -30°. Since the coordinates of the point on the terminal side of 30° and the unit circle are $(\cos 30^\circ, \sin 30^\circ) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, the corresponding reflected point of the terminal side of 150° is $(\cos 150^\circ, \sin 150^\circ) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Identifying the reference angle for an angle is used to determine the values of the trig functions of the angle. Once the reference angle is graphed, a right triangle can be drawn using the reference angle and the $x-$axis. The values of the trigonometric functions of the angle can then be determined using the resulting reference triangle.



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Create the Unit Circle for Special Angles

To be ready to completely fill in a unit circle, we need to review two special triangles. Start by finding the side lengths of a 30-60-90 triangle and a 45-45-90 triangle, each with a hypotenuse equal to 1.

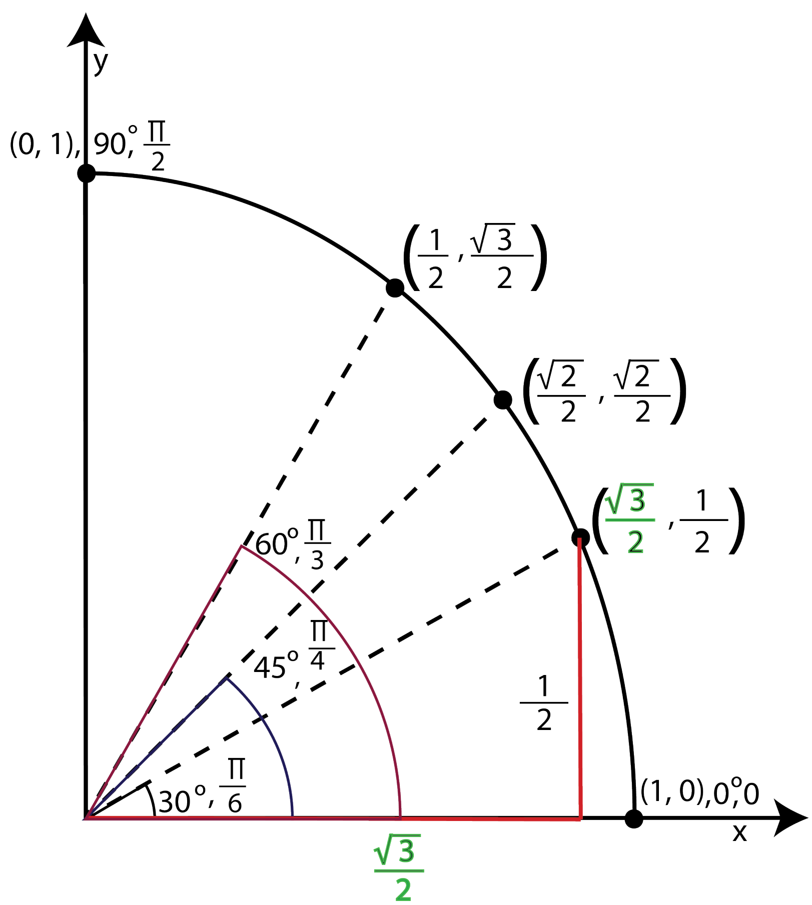
TABLE 7.1:

30°	60°	90°
x	$x\sqrt{3}$	$2x$
$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

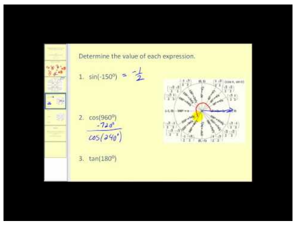
TABLE 7.2:

45°	45°	90°
x	x	$x\sqrt{2}$
$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1

This is enough information to fill out the important points in the 1st quadrant of the unit circle. The values of the x- and y-coordinates for each of the key points are shown below. Remember that the x- and y-coordinates come from the lengths of the legs of the special right triangles, as shown specifically for the 30° angle. The hypothenuse of any right triangle in the unit circle is the same length as the radius of the unit circle, $r = 1$. Always remember to measure the angle from the positive portion of the x-axis.



Knowing the 1st quadrant well is the key to knowing the entire unit circle. Every other point on the unit circle can be found by reflecting the values in the 1st quadrant.



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Examples

Example 1

Graph each angle and identify its reference angle.

a. 140°

Solution:

140° makes a 40° angle with the negative $x-$axis. Therefore, the reference angle is 40°.

b. 240°

Solution:

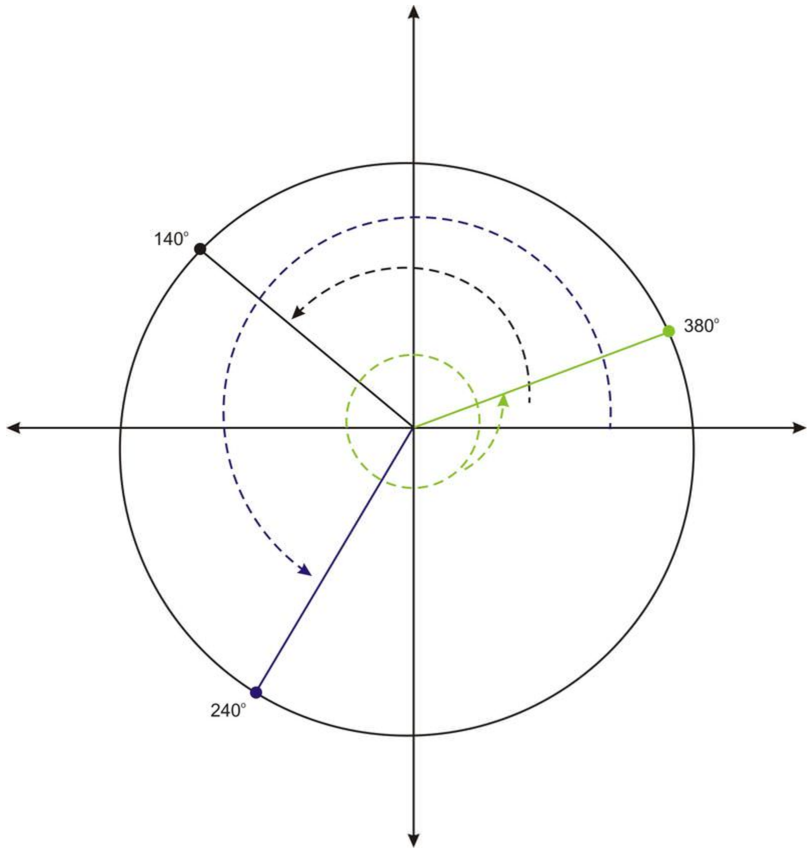
240° makes a 60° angle with the negative $x-$axis. Therefore, the reference angle is 60°.

c. 380°

Solution:

380° is a full rotation of 360°, plus an additional 20°. So this angle is coterminal with 20°, and 20° is its reference angle.

Graph for a, b, and c:

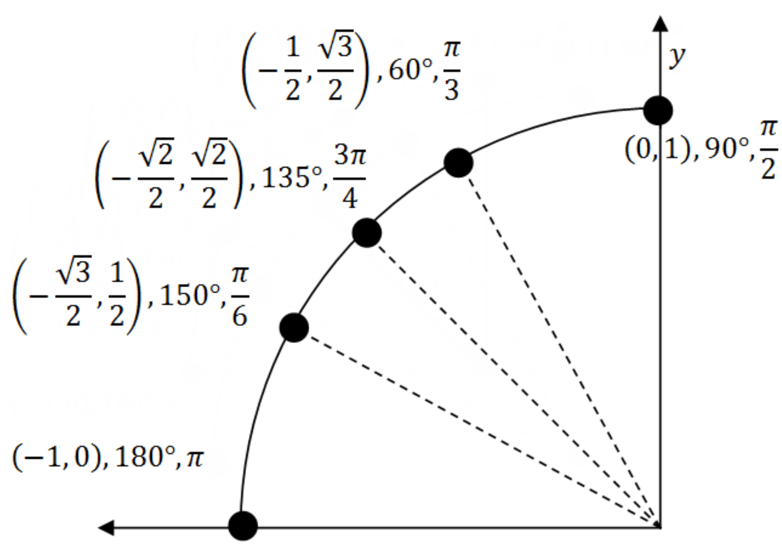


Example 2

Use the 1st quadrant of the unit circle to identify the angles and important points of the 2nd quadrant.

Solution:

The heights are equal and correspond to the y-values. The x-values are all negative.



Example 3

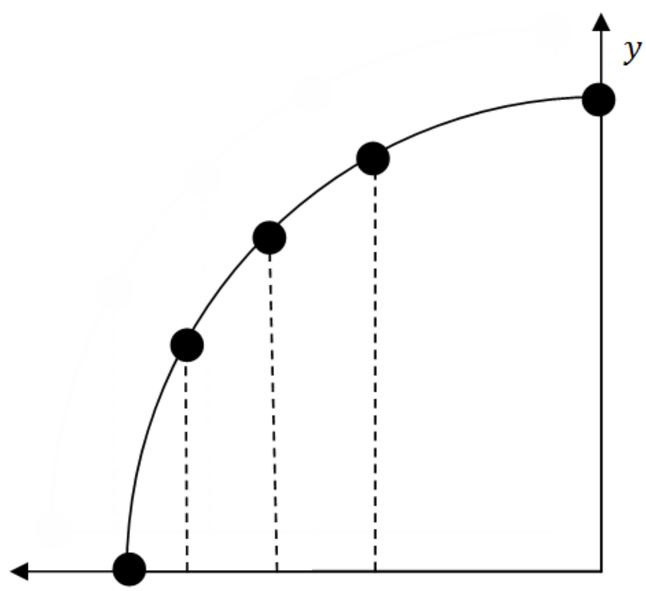
Identify a pattern in the heights of the points in the 1st quadrant as a memory device.

Solution:

The heights of the points in the 1st quadrant are the y-coordinates, which are $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$.

When rewritten, the pattern becomes clear: $\frac{0}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$.

The three points in the middle are the most often confused. This pattern illustrates how they increase in size from small $\frac{1}{2}$, to medium $\frac{\sqrt{2}}{2}$, to large $\frac{\sqrt{3}}{2}$. When completing the unit circle, look for the heights that are small, medium, and large to position trig values correctly. Notice that the heights for these five points in the 2nd quadrant are also $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$.



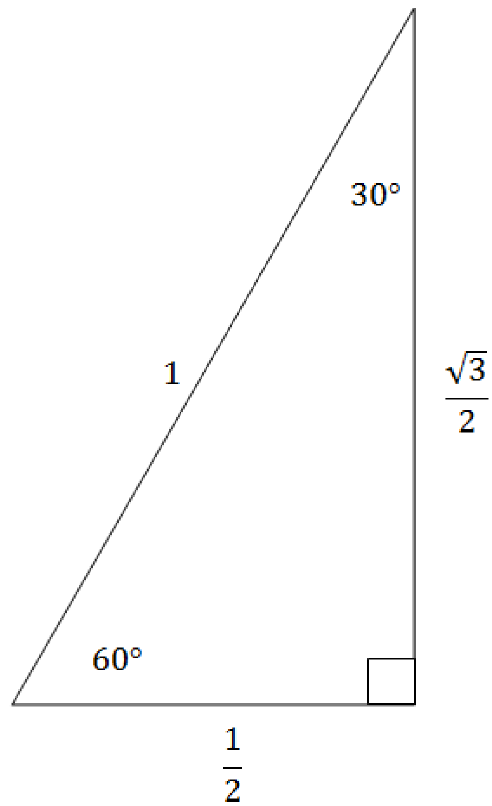
This technique also works for the widths. This can make memorizing the 16 points of the unit circle a matter of reflecting the pattern: $\frac{0}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$.

Example 4

Evaluate $\cos 60^\circ$ using the unit circle and right triangle trigonometry. What is the connection between the x-coordinate of the point and the cosine of the angle?

Solution:

The point on the unit circle for 60° is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and the point is 1 unit from the origin. This can be represented as a 30-60-90 triangle.



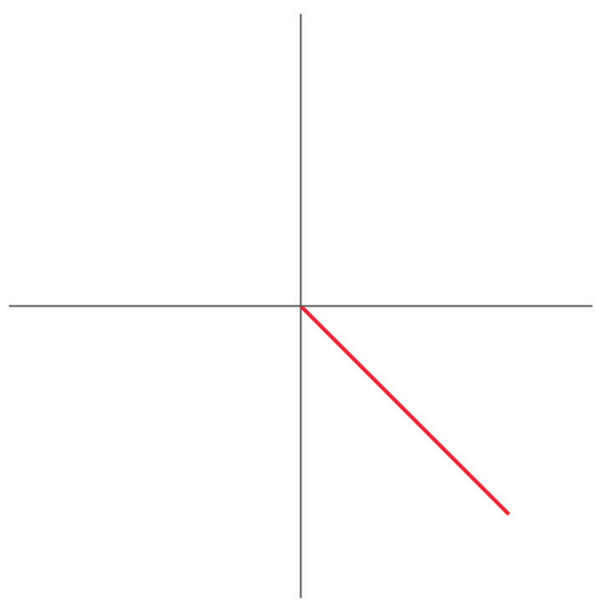
Since cosine is adjacent over hypotenuse, cosine turns out to be exactly the x coordinate $\frac{1}{2}$.

Example 5

Graph 315° and identify its reference angle. Find $\sin 315^\circ$.

Solution:

The graph of 315° is as follows:



Since the angle makes a 45° angle with the positive x-axis, the reference angle is 45°. Using the unit circle, the ordered pair associated with 315° is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, so $\sin 315^\circ = -\frac{\sqrt{2}}{2}$ in standard position.

Example 6

Find the ordered pair on the unit circle for 150° and use it to find the value of cos 150°.

Solution:

Step 1: Identify that the reference angle is 30°.

Step 2: Determine the coordinates for the point on the unit circle are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Step 3: Substitute:

$$\cos 150^\circ = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{-\frac{\sqrt{3}}{2}}{1} = -\frac{\sqrt{3}}{2}$$

Example 7

Evaluate the following trigonometric expressions using the unit circle. Remember: $\csc\left(-\frac{5\pi}{4}\right) = \frac{1}{\sin\left(-\frac{5\pi}{4}\right)} = \frac{2}{\sqrt{2}} = \sqrt{2}$

a. $\sin \frac{\pi}{2}$

Solution:

$\sin \frac{\pi}{2} = 1$

b. $\cos 210^\circ$

Solution:

$\cos 210^\circ = -\frac{\sqrt{3}}{2}$

c. $\tan 315^\circ$

Solution:

$\tan 315^\circ = -1$

d. $\cot 270^\circ$

Solution:

$\cot 270^\circ = 0$

e. $\sec \frac{11\pi}{6}$

Solution:

$\sec \frac{11\pi}{6} = \frac{1}{\cos \frac{11\pi}{6}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

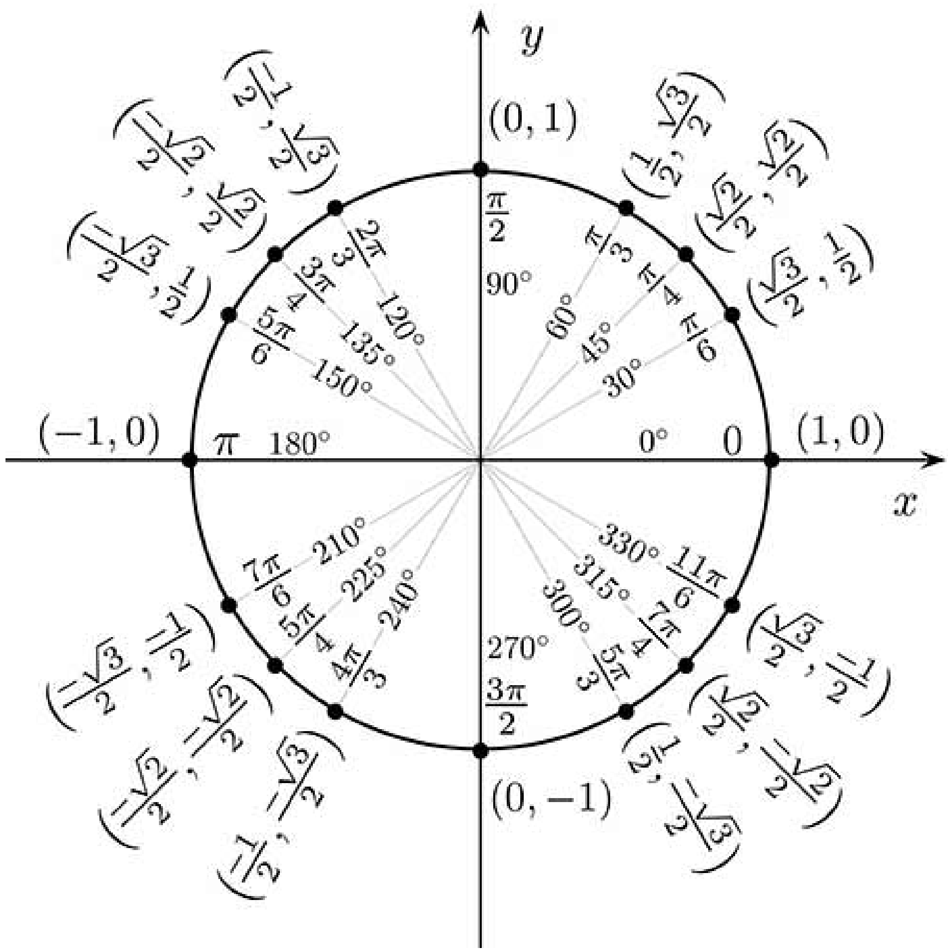
f. $\csc -\frac{5\pi}{4}$

Solution:

$\csc\left(-\frac{5\pi}{4}\right) = \frac{1}{\sin\left(-\frac{5\pi}{4}\right)} = \frac{2}{\sqrt{2}} = \sqrt{2}$

Summary

- The unit circle is a circle with its center at the origin and with a radius of 1, whose equation is $x^2 + y^2 = 1$.
- A **central angle** is the angle formed by two radii of the circle intersecting the circle.
- The **reference angle** of an angle in standard position is the angle formed between the terminal side and the closest portion of the $x-$axis.
- **Since coterminal angles end at identical points along the unit circle, trigonometric expressions involving coterminal angles are equivalent:** $\sin(-10°) = \sin 350° = \sin 710°$**.**



Review

Name an angle between 0° and 360° that is coterminal with the given angle:

1. $-20°$

2. $475°$

3. $-220°$

4. $690°$

5. $-45°$

Use your knowledge of the unit circle to help determine whether the trigonometric expression is positive or negative:

6. $\tan 143^\circ$

7. $\cos \frac{\pi}{3}$

8. $\sin 362^\circ$

9. $\csc \frac{3\pi}{4}$

Use your knowledge of the unit circle to evaluate the trigonometric expression:

10. $\cos 120^\circ$

11. $\sec \frac{\pi}{3}$

12. $\tan 225^\circ$

13. $\cot 120^\circ$

14. $\sin \frac{11\pi}{6}$

15. $\csc 240^\circ$

16. Draw the complete unit circle (all four quadrants) and label the important points.

Review (Answers)

Please see the Appendix.

7.3 Graphing Sine and Cosine

Learning Objectives

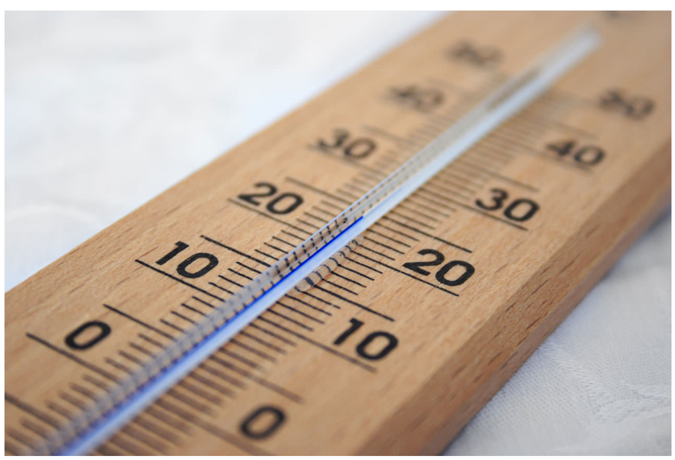
Learn how to graph and vertically stretch and shrink the sine and cosine functions.

Introduction

Now that trigonometric ratios have been established, they can be used to model cyclic behavior. Consider the set of data below. These temperatures will repeat with consistent patterns over time. The unit circle's repeating pattern helps to develop six new functions, which can be used to model periodic behavior such as this dataset.

TABLE 7.3:

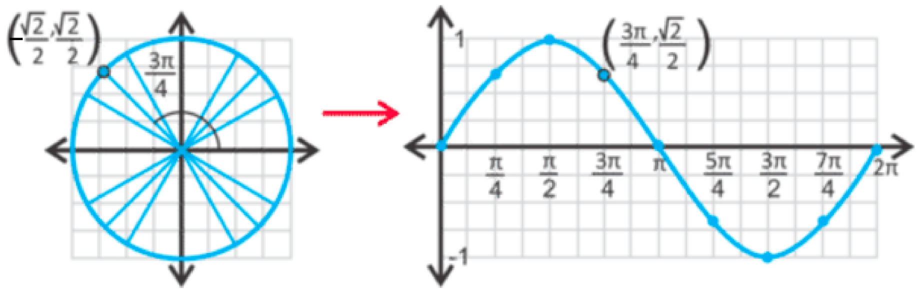
Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Average Temperature (F)	18	26	29	40	60	64	73	70	63	59	42	28



Graphing Sine and Cosine

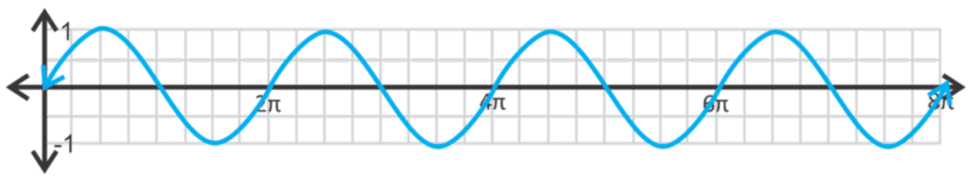
To graph the sine and cosine functions, the unit circle will be transferred in coordinate pairs (angle in radians, trigonometic ratio) to the Cartesian coordinate system. To carry out this process, the unit circle will be "unraveled."

Recall that for the unit circle, the coordinates are $(\cos\theta, \sin\theta)$, where θ is the central angle. To graph $y = \sin x$, rewrite the coordinates as $(x, \sin x)$, where x is the central angle in radians. Below the sine coordinates for $\frac{3\pi}{4}$ are labeled on the sine graph.

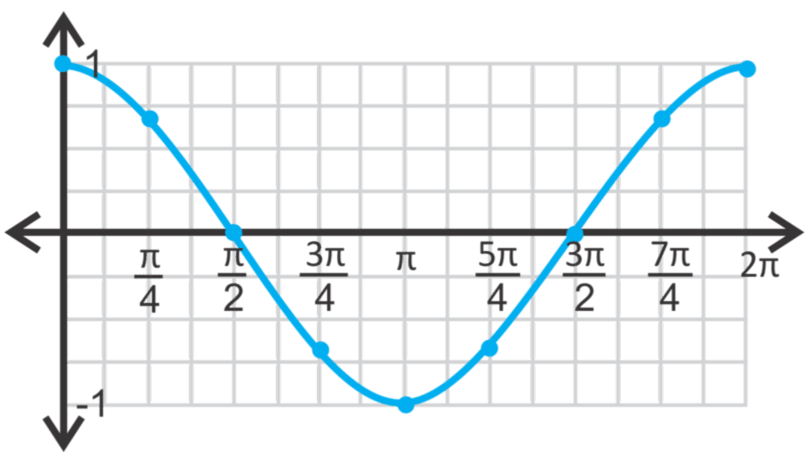


Notice that the range of $y = \sin x$ is $[-1, 1]$. The maximum value is 1, which is at $x = \frac{\pi}{2}$. The minimum value is -1 at $x = \frac{3\pi}{2}$. The value $\left(\frac{4\pi}{3}, -\frac{1}{2}\right)$ of the sine function is called the **amplitude**.

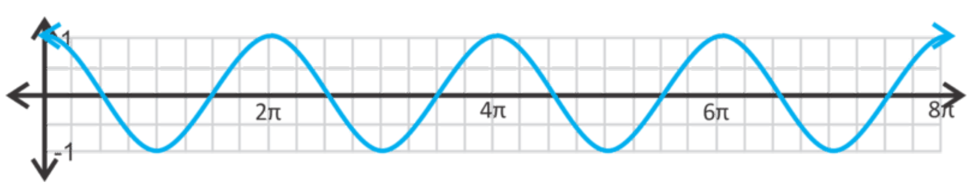
Now, consider the domain. Since a 2nd trip around the unit circle yields a new set of coordinate pairs with the same sine values, the initial pattern repeats. This means that the sine curve is **periodic**. A **periodic function** is one that has a predictable repeating pattern. If you look back at the unit circle, you see the sine values repeat every 2π radians. Therefore, the curve above will repeat every 2π units, making the **period** 2π. Since it is possible to generate both positive and negative values for the angle, x, the domain is all real numbers.



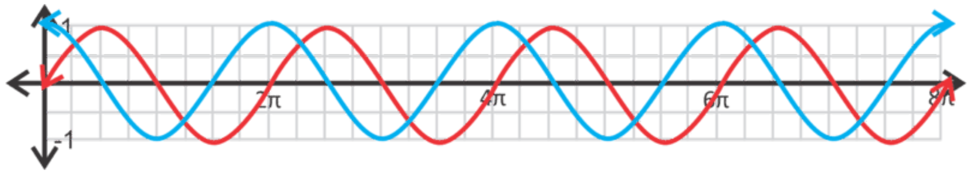
Similarly, $y = \cos x$ can be created by "unraveling" the unit circle:



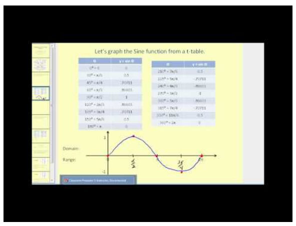
Notice that the range is also between 1 and -1, and the domain will be all real numbers. The cosine curve is also periodic, with a period of 2π.



Comparing $y = \sin x$ and $y = \cos x$ (below), notice that the curves are almost identical, except that the sine curve starts at $y = 0$, and the cosine curve starts at $y = 1$.



If the sine curve is shifted $\frac{\pi}{2}$ units to the left or the cosine curve the matching value to the right, then they will overlap. Any horizontal shift of a trigonometric function is called a **phase shift**, which is covered in another section.



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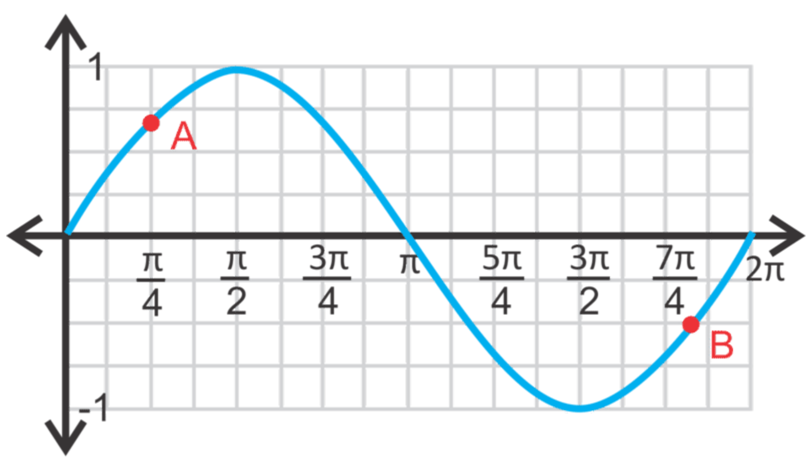
Amplitude

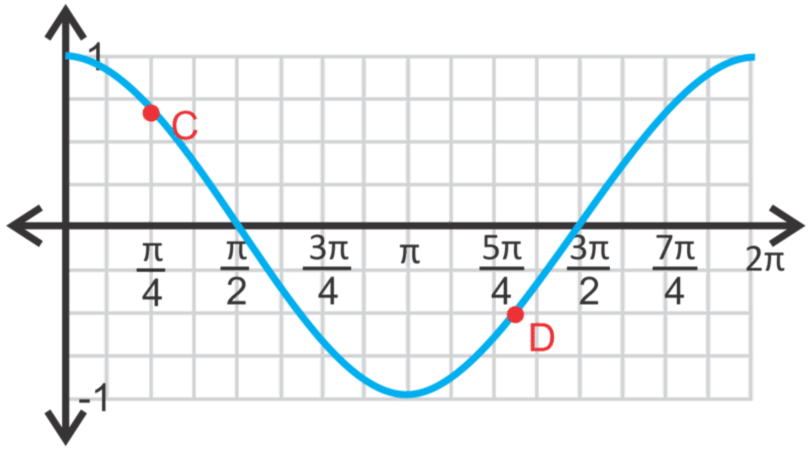
Stretching the graphs of $y = \sin x$ and $y = \cos x$ vertically is accomplished by multiplying the sine or cosine by a, where $y = a \sin x$ or $y = a \cos x$, where $|a|$ is the amplitude of the curve. As noted earlier, it equals $\frac{\text{maximum value - minimum valu}}{2}$

Examples

Example 1

Identify the highlighted points on $y = \sin x$ and $y = \cos x$.





Solution:

For each point, think about what the sine or cosine value is at those values.

For point A, $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Therefore, the point is $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$.

For point B, work backwards because it is not exactly on a vertical line, but on a horizontal one. When is $\sin x = -\frac{1}{2}$? When $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$. By looking at point B's location, it is the 2nd option. Therefore, the point is $\left(\frac{11\pi}{6}, -\frac{1}{2}\right)$.

For the cosine curve, point C is the same as point A because the sine and cosine for $\frac{\pi}{4}$ is the same.

As for point D, use the same logic as for point B. When does $\cos x = -\frac{1}{2}$? When $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Looking at the location of point D, we know it is the 2nd option. The point is $\left(\frac{4\pi}{3}, -\frac{1}{2}\right)$.

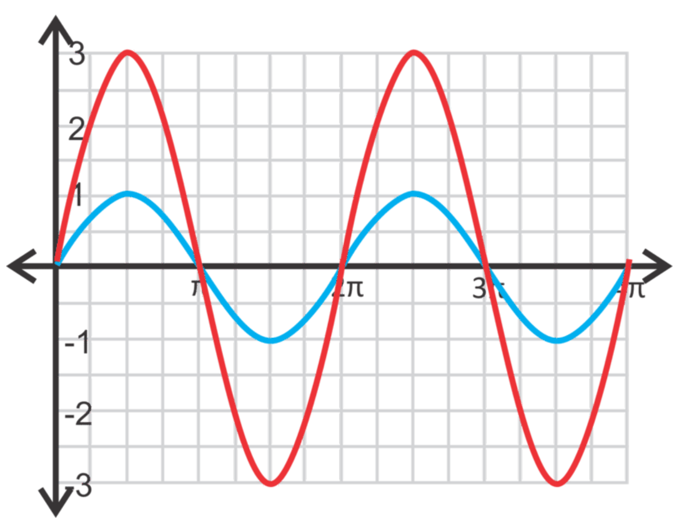
Example 2

Graph $y = 3\sin x$ over two periods.

Solution:

Step 1: Start with the basic sine curve. Recall that one period of the parent graph, $y = \sin x$, is 2π. Therefore, two periods will be 4π.

Step 2: The 3 indicates that the range will now be from 3 to -3, and the curve will be stretched so that the maximum is 3 and the minimum is -3. The stretched red curve is $y = 3\sin x$.



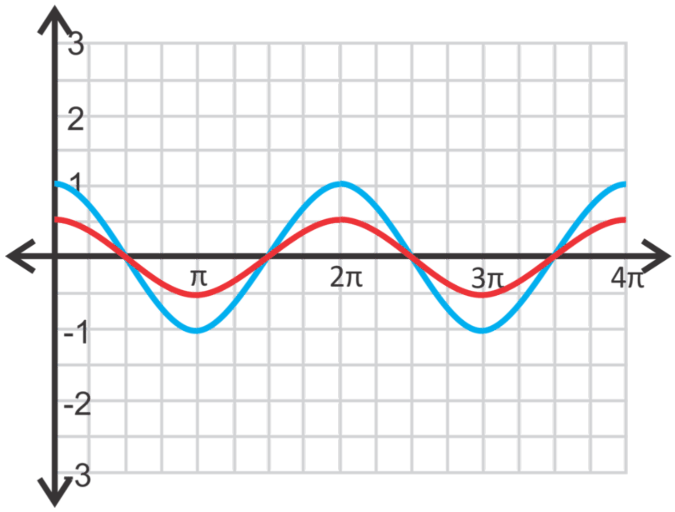
Notice that the x-intercepts are the same as the parent graph. Typically, when graphing a trigonometric function, always show two full periods of the function to indicate that it does repeat.

Example 3

Graph $y = \frac{1}{2}\cos x$ over two periods.

Solution:

The amplitude will be $\frac{1}{2}$ and the function will shrink rather than stretch.

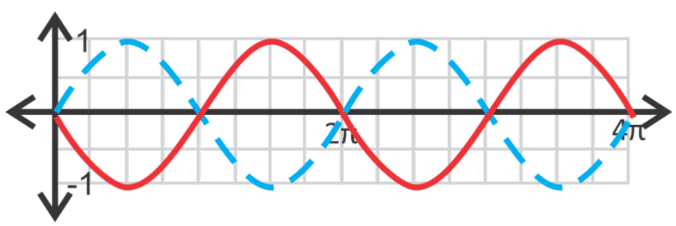


Example 4

Graph $y = -\sin x$ over two periods.

Solution:

The last two examples dealt with changing a, and a was positive. Now, a is negative. Just as with other functions, when the leading coefficient is negative, the function is reflected over the x-axis. $y = -\sin x$ is in solid red, while the original function is dotted.



Example 5

Is the point $\left(\frac{5\pi}{6}, \frac{1}{2}\right)$ on $y = \sin x$? How can that question be answered?

Solution:

Substitute in the point for x and y and see if the equation holds true.

$$\frac{1}{2} = \sin\left(\frac{5\pi}{6}\right)$$

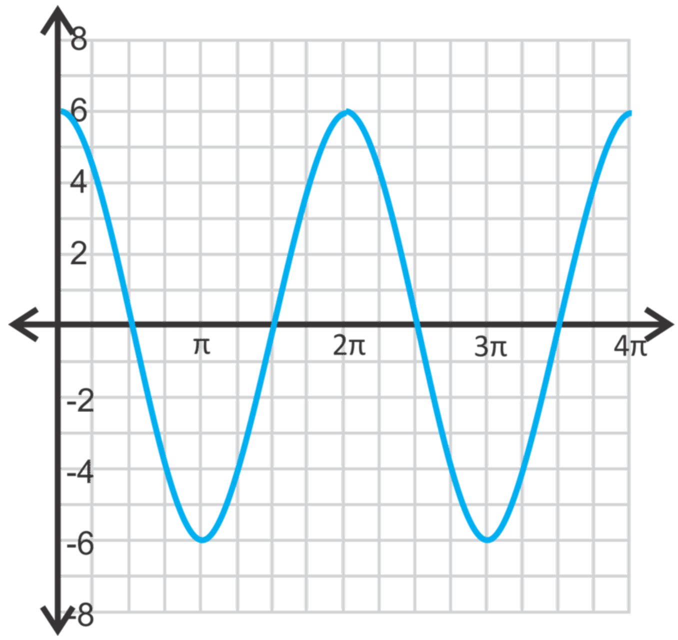
This is true, so $\left(\frac{5\pi}{6}, \frac{1}{2}\right)$ is on the graph.

Example 6

Graph the following functions for two full periods: $y = 6\cos x$.

Solution:

Stretch the cosine curve so that the maximum is 6 and the minimum is -6.

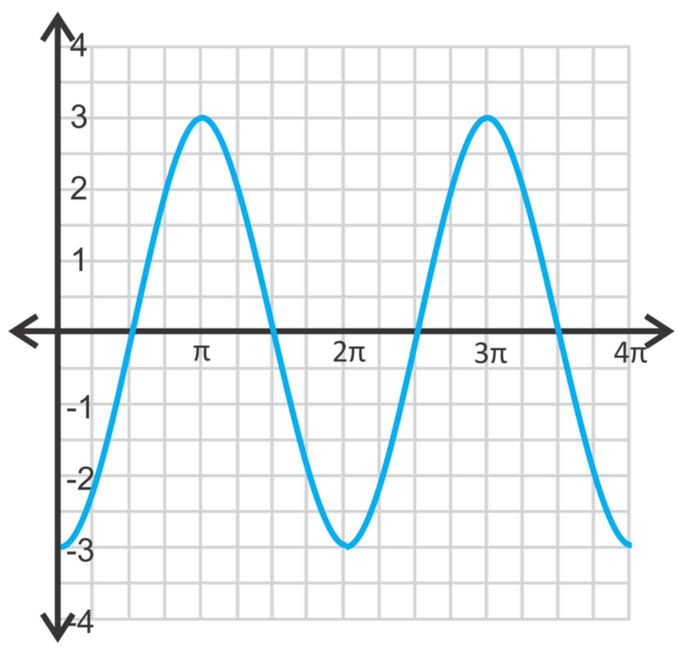


Example 7

Graph the following functions for two full periods: $y = -3\cos x$.

Solution:

The graph is reflected over the x-axis and stretched so that the amplitude is 3.

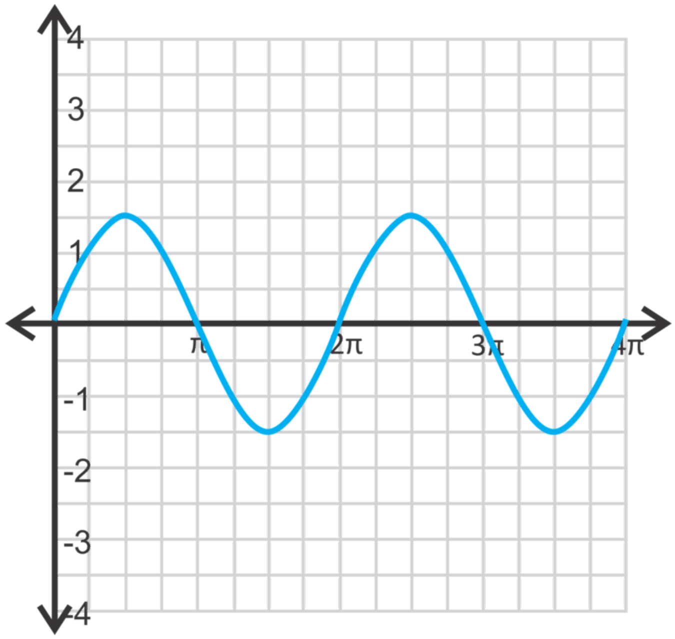


Example 8

Graph the following functions for two full periods: $y = \frac{3}{2}\sin x$.

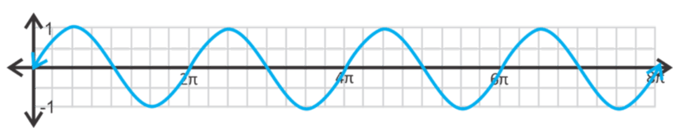
Solution:

The amplitude is $\frac{3}{2}$.

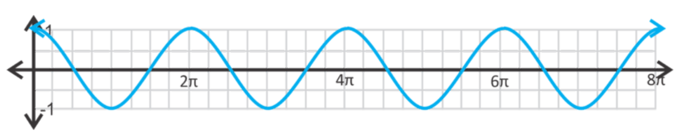


Summary

- A number of key values, such as the starting point, the period, and the amplitude, are necessary when graphing trigonometric functions.
- The behavior of the sine and cosine functions at the origin will allow you to know where to start graphing.
- In general, graphs can be interpreted and graphed using the general formulas $y = a \sin x$ and $y = a \cos x$. The height of the graph is designated by the amplitude $|a|$.
- Graph of $y = \sin x$

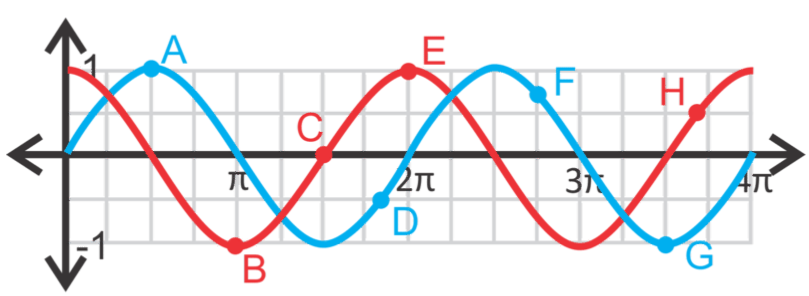


- Graph of $y = \cos x$



Review

1. Determine the exact value of each point on $y = \sin x$ or $y = \cos x$.



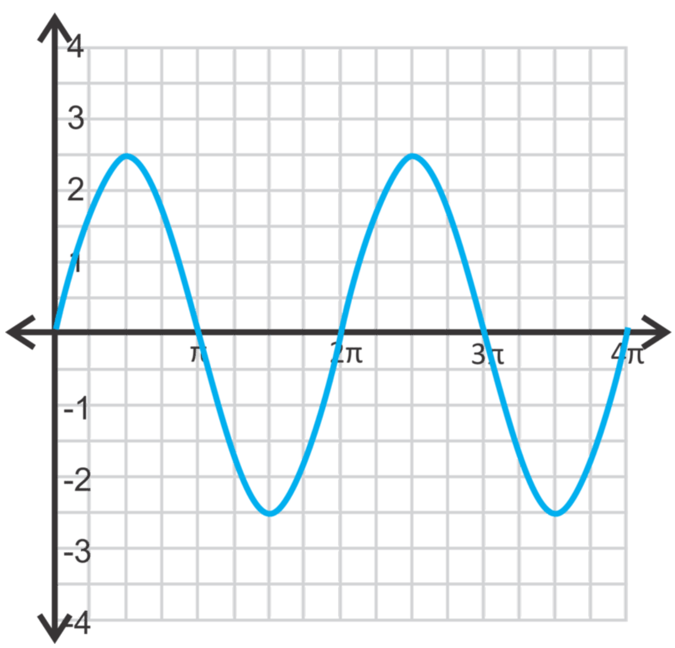
2. List all the points in the interval $[0, 4\pi]$ where $\sin x = \cos x$. Use the graph from 1 above to help you.
3. Draw $y = \sin x$ on $[0, 2\pi]$. Find $f\left(\frac{\pi}{3}\right)$ and $f\left(\frac{5\pi}{3}\right)$. Plot these values on the curve.

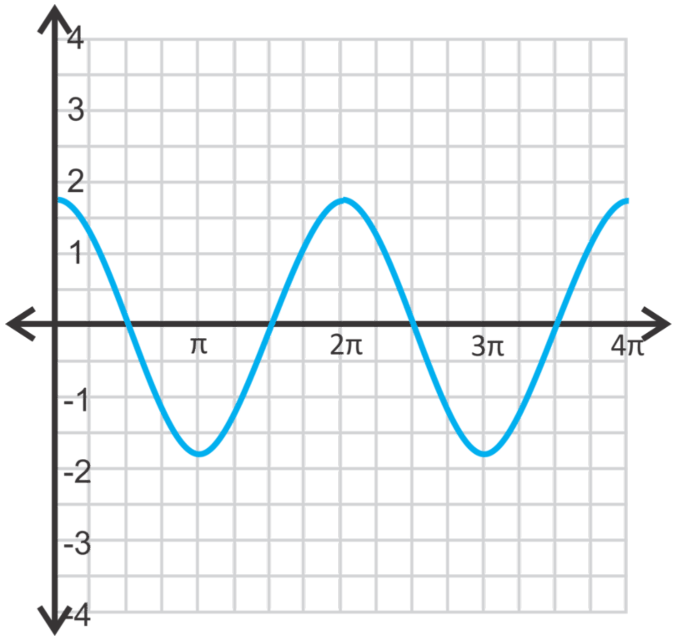
For questions 4-12, graph the sine or cosine curve over two periods.

4. $y = 2\sin x$
5. $y = -5\cos x$
6. $y = \frac{1}{4}\cos x$
7. $y = -\frac{2}{3}\sin x$
8. $y = 4\sin x$
9. $y = -1.5\cos x$
10. $y = \frac{5}{3}\cos x$
11. $y = 10\sin x$
12. $y = -7.2\sin x$
13. Graph $y = \sin x$ and $y = \cos x$ on the same set of axes. How many units would you have to shift the sine curve (to the left or right) so that it perfectly overlaps the cosine curve?
14. Graph $y = \sin x$ and $y = -\cos x$ on the same set of axes. How many units would you have to shift the sine curve (to the left or right) so that it perfectly overlaps $y = -\cos x$?

Write the equation for each sine or cosine curve below. $a > 0$ for both questions.

15.





Review (Answers)

Please see the Appendix.

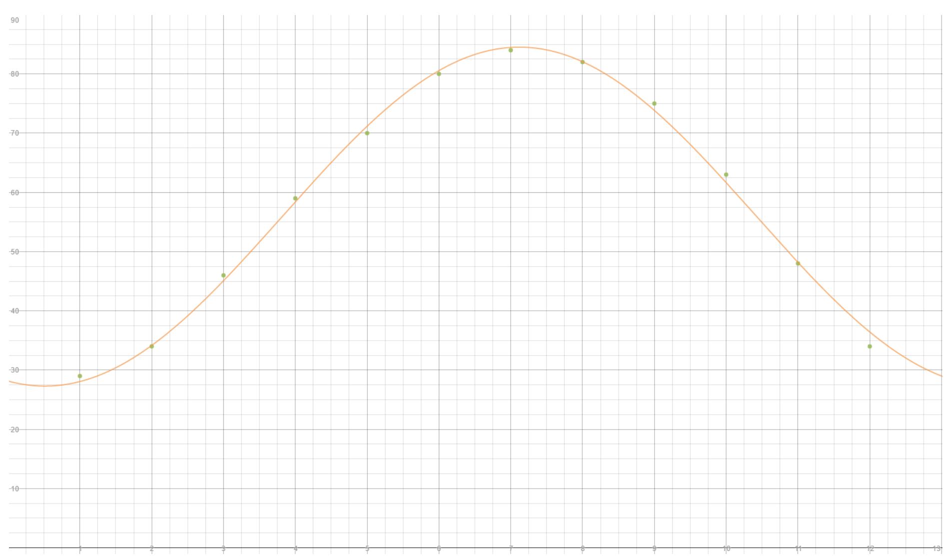
7.4 Translating Sine and Cosine Functions

Learning Objectives

Learn how to graph a translated sine or cosine function.

Introduction

Consider the graph of average monthly temperatures during one year in Chicago:



Clearly, this is a sine or cosine graph. However, several transformations have been performed. The amplitude is $\frac{\text{maximum value - minimum value}}{2} = 28$. This graph also has a vertical and horizontal shift. In this section, we will explore both vertical and horizontal shifts.

Translating Sine and Cosine Functions

Just like other functions, sine and cosine curves can be translated to the left, right, up, and down.

The general equation for a sine and cosine curve is $y = A\sin(x-h)+k$ and $y = A\cos(x-h)+k$,respectively. Similar to other function transformations, h is the horizontal shift (also called a phase shift), and k is the vertical shift.

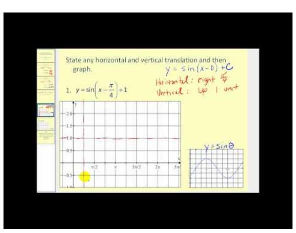
Horizontal & Vertical Shifts

$$y = \sin(x-h) \text{ is } y = \sin x \text{ shifted right } h \text{ units}$$

$$y = \sin(x+h) \text{ is } y = \sin x \text{ shifted left } h \text{ units}$$

$$y = \sin(x)+k \text{ is } y = \sin x \text{ shifted up } k \text{ units}$$

$$y = \sin(x)-k \text{ is } y = \sin x \text{ shifted down } k \text{ units}$$



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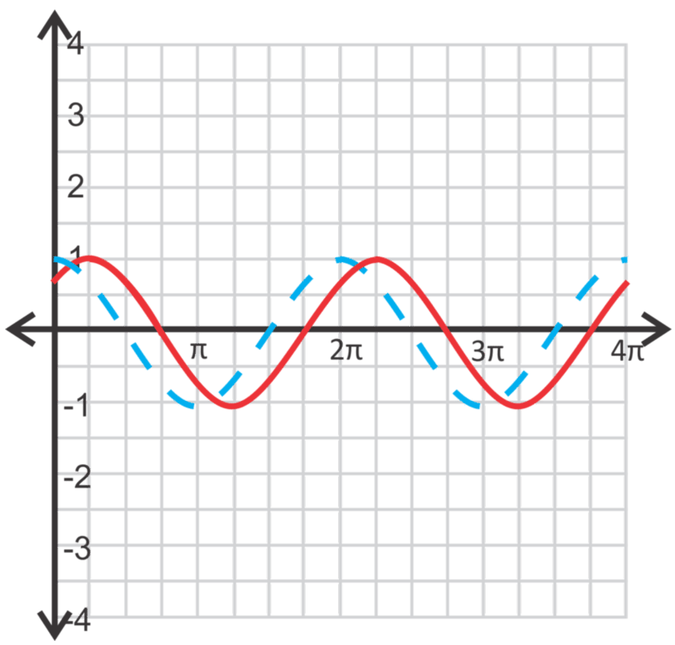
Examples

Example 1

Graph $y = \cos\left(x - \frac{\pi}{4}\right)$.

Solution:

Identify the shift. Since the x-value is reduced by $\frac{\pi}{4}$, this function will be shifted $\frac{\pi}{4}$ units to the right. The easiest way to sketch the curve is to start with the parent graph and then move it to the right the correct number of units. The parent graph ($y = \cos x$) is shown below in dotted blue. The solid graph in red is the graph of $y = \cos\left(x - \frac{\pi}{4}\right)$.



Example 2

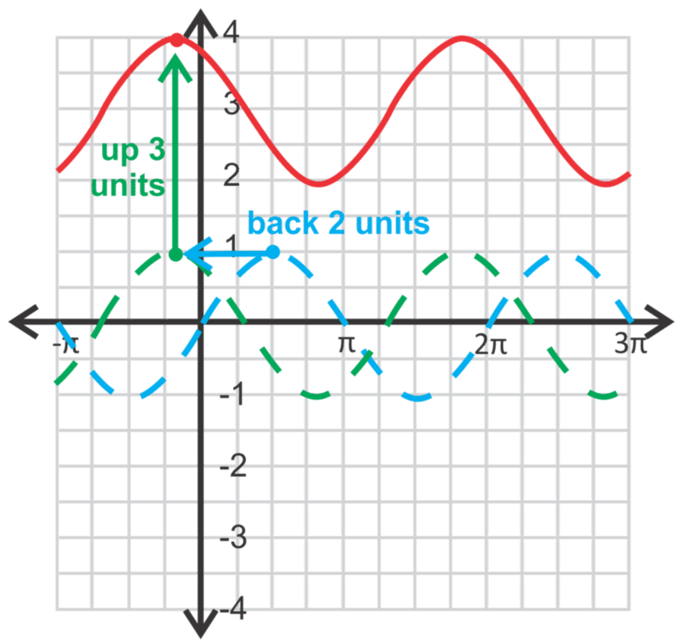
Graph $y = \sin(x+2)+3$.

Solution:

Step 1: The sine function is graphed in blue.

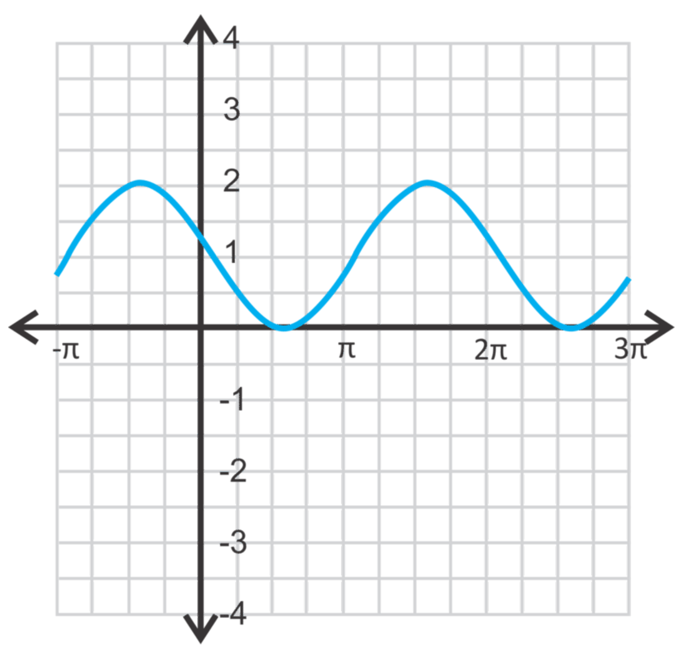
Step 2: Then the graph is shifted 2 units to the left, because the horizontal shift is -2. This graph is in green.

Step 3: Translate the green graph up 3 units, because the vertical shift is +3. The solid red graph is the result.



Example 3

Find the equation of the sine curve below.

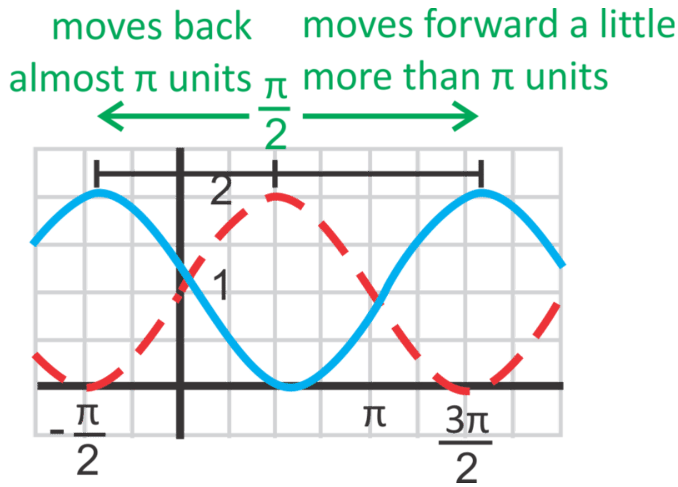


Solution:

Step 1: The amplitude is 1 because the maximum is 2 and the minimum is 0.

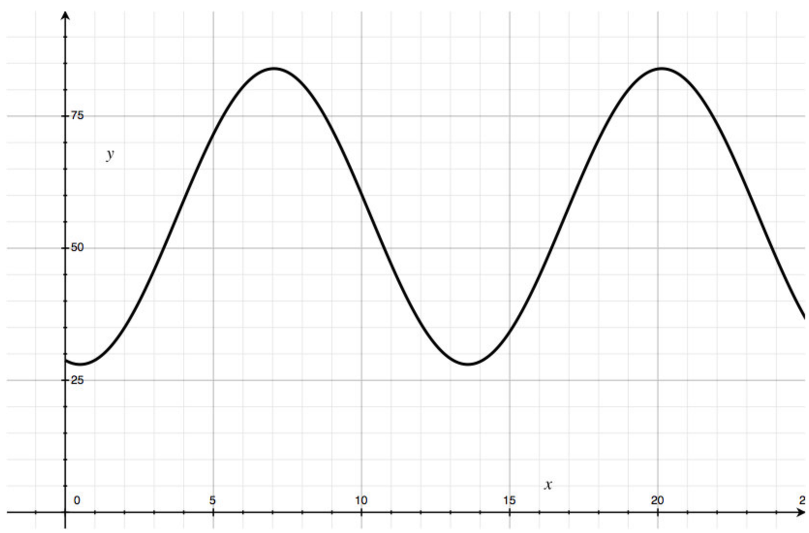
Step 2: Identify the vertical shift. The sine function is shifted up 1 unit to get the given graph.

Step 3: The horizontal shift is the hardest to find. Because sine curves are periodic, the horizontal shift can either be positive or negative. Find a convenient point that reflects the horizontal shift. $(\pi, 1)$ is close. So $h \approx \pi$, and the equation is $y \approx \sin(x - \pi) + 1$.



Example 4

Recall the problem from the Introduction. Consider the graph of average monthly temperatures during one year in Chicago. The graph of the model that fits that data is a sine or cosine graph and has an amplitude of 28. Determine the vertical shift.



Solution:

The sine function is shifted up approximately 56 units to get the given graph.

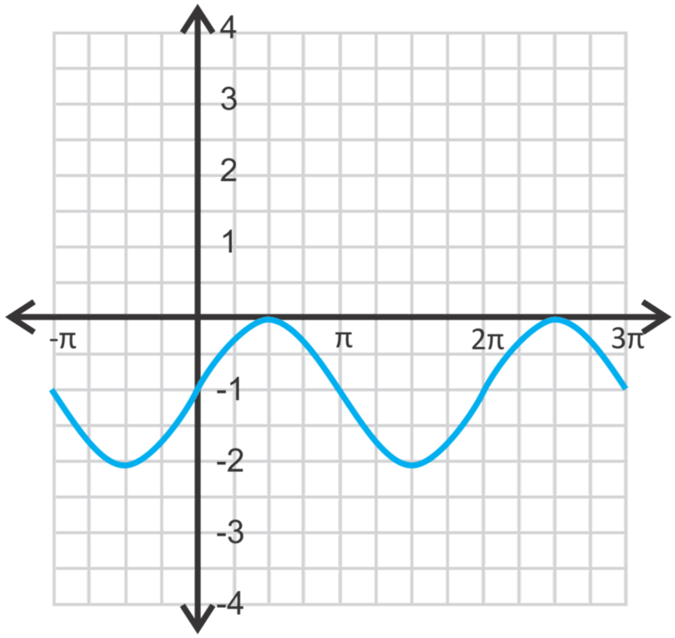
Example 5

Graph the following functions on $[-\pi, 3\pi]$:

$y = -1 + \sin x$.

Solution:

Shift the parent graph down 1 unit.



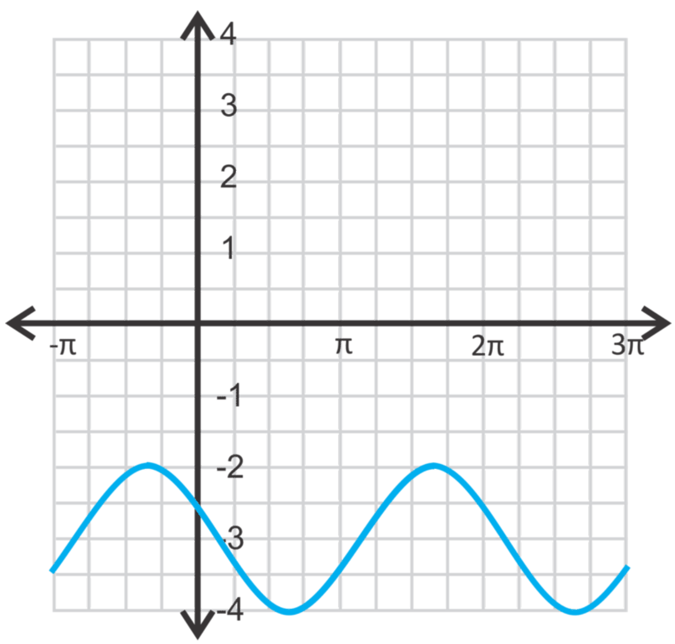
Example 6

Graph $y = \cos\left(x + \frac{\pi}{3}\right) - 2$.

Solution:

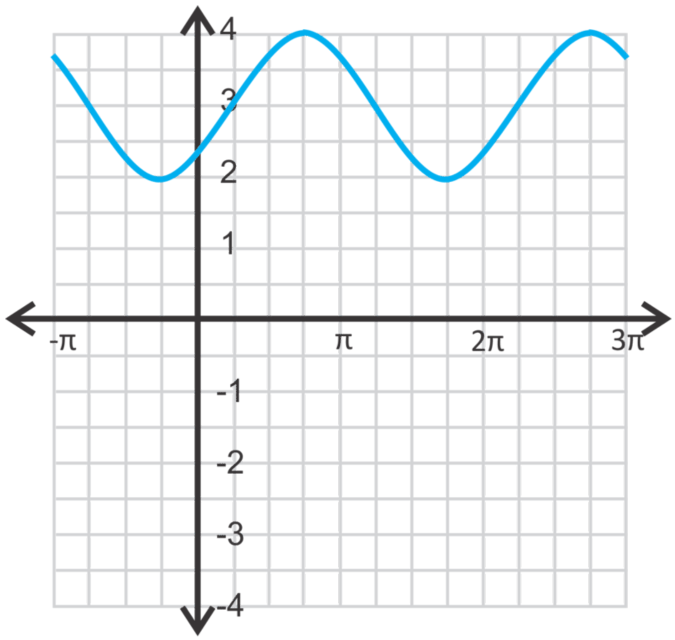
Step 1: Shift the parent graph to the left $\frac{\pi}{3}$ units

Step 2: Shift the resultant graph down 2 units.



Example 7

Find the equation of the cosine curve below.

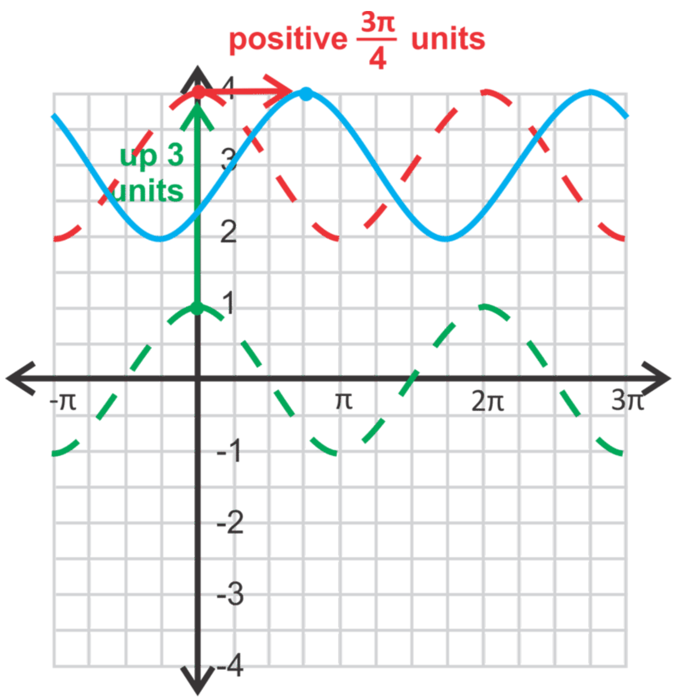


Solution:

Step 1: Identify the vertical shift. It moves up 3 units.

Step 2: Identify the horizontal shift. The graph moves to the right $\frac{3\pi}{4}$ units.

Step 3: The equation is $y = \cos\left(x - \frac{3\pi}{4}\right) + 3$.



If the equation was based on the cosine curve moving to the left, then the equation would be $y = \cos\left(x + \frac{5\pi}{4}\right) + 3$.

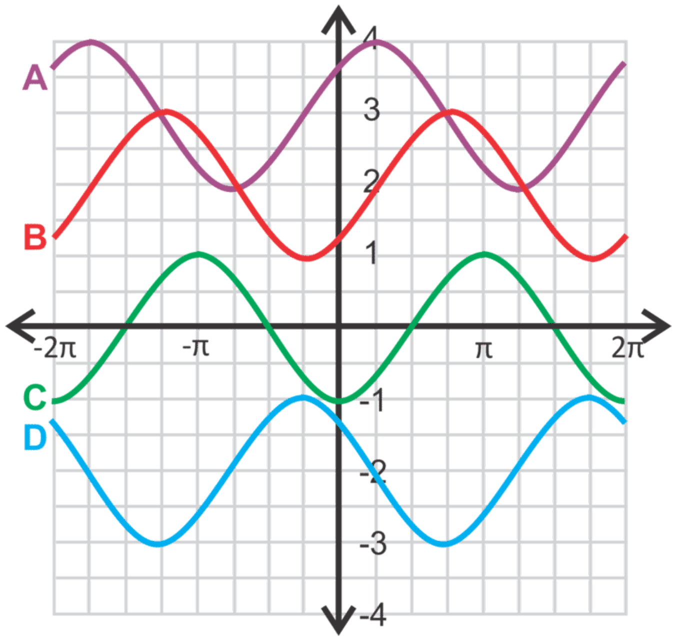
Summary

- The standard forms for the sine and cosine graphs are $y = a\sin(x-h)+k$ and $y = a\cos(x-h)+k$, where $|a|$ is the amplitude, h is the horizontal shift, and k is the vertical shift.

- The period of standard sine and cosine functions is 2π.

Review

For questions 1-4, match the equation with its graph.



1. $y = \sin\left(x - \frac{\pi}{2}\right)$
2. $y = \cos\left(x - \frac{\pi}{4}\right) + 3$
3. $y = \cos\left(x + \frac{\pi}{4}\right) - 2$
4. $y = \sin\left(x - \frac{\pi}{4}\right) + 2$

Which graph above also represents these equations?

5. $y = \cos(x - \pi)$
6. $y = \sin\left(x + \frac{3\pi}{4}\right) - 2$
7. Write another sine equation for graph A.
8. How many sine (or cosine) equations can be generated for one curve? Why?
9. Fill in the blanks below.
   a. $\sin x = \cos(x - \_\_)$
   b. $\cos x = \sin(x - \_\_)$

For questions 10-15, graph the following equations from $[-2\pi, 2\pi]$:

10. $y = \sin\left(x + \frac{\pi}{4}\right)$
11. $y = 1 + \cos x$
12. $y = \cos(x + \pi) - 2$
13. $y = \sin(x + 3) - 4$
14. $y = \sin\left(x - \frac{\pi}{6}\right)$
15. $y = \cos(x - 1) - 3$
16. Is there a difference between $y = \sin x + 1$ and $y = \sin(x + 1)$? Explain your answer.

Review (Answers)

Please see the Appendix.

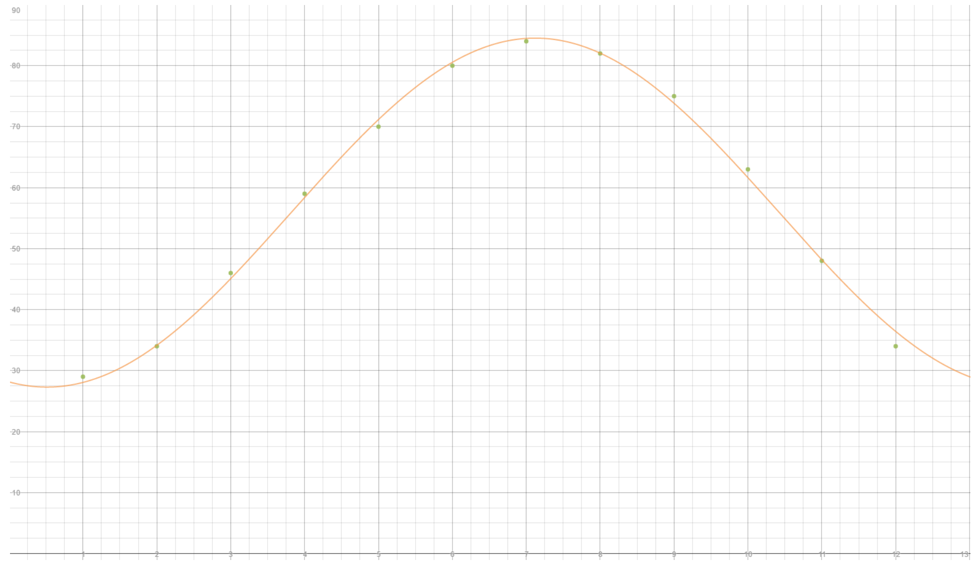
7.5 Frequency and Period of Sinusoidal Functions

Learning Objectives

Learn to apply your knowledge of horizontal stretching transformations to sine and cosine functions.

Introduction

In Chicago there is a public sculpture called *Cloud Gate* that looks like a giant bean. This sculpture can be modeled by a sinusoidal function as you walk around it. Similarly, the temperature standing next to it can be modeled with a sinusoidal function.



A mathematical model that fits this data is

$$y = 28\sin(0.48x - 1.81) + 56 = 28\sin(0.48(x - 3.77)) + 56.$$

As covered in another section, it is clear that the amplitude of the sine function is 28, and the vertical shift is up 56 units, which is the average of the maximum and minimum temperatures. From the second form of the function, the horizontal or phase shift is 3.77. How does the coefficient of $(x - 3.77)$ affect the graph?

Sinusoidal Functions

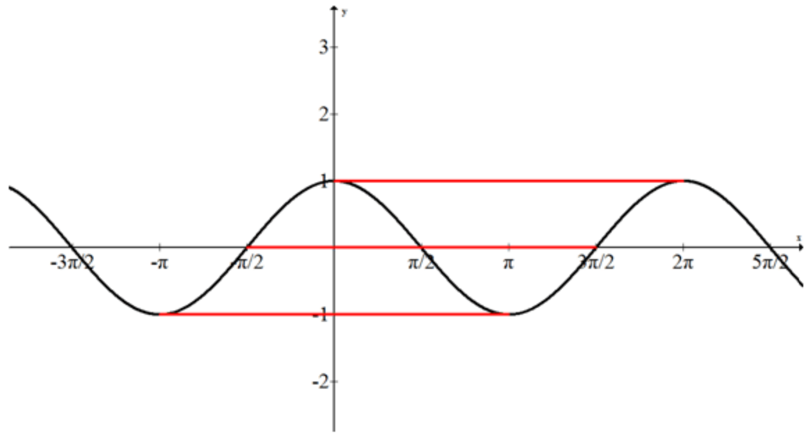
A sinusoid is the graph of the sine function. The general equation for the sine function provides the necessary information to graph the function: $f(x) = a\sin(b(x-h)) + k$. The sign on a controls the reflection across the x-axis, and the value of a controls the amplitude. The constant k controls the vertical shift and h the horizontal or phase shift. The coefficient b controls the horizontal stretch.

General Form of Sinusoidal Function:

$$f(x) = a\sin(b(x-h)) + k,$$

where $|a|$ is the amplitude, $\frac{2\pi}{|b|}$ is the period, h is the horizontal or phase shift, and k is the vertical shift.

Horizontal stretch for sinusoidal functions is connected to the function's period. Since this is the basis for the periodic function family, the period is a critical characteristic. The period of a sine graph is the length of a complete cycle. For basic sine and cosine functions, the period is 2π. This length can be measured in multiple ways. In applications, it may be most useful to measure from peak to peak.



The ability to measure the period of a function in multiple ways allows multiple equations to model the same graph. In the image above, the top red line would represent a regular cosine wave. The center red line would represent a regular sine wave with a horizontal shift. The bottom red line would represent a reflected cosine wave with a horizontal shift. Any cosine function can be written as a sine function, since $\cos(x) = \sin(x + \frac{\pi}{2})$. This flexibility in perspective means that graphs have multiple solutions. Unless you are directed otherwise, choose the function that has a period starting at $x = 0$ or the closest option.

Frequency is a different way of measuring horizontal stretch. With sinusoidal functions, frequency is the number of cycles that occur in 2π. A shorter period means more cycles can fit in 2π, thus requiring a higher frequency. Period and frequency are inversely related by the equation:

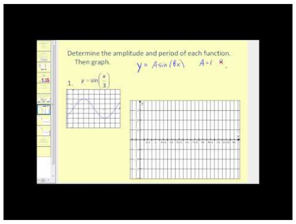
Period

$$\text{frequency} = \frac{1}{\text{period}} = \frac{|b|}{2\pi}$$
$$\text{period} = \frac{1}{\text{frequency}} = \frac{2\pi}{|b|}$$

The equation of a standard sine function is $f(x) = \sin x$. In this case, b, the frequency, is equal to 1, which means 1 cycle is completed in 2π.

A fractional frequency will increase the length of the period. Note that if you have a function where $b = \frac{1}{2}$, the period will be 4π since

$$p = \frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi.$$



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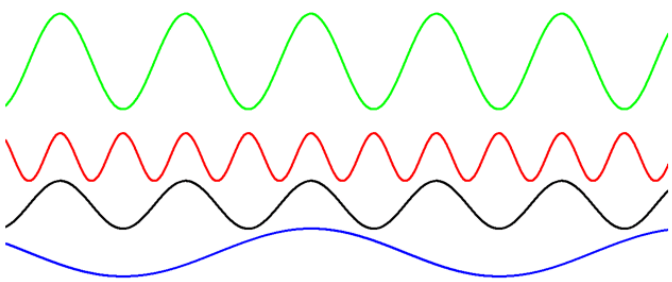
Examples

Example 1

Rank the waves from the shortest to the longest period.



Solution:

The red wave has the shortest period.

The green and black waves have equal periods. The difference between these two graphs is in their amplitude.

The blue wave has the longest period.

Example 2

Identify the amplitude, vertical shift, period, and frequency of the function below. Then graph the function: $f(x) = 2\sin\left(\frac{x}{3}\right) + 1$.

Solution:

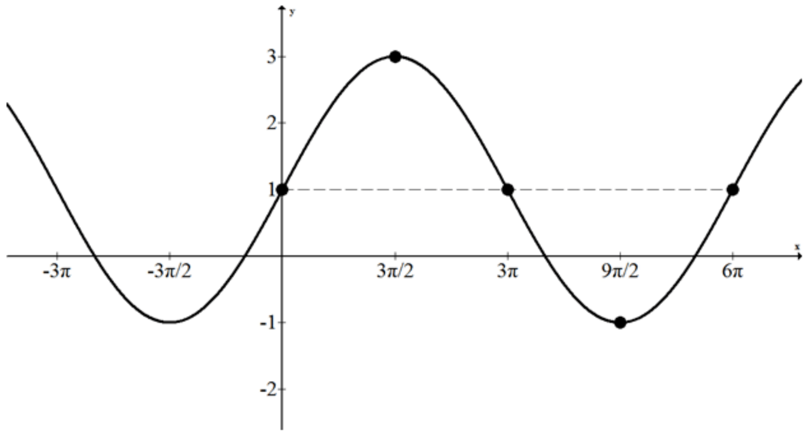
Step 1: Identify the key information from the general equation: $A = 2, b = \frac{1}{3}, k = 1$.

Step 2: Calculate the period: Since $b = \frac{1}{3}$, then the period is 6π.

Step 3: Since the period is 6π, the graph can be divided into four parts so that the five guiding points (four sections and the 1st point on the y-axis) of the sine graph can be plotted with the amplitude.

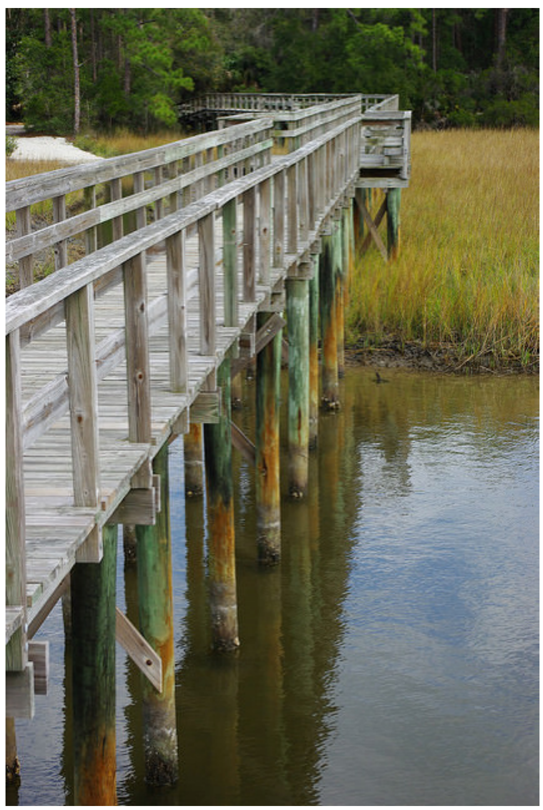
Identify key points: (0,1) is the first point and (6π, 1) is the last point to plot of our five. The midpoint of the points will be found at (3π, 1). The midpoint between the 1st and 3rd points will be the maximum, found at ($\frac{3\pi}{2}$, 3). The midpoint between the 3rd and last points will be found at ($\frac{9\pi}{2}$, -1).

Step 4: Graph the five key points and sketch the curve:



Example 3

A measuring stick on a dock measures high tide to be 18 feet and low tide to be 6 feet. It takes about 6 hours for the tide to switch between low and high tides. Determine a graphical and algebraic model for the tides, knowing that at $t = 0$ there is a high tide.



Solution:

Step 1: From the given information, the points below can be found. Notice how the sinusoidal axis can be assumed to be the average of the high and low tides.

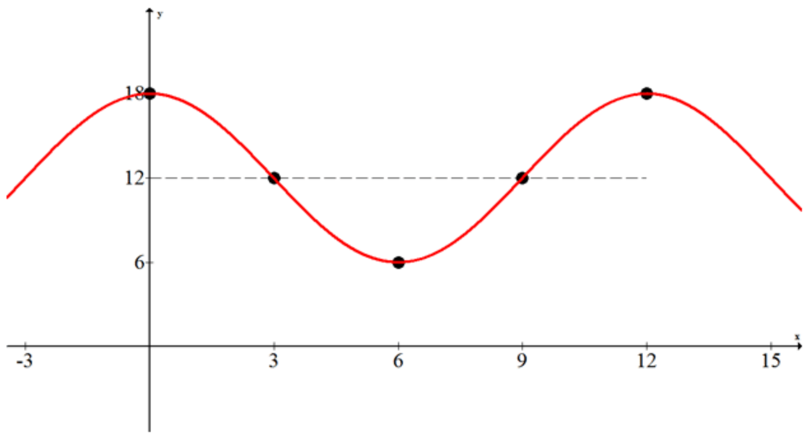
TABLE 7.4:

Time (hours)	Water level (feet)
0	18
6	6

TABLE 7.4: (continued)

12	18
$\frac{0+6}{2}=3$	$\frac{18+6}{2}=12$
$\frac{6+12}{2}=9$	$\frac{18+6}{2}=12$

Step 2: By plotting those points and filling in the sinusoidal axis, we can observe a cosine graph.



Step 3: The amplitude is 6, so $A=6$. There is no vertical reflection.

Step 4: Since the period is 12, b is found:

$$12=\frac{2\pi}{b}\rightarrow b=\frac{\pi}{6}.$$

Step 5: The vertical shift is 12, so $k=12$.

Step 6: Enter the gathered information into the general equation:

$$f(x)=6\cos\left(\frac{\pi}{6}x\right)+12.$$

Example 4

Return to the Introduction problem: How does the coefficient 0.48 on $(x-3.77)$ affect the graph of $y=28\sin(0.48(x-3.77))+56$?

Solution:

The period of the graph is

$$\text{period}=\frac{2\pi}{0.48}\approx 13.1,$$

which is the length of time for the graph to make one full cycle and return to the starting temperature.

Example 5

A fish is caught in a water wheel by the side of a river. Initially, the fish is at the bottom of the water wheel, which is 2 feet below the surface of the water. Twenty seconds later the fish is 14 feet in the air at the top of the water wheel.

Model the fish's height with a graph and an equation.

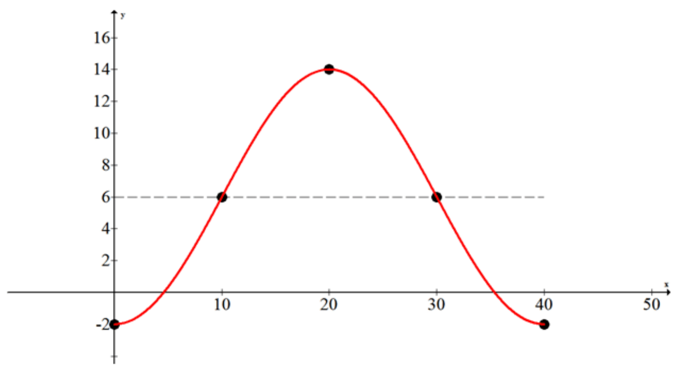
Solution:

Step 1: Use logic to identify five key points. Use those key points to come up with a sketch.

TABLE 7.5:

Time (seconds)	Fish height (feet)
0	-2
20	14
40	-2
$\frac{0+20}{2}=10$	$\frac{-2+14}{2}=6$
30	6

Step 2: Use the sketch to identify information for the equation.



Step 3: Identify the amplitude. The amplitude is 8, so $A=8$.

Step 4: Identify the function. The function looks like a reflected cosine graph, adding a negative to our final equation.

Step 5: The vertical shift is $k=6$.

Step 6: The period is 40.

$$40=\frac{2\pi}{b}\rightarrow b=\frac{\pi}{20}$$

Step 7: Insert the gathered information into the general equation:

$$f(x)=-8\cdot\cos\left(\frac{\pi}{20}x\right)+6.$$

Labeling on the graph can be helpful to identifying the graph. On both the x- and y-axes, only the most important intervals are labeled. This keeps the sketch accurate, evenly spaced, and easy to read.

Example 6

Graph the following function: $g(x)=-\cos(8x)+2$.

Solution:

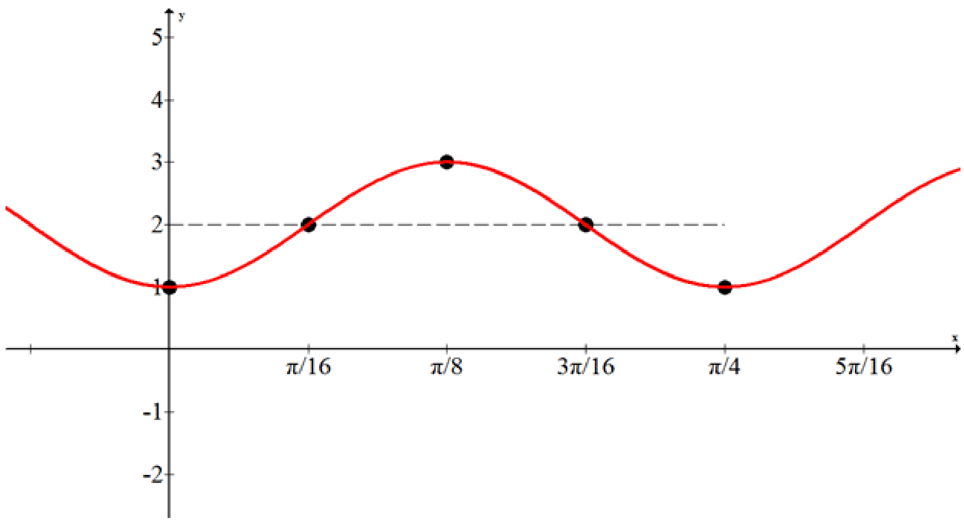
Step 1: Identify the amplitude. The amplitude is 1.

Step 2: The shape is a negative cosine.

Step 3: Identify the vertical shift. The vertical shift is up 2.

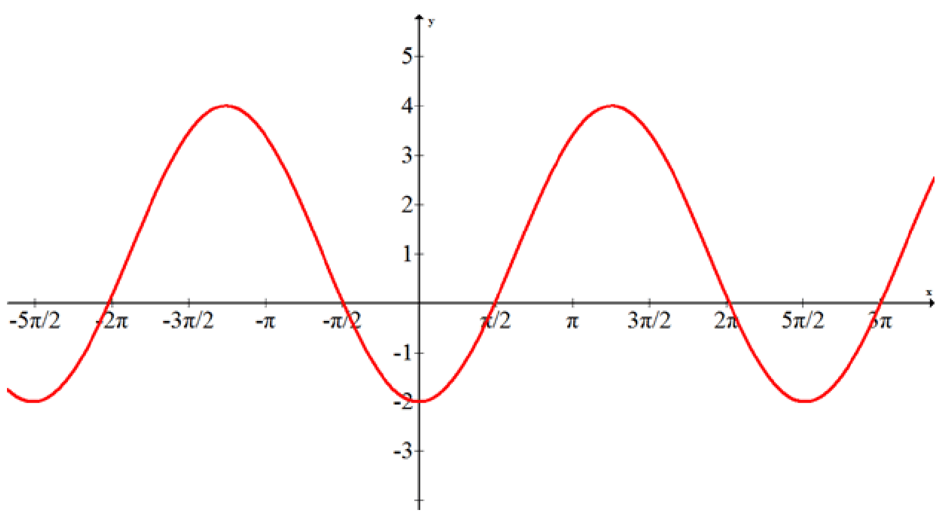
Step 4: Calculate the period. The period is $\frac{2\pi}{8} = \frac{\pi}{4}$.

Step 5: Identify five key points and graph:



Example 7

Given the following graph, identify the amplitude, period, and frequency and create an algebraic model:



Solution:

Step 1: Identify the amplitude. The amplitude is 3.

Step 2: Identify the function. The shape is a reflected cosine graph.

Step 3: Calculate the period: The period is $\frac{5\pi}{2}$, which implies that $b = \frac{4}{5}$ since:

$$\frac{5\pi}{2} = \frac{2\pi}{b}$$
$$b \cdot 5\pi = 2 \cdot 2\pi$$
$$b = \frac{4\pi}{5\pi}$$
$$b = \frac{4}{5}.$$

Step 4: Identify the vertical shift. The vertical shift is 1.

Step 5: Insert the gathered information into the general equation:

$$f(x) = -3\cos\left(\frac{4}{5}x\right) + 1.$$

.

Summary

- Sinusoidal functions have the form $f(x) = a\sin(b(x-c)) + d$.
- The sine graph has an amplitude $|a|$, period $\frac{2\pi}{|b|}$, frequency $\frac{|b|}{2\pi}$, phase shift h, and vertical shift k.

Review

Find the frequency and period of each function below.

1. $f(x) = \sin(4x) + 1$

2. $g(x) = -3\cos(2x)$

3. $h(x) = \cos\left(\frac{1}{2}x + 4\right) + 2$

4. $k(x) = -2\sin\left(\frac{3}{4}x\right) + 1$

5. $j(x) = 4\cos(3x + 6) - 1$

Graph each of the following functions:

6. $f(x) = 3\sin(2x) + 1$

7. $g(x) = 2.5\cos(\pi x) - 4$

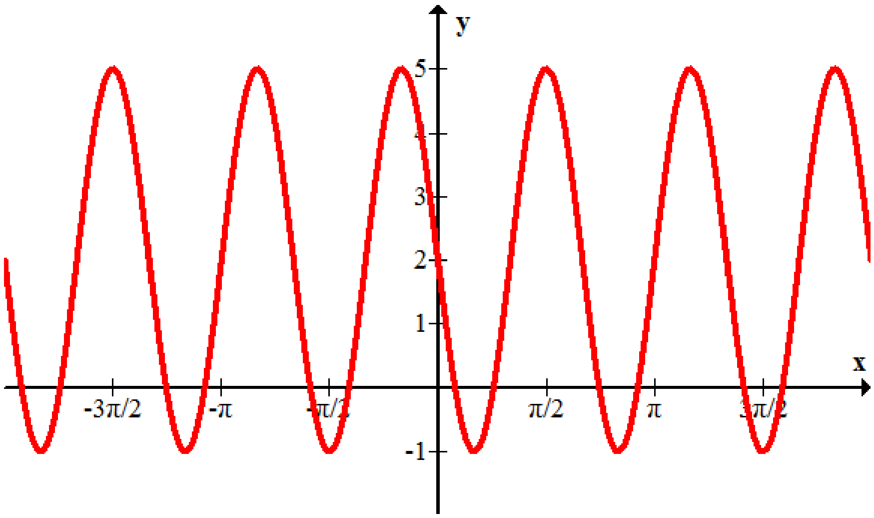
8. $h(x) = -\sin(4x + 8) - 3$

9. $k(x) = \frac{1}{2}\cos(2x + 6)$

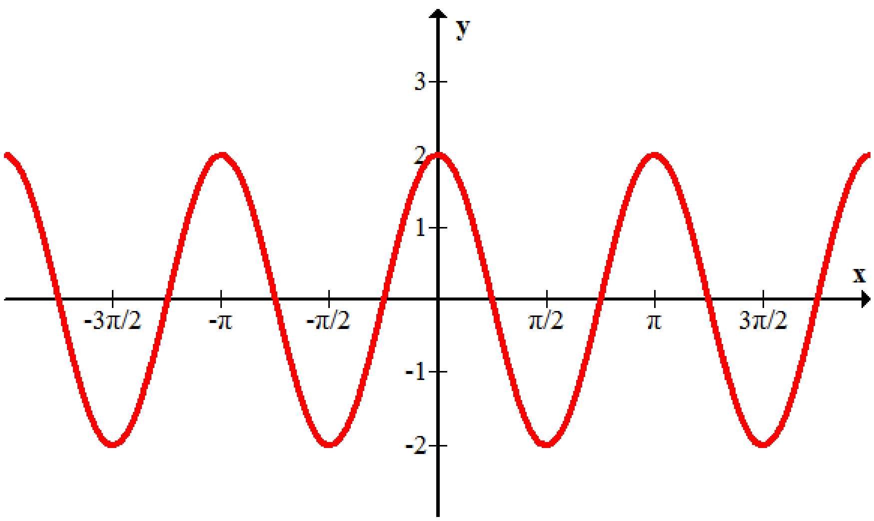
10. $j(x) = -2\sin\left(\frac{3}{4}x\right) - 1$

Create an algebraic model for each of the following graphs:

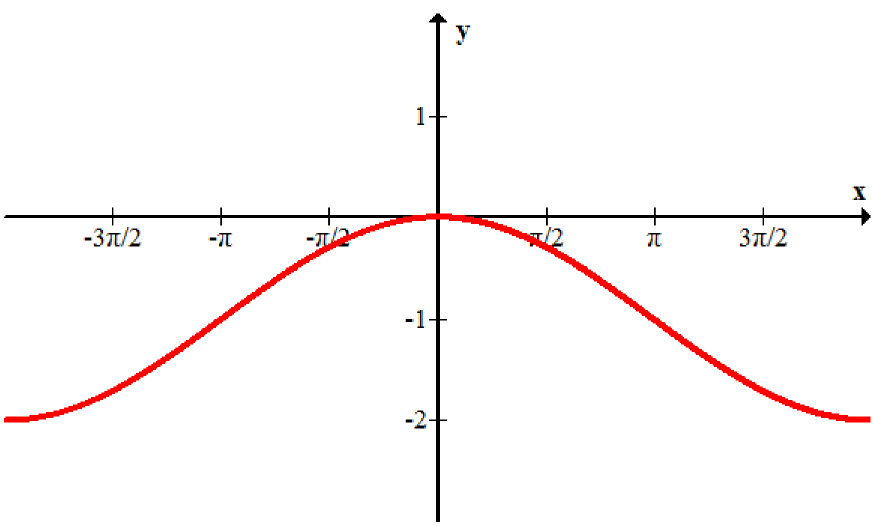
11.


y
5
4
3
2
1
-1
x
-3π/2
-π
-π/2
π/2
π
3π/2

12.


y
3
2
1
-1
-2
x
-3π/2
-π
-π/2
π/2
π
3π/2

13.


y
1
-1
-2
x
-3π/2
-π
-π/2
π/2
π
3π/2

14. At time 0 it is high tide, and the water at a certain location is 10 feet high. At low tide 6 hours later, the water is 2 feet high. Given that tides can be modeled by sinusoidal functions, find a graph that models this scenario.

15. Find the equation that models the scenario in the previous problem.

Review (Answers)

Please see the Appendix.

7.6 Graphs of Other Trigonometric Functions

Learning Objectives

Learn to graph four other trigonometric functions: tangent, secant, cosecant, and cotangent.

Introduction

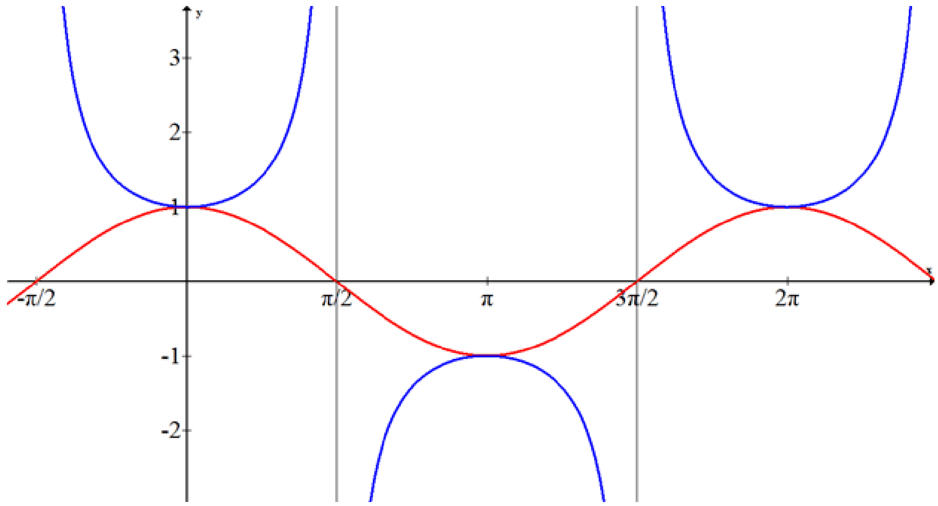
The rotating light on the front of an ambulance can be modeled by the following formula: $d = 16\tan(1.5\pi t)$. The 16 represents the 16 feet from an object it is casting light on, and d represents the distance of the beam from a point after t seconds. What would the graph of this function look like?

Knowing the relationship between the equation and graph of sine and cosine functions, we can find the other four functions (tangent, cotangent, secant, and cosecant) by identifying zeros, asymptotes, and key points.

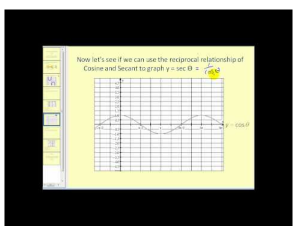
Graphing Cosecant and Secant Functions

Recall that cosecant is the reciprocal of the sine function, and secant is the reciprocal of the cosine function. Thus, their graphs are very closely related.

For instance, when the cosine function is equal to 0, the secant graph has a vertical asymptote. When $\cos x = 1$, then $\sec x = 1$ as well. Also, the period will be the same for both functions' graphs, 2π. The cosine function is the continuous red graph and the secant function is the blue one with vertical asymptotes wherever the cosine function equals 0.



We can utilize this technique of graphing by using the reciprocal relationship to graph the cosecant function as well.



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Graphing the Tangent Function

Graphing the tangent function is more difficult than graphing the sine and cosine functions, because the tangent is composed of a ratio of the sine and cosine functions.

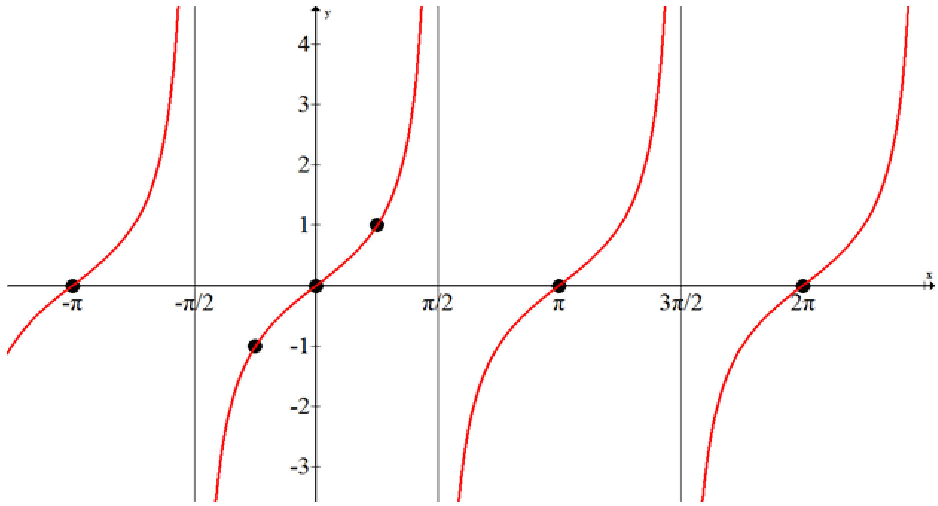
$$\tan x = \frac{\sin x}{\cos x}$$

To graph the tangent function, first determine its asymptotes. When the cosine function is equal to 0, the tangent graph has a vertical asymptote. This happens when $\cos x = 0$ at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$.

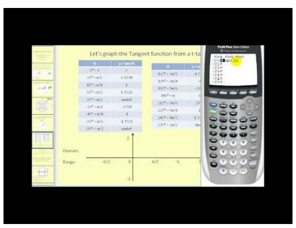
Next, plot the zeros or x-intercepts of the tangent function. This occurs when the numerator, $\sin x$, is equal to 0, which happens at $0, \pm\pi, \pm, 2\pi, \ldots$.

Also, note from the unit circle that $\tan\frac{\pi}{4} = 1$ and $\tan\left(-\frac{\pi}{4}\right) = -1$.

By plotting all this information, the graph of tangent can be sketched:



Notice that the period of tangent is π, **not** 2π**,** because it has a shorter cycle before it repeats.



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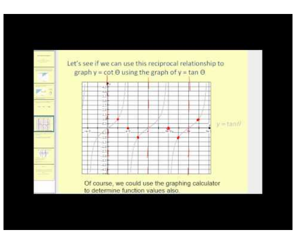
Graphing the Cotangent Function

Recall that cotangent is the reciprocal of the tangent function. Thus, their graphs are very closely related. When the tangent function is equal to 0, the cotangent graph has a vertical asymptote. When the tangent graph has a vertical asymptote, the cotangent function is equal to 0. Also, the period will be the same for both functions' graphs, π.



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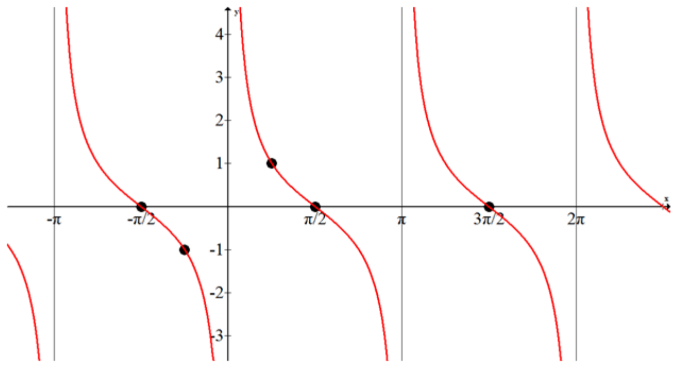
Examples

Example 1

Graph $f(x) = \cot x$.

Solution:

Since $\cot x = \frac{1}{\tan x}$, the graph of cotangent will have zeros wherever tangent has asymptotes, and asymptotes wherever tangent has zeros. Where tangent is 1, cotangent is also 1.



Example 2

Graph the function $f(x) = -2\csc(\pi(x-1))+1$.

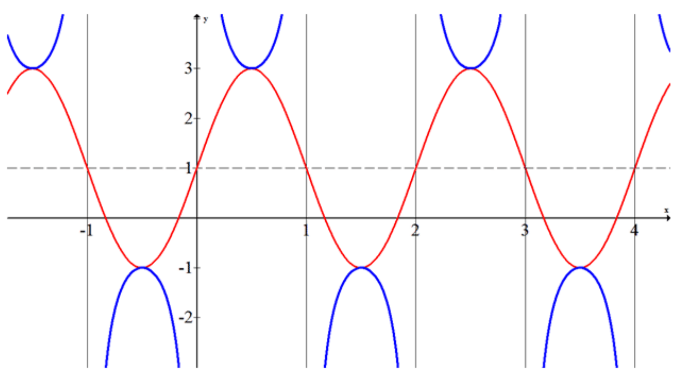
Solution:

Step 1: Graph the function as if it were a sine function. Then insert asymptotes wherever the sine function crosses the sinusoidal axis. Lastly, add in the cosecant curves.

Step 2: The amplitude is 2. The shape is a reflected sine curve.

Step 3: The function is shifted up 1 unit and to the right 1 unit.

Step 4: The period is $\frac{2\pi}{\pi} = 2$.



Note that continuous red graph is not part of the given function but a guideline to help you draw the graph.

Example 3

How can the cotangent function be expressed as a set of transformations applied to the tangent function $f(x) = \tan x$?

Solution:

Step 1: Start by reflecting across the x or the y axis.

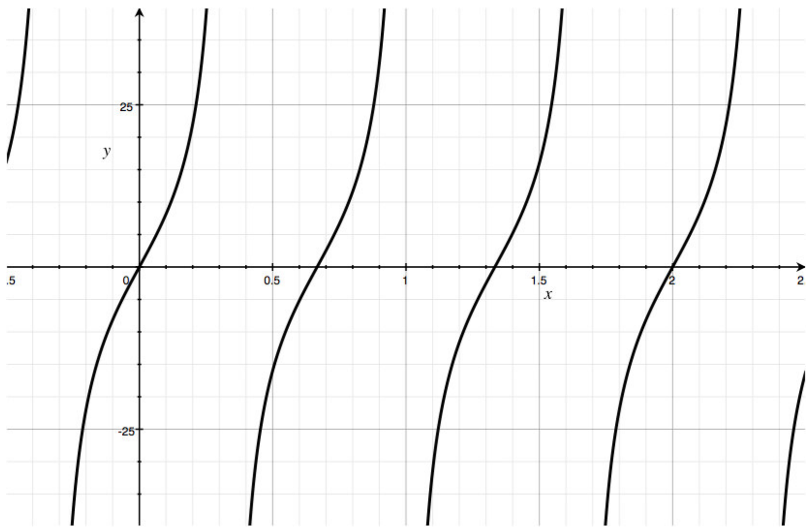
Step 2: Shift the function to the right or left by $\frac{\pi}{2}$.

Step 3: $f(x) = \tan x = -\cot\left(x - \frac{\pi}{2}\right)$.

Example 4

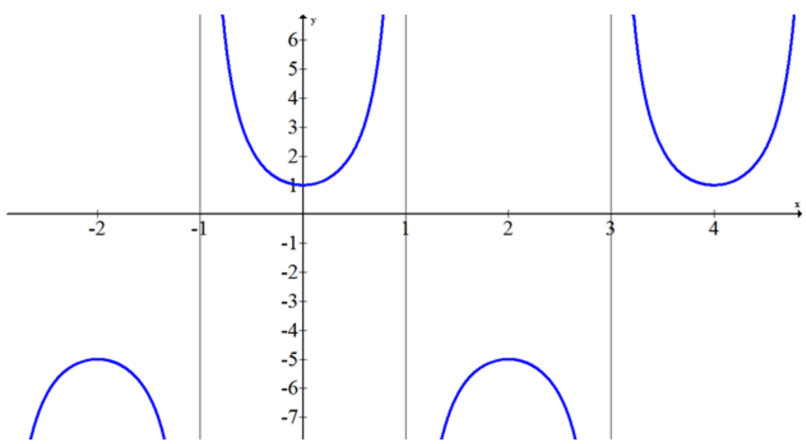
Return to the Introduction problem: What would the graph of $d = 16\tan(1.5\pi t)$ look like?

Solution:



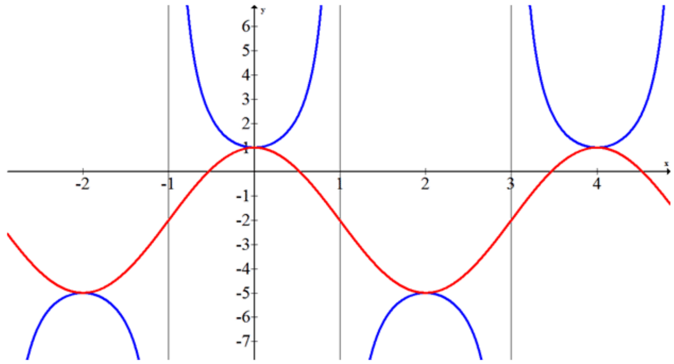
Example 5

Find the equation of the function in the following graph:



Solution:

Connecting the relative maximums and minimums of the function produces a shifted cosine curve that is easier to work with.



Step 1: Identify the amplitude. The amplitude is 3.

Step 2: Identify the vertical shift. The vertical shift is 2 down.

Step 3: The period is 4, which implies that $b = \frac{\pi}{2}$.

Step 4: Identify the shape. The shape is positive cosine. At $x = 0$, there is no phase shift to the standard cosine function.

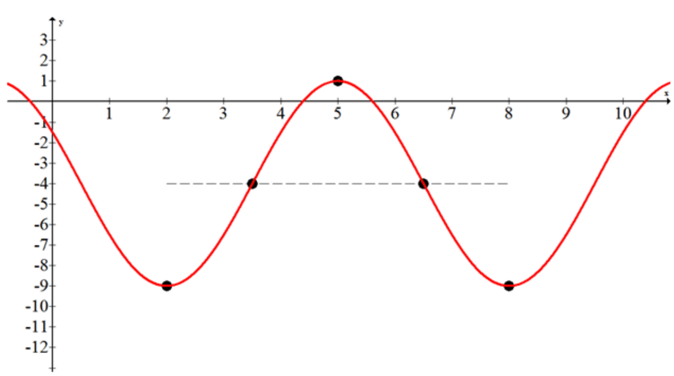
Step 5: Insert all information into the general formula: $f(x) = 3\sec\left(\frac{\pi}{2}x\right) - 2$.

Example 6

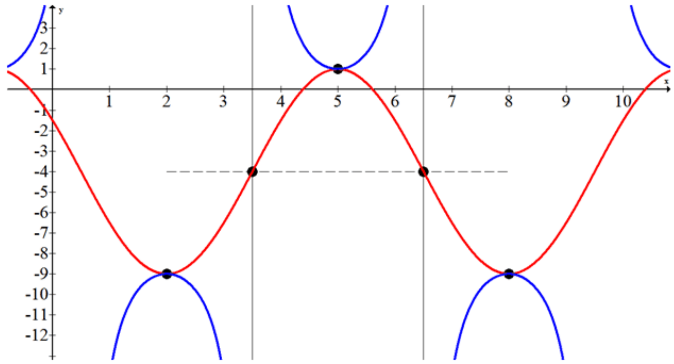
Graph the function $f(x) = -5\sec\left(\frac{\pi}{3}(x-2)\right) - 4$.

Solution:

Step 1: Graph the function as if it were a cosine. The vertical shift is -4. The horizontal shift is to the right 2. This gives a starting point for one period. Since $b = \frac{\pi}{3}$ the period must be 6. The amplitude is 5.



Step 2: Add in the asymptotes and secant curves. Note that the solution does not include the continuous red cosine curve.



Example 7

Where are the asymptotes for tangent, and why do they occur?

Solution:

Since $\tan x = \frac{\sin x}{\cos x}$, the asymptotes occur whenever $\cos x = 0$, which is $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Summary

- Since cosecant is the reciprocal of the sine function, and secant is the reciprocal of the cosine function, the secant and cosecant graphs can be sketched by sketching a faint sine or cosine graph, and using that graph to establish asymptotes.
- The tangent and cotangent functions are composed of a ratio of the sine and cosine functions, $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x}$, so their graphs can be sketched using knowledge of the sine and cosine functions.
- To graph the tangent and cotangent functions, determine the asymptotes and zeros of the functions.

Review

1. What function can you use to help you make a sketch of $f(x) = \sec x$? Why?

2. What function can you use to help you make a sketch of $g(x) = \csc x$? Why?

Make a sketch of each of the following from memory:

3. $f(x) = \sec x$

4. $g(x) = \csc x$

5. $h(x) = \tan x$

6. $k(x) = \cot x$

Graph each of the following:

7. $f(x) = 2\csc(x) + 1$

8. $g(x) = 2\csc\left(\frac{\pi}{2}x\right) + 1$

9. $h(x) = 2\csc\left(\frac{\pi}{2}(x-3)\right) + 1$

10. $j(x) = \cot\left(\frac{\pi}{2}x\right) + 3$

11. $k(x) = -\sec\left(\frac{\pi}{3}(x+1)\right) - 4$

12. $m(x) = -\tan(x) + 1$

13. $p(x) = -2\tan\left(x - \frac{\pi}{2}\right) + 1$

14. Find two ways to write $\sec x$ in terms of other trigonometric functions.
15. Find two ways to write $\csc x$ in terms of other trigonometric functions.

Review (Answers)

Please see the Appendix.

7.7 Graphs of Inverse Trigonometric Functions

Learning Objectives

Learn how to graph inverse trigonometric functions.

Introduction

Have you ever contemplated the ideal spot to sit in a movie theater? Some people like to be as close as possible to the movie, while others prefer a full view from the farthest seat. Some people prefer to sit on an aisle seat, and others in the middle. If a movie theater has a 30-foot-high screen that is 6 feet above eye level, your viewing angle can be modeled by the following equation: $\theta = \tan^{-1}\frac{36}{x} - \tan^{-1}\frac{6}{x}$.

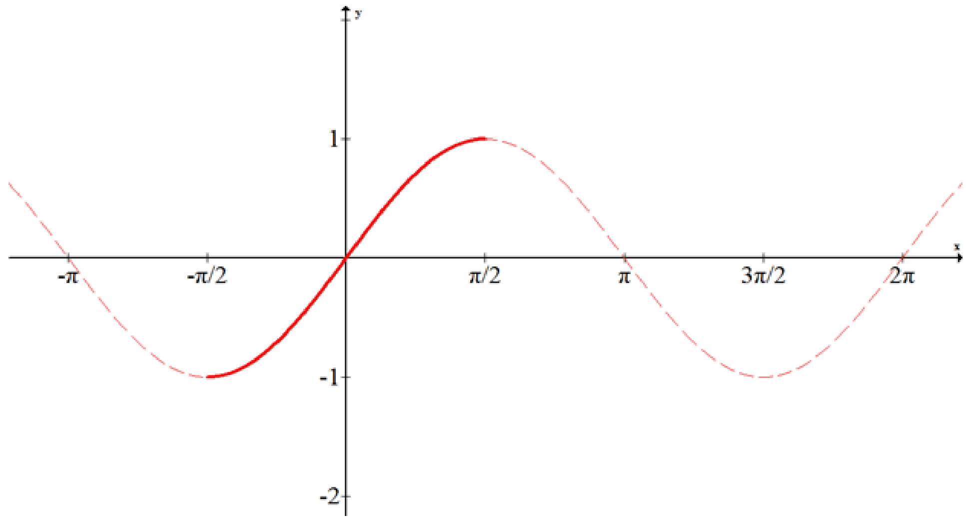
Notice in this problem we have an inverse of a tangent, designated by $\arctan x$ or $\tan^{-1} x$. This lesson will explore the graphs of inverse trigonometric functions, and will address why $\sin^{-1}(\sin 370^\circ) \neq 370^\circ$.

Inverse Trigonometric Functions

In order for inverses of functions to be functions, the original function must pass the horizontal line test. Recall that the horizontal line test can be used to determine if a function is one-to-one, which is a requirement for a function to have an inverse.

Even though none of the trigonometric functions pass the horizontal line test over the entire set of real numbers, their domains can be restricted for the purpose of studying and applying inverses. Inverses are produced by reflecting that portion over $y = x$, and solving a function $f(x) = y$ for $f(y)$.

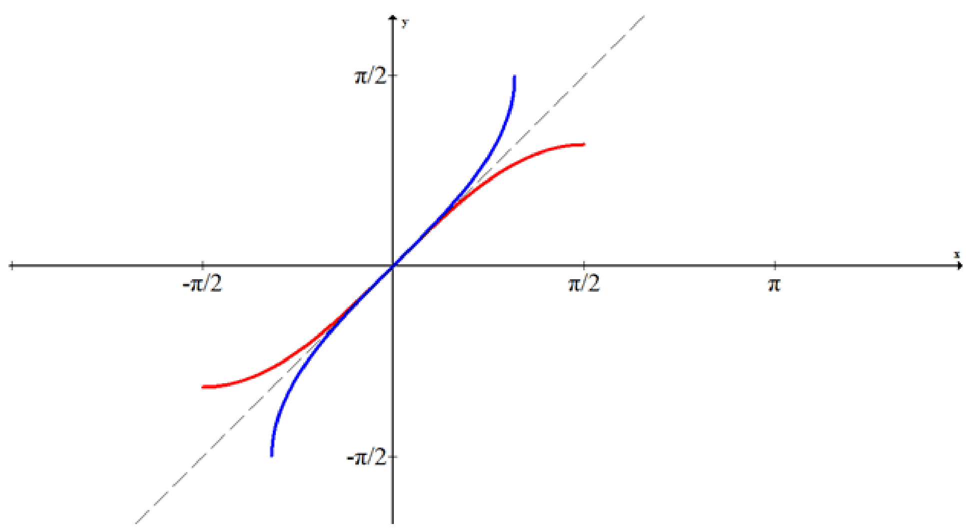
Consider the sine graph:



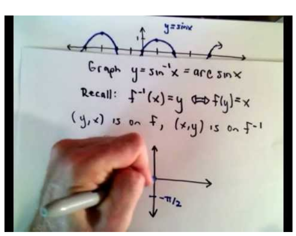
As a general rule, the restrictions to the domain of $y = \sin x$ is the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $y = \cos x$ the interval $[0, \pi]$. The 1st interval is appropriate for sine and cosecant functions, because half of its period is captured, and the graphs still pass the horizontal line test. The sine function restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is shown above. This interval is also appropriate for the tangent function because its entire period is captured, and the graph passes the horizontal line test.

The interval $[0, \pi]$ is appropriate for cosine, secant, and cotangent functions for the same reasons, because the entire period for these functions is captured and their graphs pass the horizontal line test.

Like other functions, to graph the inverse of a trigonometric function, reflect the restricted portion of the graph across the line $y = x$. For the sine function above (the red curve), the blue curve below shows $f(x) = \sin^{-1} x$:



The result of this inversion is that arcsine function will only produce angles between and including $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. These values are used in context to evaluate a wide group of function values.



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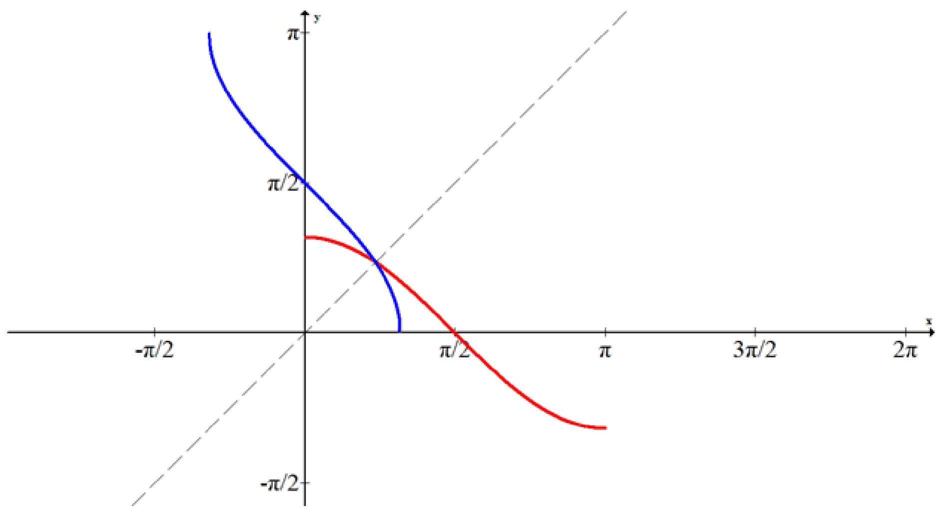
Examples

Example 1

What is the graph of $f(x) = \cos^{-1} x$?

Solution:

Graph the portion of cosine that fits the horizontal line test (the interval $[0, \pi]$) and reflect across the line $y = x$.

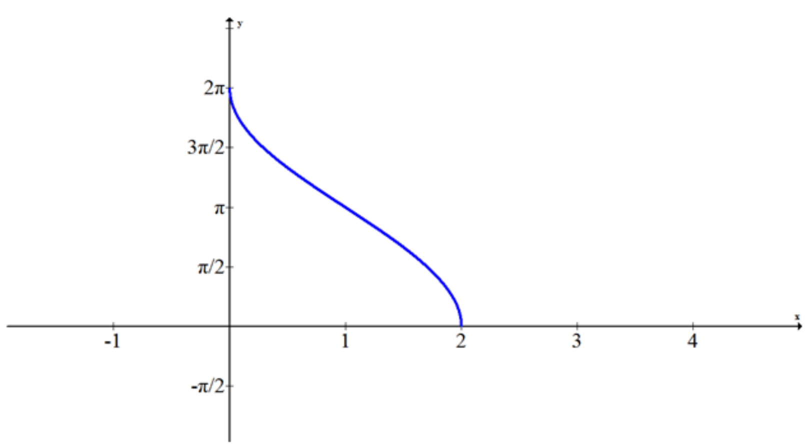


Example 2

Graph the function $f(x) = 2\cos^{-1}(x-1)$.

Solution:

Since the graph of $f(x) = \cos^{-1} x$ was sketched in Example 1, simply shift it right 1 unit and stretch it vertically by a factor of 2. It intersected the x axis at 1 before, and now it will intersect at 2. It reached a height of π before, and now it will reach a height of 2π.



Example 3

Evaluate the following expression with and without a calculator, using right triangles and the basic inverse trigonometric functions:

$$\cot\left(\csc^{-1}\left(-\frac{13}{5}\right)\right).$$

Solution:

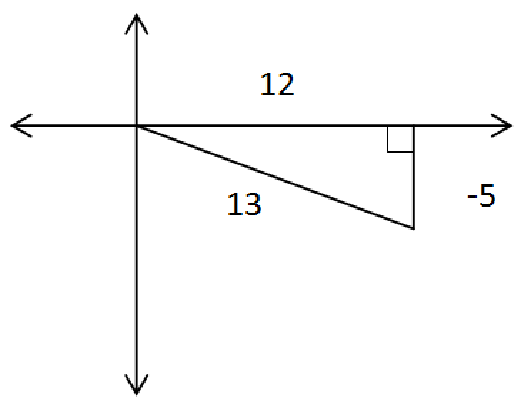
Recall that the cosecant is a function that produces an angle of a ratio of a hypotenuse of 13, in this case, and an opposite side of -5. The sine of the inverse ratio must produce the same angle, so start there:

$$\csc^{-1}\left(-\frac{13}{5}\right) = \sin^{-1}\left(-\frac{5}{13}\right).$$

Apply $\cot(\theta) = \frac{1}{\tan\theta}$

$$\cot\left(\csc^{-1}\left(-\frac{13}{5}\right)\right) = \frac{1}{\tan\left(\sin^{-1}\left(-\frac{5}{13}\right)\right)} = -\frac{12}{5}.$$

Graphically, you can recall that $\csc^{-1}\left(-\frac{13}{5}\right)$ describes an angle in the 4th or 2nd quadrant because those are the two quadrants where cosecant is negative. Since the inverse cosecant function has a range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it only produces angles in quadrant I or quadrant IV. This reference triangle must then be in the 4th quadrant. Just draw the triangle and identify the cotangent ratio.



Example 4

Return to the Introduction problem: Explain why $\sin^{-1}(\sin 370°) \neq 370°$.

Solution:

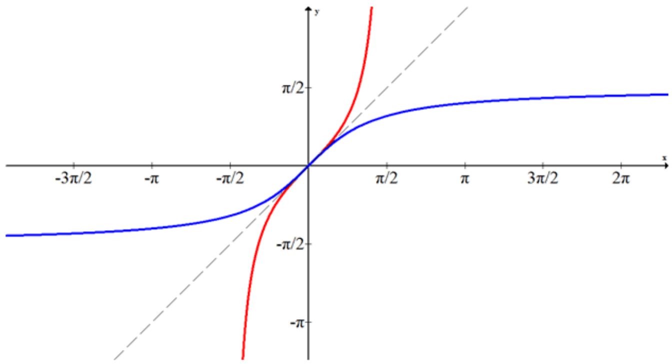
The initial angle of 370° is not in the restricted domain of $[-90°, 90°]$. However, the resulting angle of the inverse sine of sine of 370° will fall in this restricted domain. Thus, the left and right sides of the given equation are not equal.

Example 5

What is the graph of $y = \tan^{-1} x$?

Solution:

Graph the portion of tangent that fits the horizontal line test and reflect across the line $y = x$. Note that the graph of the inverse tangent function is in blue.

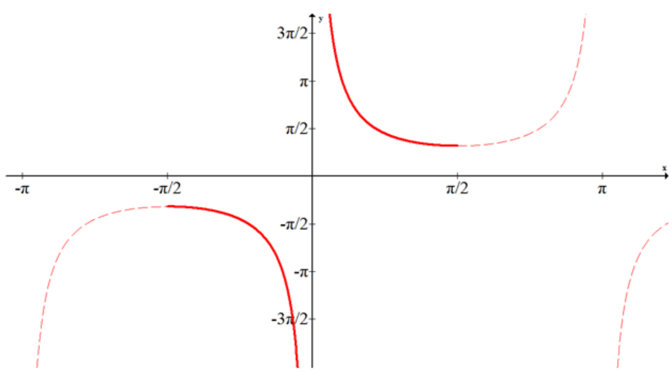


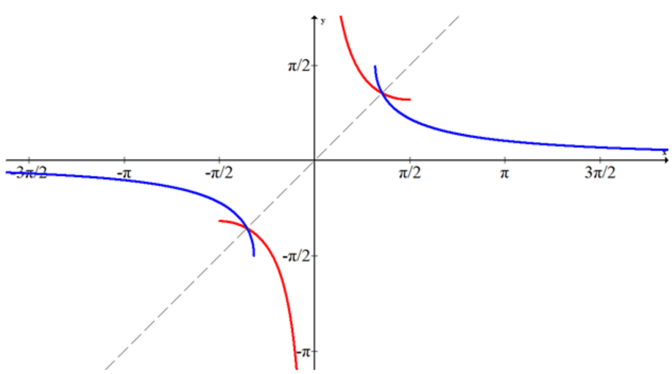
Example 6

What is the graph of $y = \csc^{-1} x$?

Solution:

Graph the portion of cosecant that fits the horizontal line test and reflect across the line $y = x$.





Note that $f(x) = \csc^{-1} x$ is the added portion in blue.

Example 7

Evaluate the expression $\csc\left(\cot^{-1}\left[-\frac{8}{6}\right]\right)$.

Solution:

Note that $\csc x = \frac{1}{\sin x}$ and $\cot^{-1}\left(\frac{a}{b}\right) = \tan^{-1}\left(\frac{b}{a}\right)$.

$$\begin{aligned}\csc\left(\cot^{-1}\left[-\frac{8}{6}\right]\right) &= \frac{1}{\sin\left(\cot^{-1}\left[-\frac{8}{6}\right]\right)}\\ &= \frac{1}{\sin\left(\tan^{-1}\left(-\frac{6}{8}\right)\right)}\\ &= -\frac{10}{6}\\ &= -\frac{5}{3}\end{aligned}$$

Summary

- Even though none of the trigonometric functions pass the horizontal line test over the entire set of real numbers, their domains can be restricted for the purpose of studying and applying inverses.
- There are two conventions used to identify inverse trigonometric functions: using the prefix of arc or the superscript of -1.
- A summary of the domains and ranges of all the inverse trig functions:

TABLE 7.6:

Notation 1	Notation 2	Definition	Domain	Range
$y = \arcsin x$	$y = sin^{-1}(x)$	$x = \sin y$	[-1, 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \arccos x$	$y = cos^{-1}(x)$	$x = \cos y$	[-1, 1]	$[0, \pi]$
$y = \arctan x$	$y = tan^{-1}(x)$	$x = \tan y$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \text{arccot}\, x$	$y = cot^{-1}(x)$	$x = \cot y$	$(-\infty, \infty)$	$(0, \pi)$
$y = \text{arccsc}\, x$	$y = csc^{-1}(x)$	$x = \csc y$	$(-\infty, -1] \bigcup [1, \infty)$	$\left(-\frac{\pi}{2}, 0\right) \bigcup \left(0, \frac{\pi}{2}\right)$
$y = \text{arcsec}\, x$	$y = sec^{-1}(x)$	$x = \sec y$	$(-\infty, -1] \bigcup [1, \infty)$	$(0, \pi)$

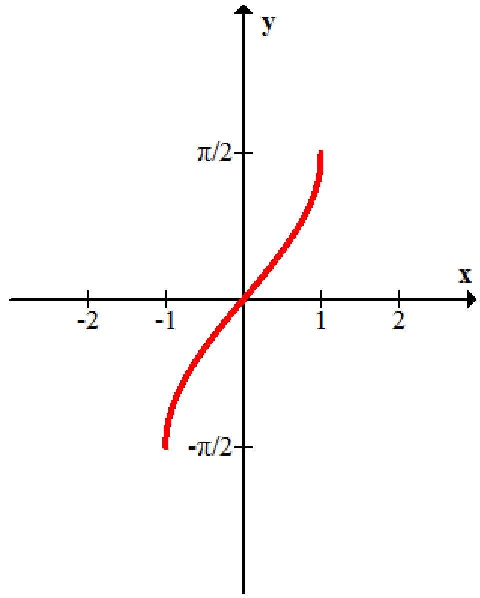
Review

1. Graph $f(x) = \cot^{-1} x$.

2. Graph $g(x) = \sec^{-1} x$.

Name each of the following graphs:

3.



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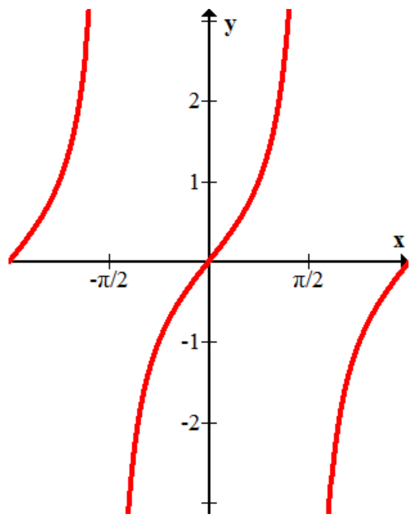

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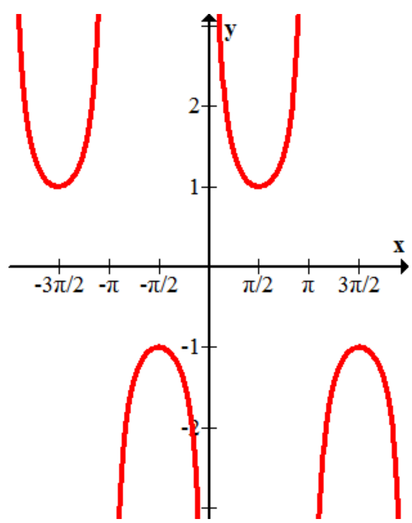

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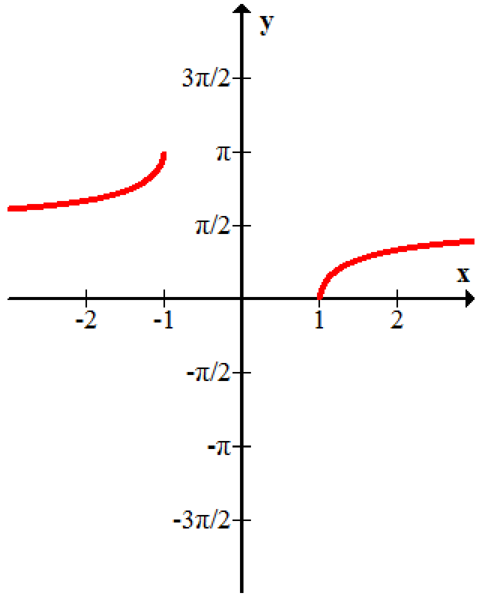

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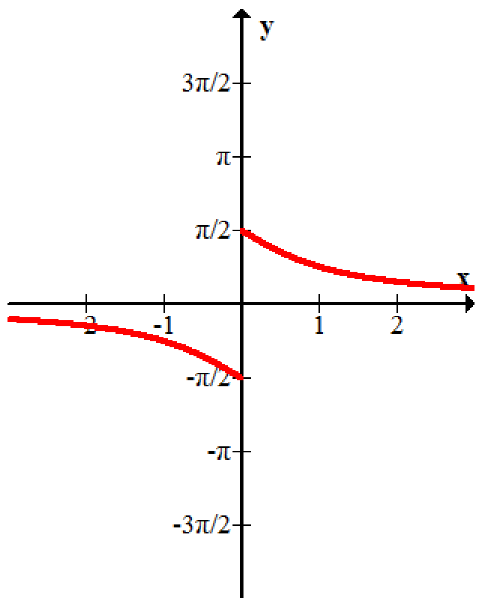


Graph each of the functions using function transformations:

8. $h(x) = 3\sin^{-1}(x+1)$

9. $k(x) = 2\sin^{-1}(x) + \frac{\pi}{2}$

10. $m(x) = -\cos^{-1}(x-2)$

11. $j(x) = \cot^{-1}(x) + \pi$

12. $p(x) = -2\tan^{-1}(x-1)$

13. $q(x) = \csc^{-1}(x-2)$

14. $r(x) = -\sec^{-1}(x) + 4$

15. $t(x) = \csc^{-1}(x+1) - \frac{3\pi}{2}$

16. $v(x) = 2\sec^{-1}(x+2) + \frac{\pi}{2}$

17. $w(x) = -\cot^{-1}(x) - \frac{\pi}{2}$

Evaluate each expression:

18. $\sec\left(\tan^{-1}\left[\frac{3}{4}\right]\right)$

19. $\cot\left(\csc^{-1}\left[\frac{13}{12}\right]\right)$

20. $\csc\left(\tan^{-1}\left[\frac{4}{3}\right]\right)$

Review (Answers)

Please see the Appendix.

7.8 Applications of Trigonometric Functions

Learning Objectives

Learn how to apply inverse and regular trig functions to solve problems you would see in real life situations.

Introduction

On a sunny spring day, you decide to map out a hiking trip to a new spot in the state park near your home. According to your map, you are supposed to go on a course that ends with you having moved 2.5 miles east and 3 miles south. Your starting and finishing points on your map are shown here:

Now you need to calculate the angle you need to walk with respect to due east. Can you find a way to calculate this angle using inverse trig functions?

Applying Trigonometric Functions

The following are real-world problems that can be solved using the trigonometric functions. In everyday life, indirect measurement is used to obtain answers to problems that are impossible to solve using measurement tools. However, mathematics will come to the rescue in the form of trigonometry to calculate these unknown measurements.

On a cold winter day, the sun streams through your living room window, creating a warm, toasty atmosphere. This is due to the angle of inclination of the sun, which directly affects the heating and cooling of buildings. At noon, the sun is at its maximum height in the sky, with the angle greater in the summer than in the winter. Because of this, buildings are constructed so that the overhang of the roof shades the windows for cooling in the summer, and yet allows sun rays to provide heat in the winter. The angle of inclination of the sun varies according to the latitude of the building's location.

If the latitude of the location is known, the following formula can be used to calculate the angle of inclination of the sun on any given date of the year:

Angle of sun $= 90^\circ - \text{latitude} + -23.5^\circ \cdot \cos\left[(N+10)\frac{360}{365}\right]$, where N represents the number of the day of the year that corresponds to the date of the year. Note: This formula is accurate to $\pm\frac{1^\circ}{2}$.

Determine the measurement of the sun's angle of inclination for a building located at a latitude of 42°, March 10, the 69*th* day of the year.

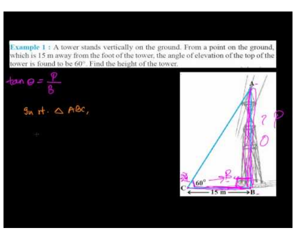
$$\begin{aligned}
\text{Angle of sun} &= 90^\circ - 42^\circ + -23.5^\circ \cdot \cos\left[(69+10)\frac{360}{365}\right] \\
\text{Angle of sun} &= 48^\circ + -23.5^\circ(0.2093) \\
\text{Angle of sun} &= 48^\circ - 4.92^\circ \\
\text{Angle of sun} &= 43.08^\circ
\end{aligned}$$

Determine the measurement of the sun's angle of inclination for a building located at a latitude of 20°, September 21.

$$\begin{aligned}
\theta_2 &= \theta - \theta_1 \\
\tan\theta = \frac{10}{x} &\text{ and } \tan\theta_1 = \frac{3}{x} \\
\theta_2 &= \tan^{-1}\left(\frac{10}{x}\right) - \tan^{-1}\left(\frac{3}{x}\right)
\end{aligned}$$

Note that if you didn't know the day of the year and had to find the day given the angle of the sun, you would need to use inverse cosine to solve.

Another application example can be seen in the following video:



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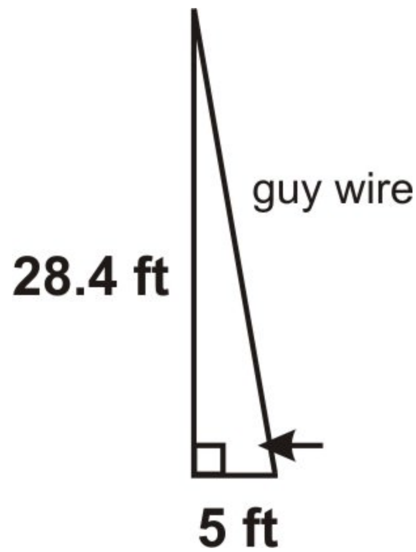
Examples

Example 1

A 28.4-foot tower must be secured with a guy-wire anchored 5 feet from the base of the tower. What angle will the guy-wire make with the ground?

Solution:



$$\tan\theta = \frac{opp.}{adj.}$$
$$\tan\theta = \frac{28.4}{5}$$
$$\tan\theta = 5.68$$
$$\tan^{-1}(\tan\theta) = \tan^{-1}(5.68)$$
$$\theta = 80.02^\circ$$

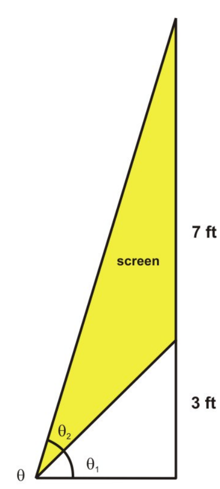
Example 2

Note that the following problem that involves functions and their inverses will be solved using the property $f(f^{-1}(x)) = f^{-1}(f(x))$.

In the main concourse of the local ice arena, several viewing screens are available to watch so you don't miss any of the action on the ice. The bottom of one screen is 3 feet above eye level, and the screen itself is 7 feet high. The angle of vision (inclination) is formed by looking at both the bottom and top of the screen. Calculate the measure of the angle of vision that results from looking at the bottom and then the top of the screen. At what distance from the screen does the maximum value for the angle of vision occur?

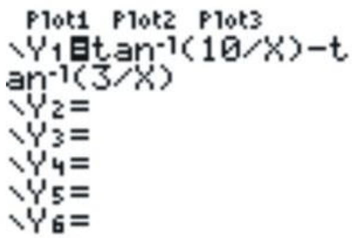
Solution:

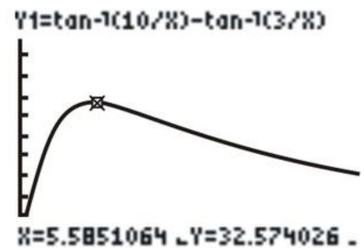
Sketch a picture to represent this problem:



$$\theta_2 = \tan\theta - \tan\theta_1$$
$$\tan\theta = \frac{10}{x} \text{ and } \tan\theta_1 = \frac{3}{x}$$
$$\theta_2 = \tan^{-1}\left(\frac{10}{x}\right) - \tan^{-1}\left(\frac{3}{x}\right)$$

To determine these values, use a graphing calculator and the trace function to determine when the actual maximum occurs.





From the graph, we can see that the maximum occurs just before $x \approx 5.59\ ft$ and $\theta_2 \approx 32.57°$. If you calculate the actual maximum there, it is approximately 5.477 at 32. 58°.

Example 3

A carnival Ferris wheel with a radius of 14 m makes one complete revolution every 16 seconds. The bottom of the wheel is 1.5 m above the ground. If a person is at the top of the wheel when a stopwatch is started, how high above the ground will that person be after 1 minute and 7 seconds?

Solution:

Amplitude is 14 m.

Since the Ferris wheel completes one revolution every 16 seconds,

$$b = \frac{360°}{16} = 22.5.$$

There is no horizontal shift for this problem, so $h = 0$.

The vertical shift is the radius of the Ferris wheel, plus the distance the wheel is from the ground.

$$k = 14 + 1.5 = 15.5 \text{ m}$$

The resulting equation is $y = 14\cos(22.5(x+0) + 15.5$.

In this problem, $x = 67$ s, so with the angle in degrees,

$$y = 14\cos(22.5(67)) + 15.5$$
$$y = 20.86 \text{ m}.$$

Thus, the person on the Ferris wheel will be 20.86 m above the ground after 1 minute and 7 seconds.

Example 4

Recall the problem from the Introduction: You are planning a hiking trip and have to calculate the angle you need to walk with respect to due east.

Solution:

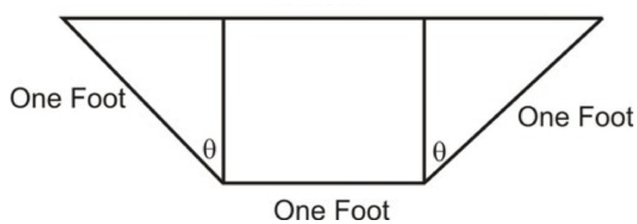
You can set up a triangle that matches the physical situation of this problem. Here's what it should look like:

Using the tangent function, you can solve for the angle you need to find:

$$\theta = \tan^{-1}\left(\frac{-3}{2.5}\right)$$
$$\theta = -50.19^\circ$$

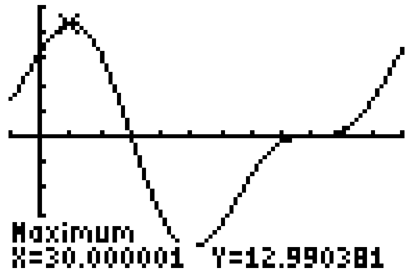
Example 5

The diagram below represents the end of a water trough. The ends are actually isosceles trapezoids, and the length of the trough from end to end is 10 feet. Determine the maximum volume of the trough and the value of θ that maximizes that volume.



Solution:

The volume is 10 feet times the area of the end. The end consists of two congruent right triangles and one rectangle. The area of each right triangle is $\frac{1}{2}(\sin\theta)(\cos\theta)$, and that of the rectangle is $(1)(\cos\theta)$. This means the volume can be determined by the function $V(\theta) = 10(\cos\theta + \sin\theta\cos\theta)$. This function can be graphed as follows to find the maximum volume and the angle θ where it occurs:



Therefore, the maximum volume is approximately 13 cubic feet and occurs when θ is about 30°.

Review

1. The distance from a boat to a lighthouse is 100 feet, and the lighthouse is 120 feet tall. What is the angle of depression from the top of the lighthouse to the boat?
2. You are standing 100 feet from an arch that is 68 feet tall. At what angle do you have to look up to see the top of the arch? Assume you are 5 feet tall.
3. The angle of elevation of the top of a church to a point 100 feet away from the base is 60°. Find the height of the church.

You are standing and looking at a large painting on the wall. The bottom of the painting is 1 foot above your eye level. The painting is 10 feet tall. Assume you are standing x feet from the painting, and that angle θ is formed by the lines of vision to the bottom and to the top of the painting.

4. Draw a picture to represent this situation.
5. Solve for θ in terms of x.
6. If you are standing 10 feet from the painting, what is θ?
7. If $\theta = 30^\circ$, how far are you standing from the wall (to the nearest foot)?

8. A 50-foot-high tower is secured with a guy-wire anchored 8 feet from the base of the tower. What angle will the guy-wire make with the ground?
9. A 30-foot-tall flagpole casts a 12-foot shadow. What is the angle that the sun hits the flagpole?

A water wheel with a radius of 10 m is partially submerged 3 m under water. The wheel makes one revolution in 360°, and the bucket starts at the center and initially moves upward.

10. Write a sine function to model the height of the bucket.
11. If $x = 40^\circ$, determine the height of the bucket.

Recall that if the latitude of the location is known, then the following formula can be used to calculate the angle of inclination of the sun on any given date of the year:

Angle of sun $= 90^\circ - \text{latitude} + -23.5^\circ \cdot \cos\left[(N+10)\frac{360}{365}\right]$, where N represents the number of the day of the year that corresponds to the date of the year.

12. Determine the measurement of the sun's angle of inclination for a building located at a latitude of 30°, April 12, the $102th$ day of the year.
13. Determine the measurement of the sun's angle of inclination for a building located at a latitude of 50°, August 14, the $226th$ day of the year.

The alternating half-daily cycles of the rise and fall of the ocean are called tides. In one bay, the tides caused the water level to rise 6.5 m above and fall 6.5 m below the average sea level. The tide completes one cycle every 12 hours.

14. Write a sine function to model the height of the water with respect to average sea level.
15. If high tide is at 8 a.m., determine where the water level would be at 2:30 p.m.

Review (Answers)

Please see the Appendix.

7.9 Project: The Unit Circle and Trigonometric Functions

Simple Harmonic Motion

The trigonometric functions sine and cosine are used to describe the behavior of objects undergoing simple harmonic motion—a motion that repeats itself in a predictable manner. For example, a pendulum swinging back and forth is an example of an object that exhibits simple harmonic motion. A buoy's fluctuating distance from the ocean floor is another example of harmonic motion.

For this assignment, you will plot data points and create an equation that can be used to predict the behavior of the object.

1. Review the following data:

TABLE 7.7:

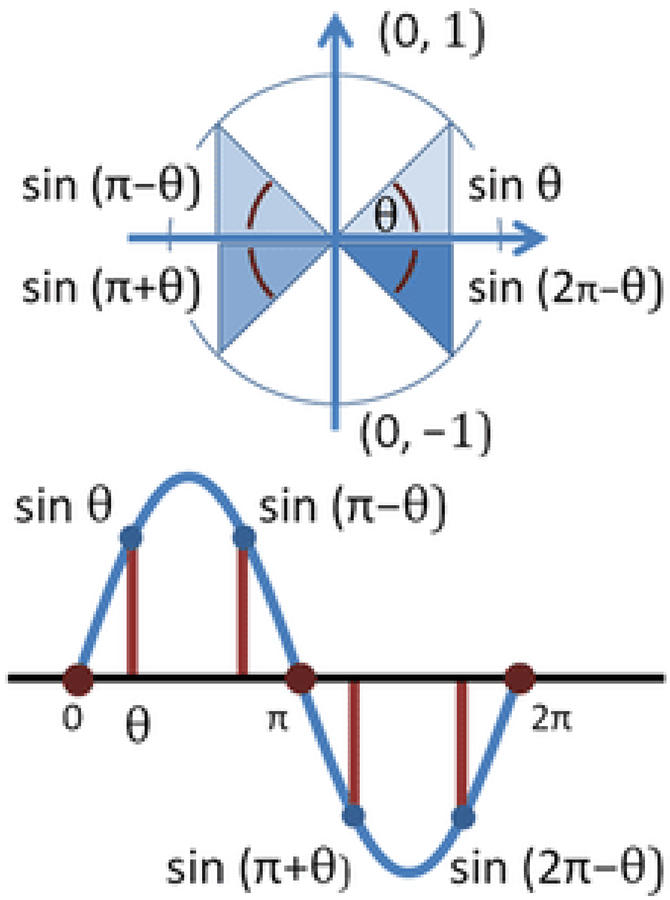
Seconds	1	2	3	4	5	6	7	8	9	10	11	12
Buoy (ft)	3.51	3.01	1.27	0.79	0	0.89	1.25	1.75	2.89	3.49	2.79	1.68
Spring (ft)	13.5	11.11	8.7	4.99	2.45	-06.5	-6.69	-10	-13.5	- 14.86	-14.9	-14
Monuments Shadow (ft)	9.77	10.27	11.29	13.52	14.77	15.01	14.99	13.02	12.11	11.19	10.22	9.28

2. Use the above data on each motion to create a smooth graph for each object.
   a. You can use as many data points as you would like, but at a minimum the data should be sufficient to complete one full period.
      a. Sketch a graph of the unit circle, and label at least 8 $(t, x(t))$ pairs.
      b. Sketch one period of $x(t) = a\cos bt + k$.
   b. Label the t and $x(t)$ axes.
   c. Determine the properties of the function. Explain each value in context.
      a. The amplitude, A.
      b. The period, T.
      c. The frequency, f
      d. The phase shift.
   d. Write the general equation for the function.

7.10 Summary: The Unit Circle and Trigonometric Functions

Chapter Summary

We've learned many tools in this chapter to understand how to graph trigonometric functions.



We've learned about:

The Unit Circle

- The unit circle provides a basis for understanding how the sine, cosine, and tangent ratios relate.
- The unit circle is a circle with its center at the origin and with a radius of 1, whose equation is $x^2+y^2=1$.

Graphing Trigonometric Functions

- Sinusoidal functions have the form $f(x)=a\sin(b(x-h))+k$, where $|a|$ is the amplitude, h is the horizontal or phase shift, and k is the vertical shift. The period is $\frac{2\pi}{|b|}$ and the frequency is $\frac{|b|}{2\pi}$.
- Since cosecant is the reciprocal of the sine function, and secant is the reciprocal of the cosine function, the secant and cosecant graphs can be sketched by sketching a faint sine or cosine graph and using that graph to establish asymptotes.

- To graph the tangent and cotangent functions, determine the asymptotes and zeros of the functions using the knowledge of the ratio of sine and cosine functions.

Inverse Trigonometric Functions

- There are two conventions used to identify inverse trigonometric functions: using the prefix of arc, or the superscript of -1.
- Even though none of the trigonometric functions pass the horizontal line test over the entire set of real numbers, their domains can be restricted for the purpose of studying and applying inverses.

Review

Try the following cumulative review problems to practice the concepts in this chapter:

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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/195773

7.11 References

1. jcesar2015. https://pixabay.com/photo-1331667/ .
2. CK-12 Foundation. .
3. CK-12. Reference Angles .
4. CK-12 Foundation. .
5. . Example 1.
6. . Second Quadrant.
7. . Structure Second Quadrant.
8. . 30-60-90 Triangle.
9. . 315 Degree Angle.
10. Gustavb. Completed Unit Circle .
11. Petr Kratochvil. http://www.publicdomainpictures.net/view-image.php?image=2225 .
12. . Derived Sine Graph.
13. CK-12. Basic Sine Graph .
14. CK-12. One Cycle Cosine Graph .
15. CK-12. Cosine Graph .
16. CK-12. Compare Sine and Cosine Graphs .
17. . Sine Graph.
18. Allen McGregor. https://commons.wikimedia.org/wiki/File:Chicago_skyline,_viewed_from_John_Hancock_-Center.jpg .
19. Paula Evans. Chicago Temperatures .
20. . Chicago Temperatures.
21. Robert Lowe. https://www.flickr.com/photos/rmlowe/6144649661 .
22. Paula Evans. Graph Created by Desmos .
23. CK-12. Sine vs Cosine Graph .
24. Chris Luczkow. https://www.flickr.com/photos/cluczkow/3498636003/in/photolist-mTiFR-9BpHRZ-6kaqAp-gcqNf-2XAzkX-ouixoK-3Khq-csZEv-5s2qiN-7kLrUf-6fbPrX-5s2r8j-foSoXJ-S7XUV3-emtgW5-D32Q-8sBuo8-gTbFUj-rZDmNL-6LbKuc-ebinUv-7LtVXw-mTkLB-7CTa7-eRNFr5-cTmvN-2tv8md-csYHq-9jQVeD-3a1Y7-oAj7LZ-qFM82q-oJteZt-mDk5UH-pBDcqi-8WDfb1-5s2reA-qcbzS7-QHA7Rm-5j9PtW-3ZnLG-S7XPTj-u7eeV-gjkwer-mmt9B-pDwN7-qwv3Mh-53JRGp-yTyg4f-7hm5vC/ .
25. Andrew Malone. https://www.flickr.com/photos/andrewmalone/236339735/in/photolist-mTiz2-33m6bx-S7XUFW-4iNfwy-Sb1UgD-S7XVmo-7ho7oP-SvtLiT-5fRTgd-eUzzk-8o3hXp-mTiFR-9BpHRZ-6kaqAp-gcqNf-2XAzkX-ouixoK-3Khq-csZEv-5s2qiN-7kLrUf-6fbPrX-5s2r8j-foSoXJ-S7XUV3-emtgW5-D32Q-8sBuo8-gTbFUj-rZDmNL-6LbKuc-ebinUv-7LtVXw-mTkLB-7CTa7-eRNFr5-cTmvN-2tv8md-csYHq-9jQVeD-3a1Y7-oAj7LZ-qFM82q-oJteZt-mDk5UH-pBDcqi-8WDfb1-5s2reA-qcbzS7-QHA7Rm .
26. fsHH. https://pixabay.com/en/first-aid-fire-rescue-112-1030700/ .
27. . Definition of Secant Graph.
28. . Definition of Tangent Graph.
29. John Fellner. https://flic.kr/p/daNkcd .
30. CK-12. Basic Arcsine Graph .
31. CK-12 Foundation. CK-12 Foundation .
32. CK-12 Foundation. CK-12 Foundation .

CHAPTER 8 Analytic Trigonometry

Chapter Outline

8.1 Introduction: Analytic Trigonometry

Trigonometric functions can be disguised in many ways. Their ability to repeat a pattern infinitely allows their forms to be represented with different functions. The art and science of showing that two trigonometric functions are either the same or not uses "trigonometric identities." To be able to answer any question that can be modeled by trigonometric functions, we must solve equations. For example, in another chapter we discussed $y = 28\sin(0.48x - 1.81) + 56$, which is a function that predicts the average monthly temperature for Chicago. For this model, x is the month number. A meteorologist would use this model to find the times of the year when the average monthly temperature is 48 degrees. To do that, the meteorologist could use identities. In this chapter, we will explore the algebraic manipulations of trigonometric functions and the trigonometric identities. These will serve as a toolbox that, together with algebra, will help simplify and solve trigonometric equations.

8.2 Basic Trigonometric Identities

Learning Objectives

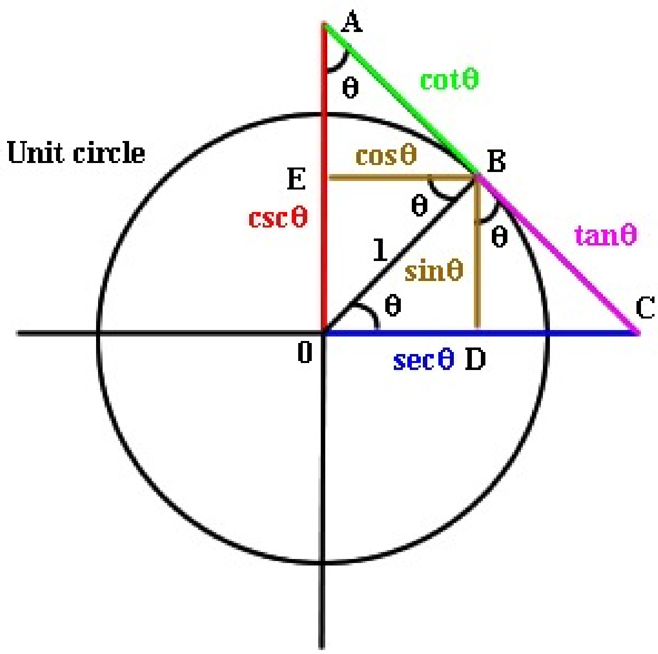
Learn to simplify trigonometric expressions using the reciprocal, quotient, odd-even and cofunction identities. You will also apply these simplification techniques in trigonometric proofs.

Introduction

A manager of a temporary employment agency is responsible for knowing when to staff temporary vocational nurses for area hospitals. After tracking data about the requests for nurses, the manager developed a mathematical function to predict the hospitals' needs: $N(t) = 20 + 4\sin(2t)$, where $t \geq 0$ is in hours. At a nursing conference, the manager attended a session where a staffing expert used the formula $N(t) = 20 + 4\cos\left(2t - \frac{\pi}{2}\right)$, $t \geq \frac{\pi}{4}$. If the manager and the expert both used the same data, why are the functions so different?

Trigonometric Identities

Trigonometric identities are used to rewrite trigonometric expressions to manipulate the form of the expression, often to be able to solve equations. The basic trigonometric identities can be logically deduced from the definitions and graphs of the six trigonometric functions. We have previously used some of these identities to develop the families of trigonometric graphs. Now we will formalize them and begin to create a toolbox of trigonometric identities.



Unit Circle Definition of Trigonometric Functions

On the unit circle with the radius $r = 1$, trigonometric functions can be defined using right triangle trigonometry:

$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y \qquad \csc\theta = \frac{r}{y} = \frac{1}{y}$$

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x \qquad \sec\theta = \frac{r}{x} = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

The reciprocal identities refer to the relationships between the trigonometric functions, like sine and cosecant, based on their definitions.

Reciprocal Identities

These identities are manipulations of the unit circle trig function definitions:

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

The quotient identities also follow from the unit circle definition of sine, cosine, and tangent.

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Cofunction Identities

The cofunction identities make the connection between trigonometric functions and their counterparts. A function f is the **cofunction** of a function g if $f(A) = g(B)$, when A and B are complementary angles. Note that for a right triangle, the sum of the two other acute angles is 90° or $\frac{\pi}{2}$ radians.

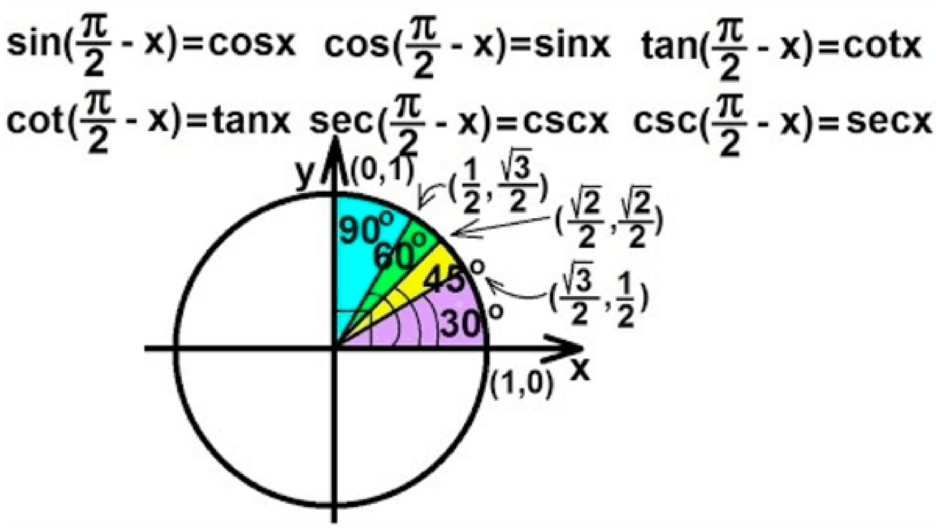
Cofunction Identities

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

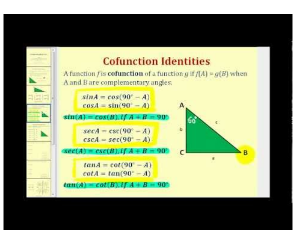
$$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta \qquad \cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$$

$$\sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta \qquad \csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta$$

Graphically, each pair of cofunctions are reflections and horizontal transformations.



A summary of cofunction identities and a few examples can be seen in the following video.



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Even and Odd Identities

Even and odd refer to the symmetry of the graphs of functions.

An **even function** is a function where the resultant value (or output) of a function is the same as the resultant value when the opposite of the value is inputted. Even functions are symmetrical across the y-axis.

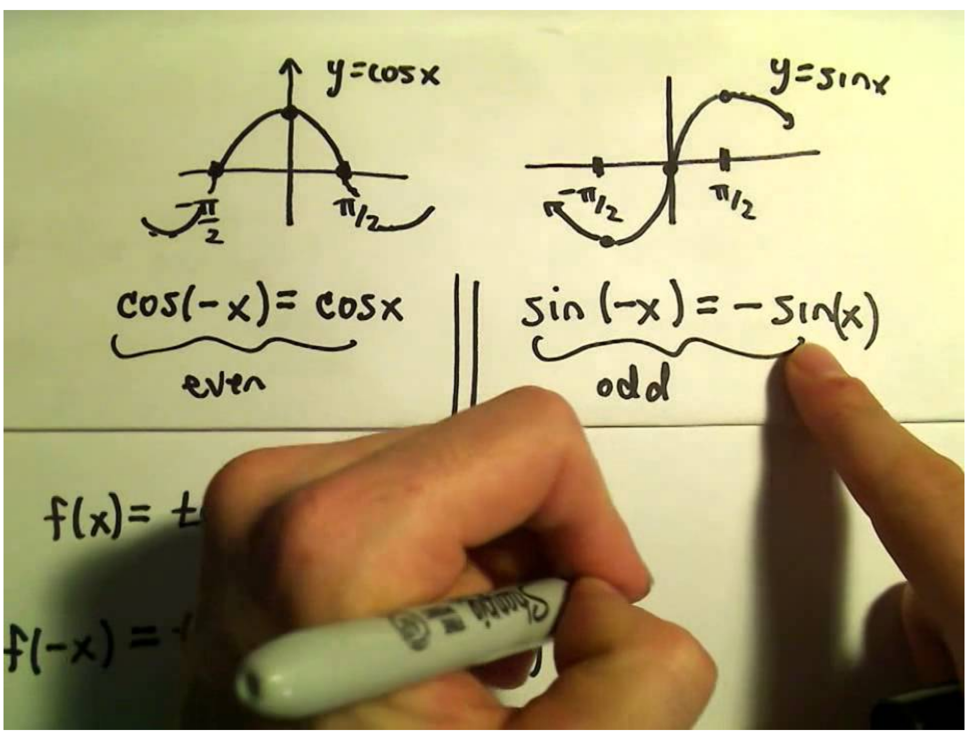
Even Function

$$f(x) = f(-x)$$

The only trigonometric functions that are even are cosine and its reciprocal secant.

Even Trigonometric Identities

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$



An **odd function** is a function where the negative of the resultant value is the same as the function performed on the opposite of the value. Odd functions are symmetrical about the origin.

Odd Function

$$-f(x) = f(-x)$$

Four basic trigonometric functions are odd: sine and its reciprocal cosecant, and tangent and its reciprocal cotangent.

Odd Trigonometric Identities

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

Examples

Example 1

Let $\sin\theta = 0.87$. Determine $\cos\left(\theta - \frac{\pi}{2}\right)$.

Solution:

While it is possible to use a calculator to find θ, practice using the identities will help you develop an intuition about the patterns of trigonometric functions, which will be useful when solving equations.

When you work with identities, your process should always include a goal and a sequence of steps to approach the goal.

$$\begin{aligned} \cos\left(\theta - \frac{\pi}{2}\right) &= \cos\left(-\left(\frac{\pi}{2} - \theta\right)\right) && \text{Factor the argument.} \\ &= \cos\left(\frac{\pi}{2} - \theta\right) && \text{Apply the property that cosine is an even function.} \\ &= \sin\theta && \text{Apply cofunction identity.} \\ &= 0.87 \end{aligned}$$

Example 2

Use trigonometric identities to simplify the expression $\tan x \cot x + \frac{\sin x \cot x (\sec x)^2}{\sec x}$.

Solution:

$$\begin{aligned}
\tan x \cdot \cot x + \frac{\sin x \cdot \cot x \cdot (\sec x)^2}{\sec x} &= \tan x \cdot \cot x + \sin x \cdot \cot \cdot \sec x \\
&= \tan x \cdot \frac{1}{\tan x} + \sin x \cdot \frac{\cos x}{\sin x} \cdot \sec x \\
&= 1 + \cos x \cdot \sec x \\
&= 1 + \cos x \cdot \frac{1}{\cos x} \\
&= 1 + 1 \\
&= 2
\end{aligned}$$

Example 3

Use trigonometric identities to prove $\cot(-\beta)\cot\left(\frac{\pi}{2}-\beta\right)\sin(-\beta) = \cos\left(\beta - \frac{\pi}{2}\right)$.

Solution:

When doing trigonometric proofs, it is vital to start on one side and work with that side until the derivation leads to the other side. In this case, work with the left side and keep rewriting to reach $\cos\left(\beta - \frac{\pi}{2}\right)$.

$$\begin{aligned}
\cot(-\beta) \cdot \cot\left(\frac{\pi}{2} - \beta\right) \cdot \sin(-\beta) &= -\cot\beta \cdot \tan\beta \cdot -\sin\beta \\
&= -1 \cdot -\sin\beta \\
&= \sin\beta \\
&= \cos\left(\frac{\pi}{2} - \beta\right) \\
&= \cos\left(-\left(\beta - \frac{\pi}{2}\right)\right) \\
&= \cos\left(\beta - \frac{\pi}{2}\right)
\end{aligned}$$

Example 4

Return to the problem from the Introduction: The manager developed a mathematical function to predict the hospitals' needs: $N(t) = 20 + 4\sin(2t),\ t \geq 0$. However, an expert at a conference used the formula $N(t) = 20 + 4\cos\left(2t - \frac{\pi}{2}\right),\ t \geq \frac{\pi}{4}$.If the manager and the expert both used the same data, why are the functions so different?

Solution:

The functions look different but are exactly the same or equivalent.

$$\begin{aligned}
N(t) &= 20 + 4\sin(2t) \\
&= 20 + 4\cos\left(\frac{\pi}{2} - 2t\right) \\
&= 20 + 4\cos\left(-(2t - \frac{\pi}{2})\right) \\
&= 20 + 4\cos\left(2t - \frac{\pi}{2}\right)
\end{aligned}$$

Example 5

Prove the quotient identity for tangent using the definition of sine, cosine, and tangent.

Solution:

When sine, cosine, and tangent are replaced with the shorthand for side ratios, the equivalence becomes a matter of algebra.

$$\begin{aligned}\tan\theta &= \frac{y}{x}\\ &= \frac{\sin\theta}{\cos\theta}\end{aligned}$$

Example 6

Let $\cos\left(\theta-\frac{\pi}{2}\right)=0.68$. Determine $\csc(-\theta)$.

Solution:

$$\begin{aligned}\cos\left(\theta-\frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{2}-\theta\right)\\ &= \sin\theta\\ &= \frac{1}{\csc\theta}\\ &= -\frac{1}{\csc(-\theta)}\end{aligned}$$

Therefore,

$$\begin{aligned}0.68 &= -\frac{1}{\csc(-\theta)}\\ \csc(-\theta) &= -\frac{1}{0.68} \approx -1.471.\end{aligned}$$

Example 7

Prove the following trigonometric identity by working with only one side:

$$\cos x\cdot\sin x\cdot\tan x\cdot\cot x\cdot\sec x\cdot\csc x = 1.$$

Solution:

Replace the second half of the trigonometric functions with their equivalent reciprocals.

$$\begin{aligned}\cos x\cdot\sin x\cdot\tan x\cdot\cot x\cdot\sec x\cdot\csc x &= 1\\ \cos x\cdot\sin x\cdot\tan x\cdot\frac{1}{\tan x}\cdot\frac{1}{\cos x}\cdot\frac{1}{\sin x} &= 1\end{aligned}$$

Since multiplication is commutative, rearrange the product so that each trigonometric function is matched with its reciprocal.

$$\begin{aligned}\cos x\cdot\frac{1}{\cos x}\cdot\sin x\cdot\frac{1}{\sin x}\cdot\tan x\cdot\frac{1}{\tan x} &= 1\\ 1\cdot 1\cdot 1 &= 1\end{aligned}$$

Summary

- Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

- Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

- Cofunction Identities

$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta \quad \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta \quad \cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$$
$$\sec\left(\frac{\pi}{2}-\theta\right) = \csc\theta \quad \csc\left(\frac{\pi}{2}-\theta\right) = \sec\theta \quad \tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$$

- Even-Odd Trigonometric Identities

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$
$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$
$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

Review

1. Prove the quotient identity for cotangent using sine and cosine.
2. Explain why $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$, using graphs and transformations.
3. Explain why $\sec\theta = \frac{1}{\cos\theta}$.
4. Prove that $\tan\theta \cdot \cot\theta = 1$.
5. Prove that $\sin\theta \cdot \csc\theta = 1$.
6. Prove that $\sin\theta \cdot \sec\theta = \tan\theta$.
7. Prove that $\cos\theta \cdot \csc\theta = \cot\theta$.
8. If $\sin\theta = 0.81$, what is $\sin(-\theta)$?
9. If $\cos\theta = 0.5$, what is $\cos(-\theta)$?
10. If $\cos\theta = 0.25$, what is $\sec(-\theta)$?
11. If $\csc\theta = 0.7$, what is $\sin(-\theta)$?

12. How can you tell from a graph if a function is even or odd?
13. Prove $\frac{\tan x \cdot \sec x}{\csc x} \cdot \cot x = \tan x$.
14. Prove $\frac{\sin^2 x \cdot \sec x}{\tan x} \cdot \csc x = 1$.
15. Prove $\cos x \cdot \tan x = \sin x$.

Review (Answers)

Please see the Appendix.

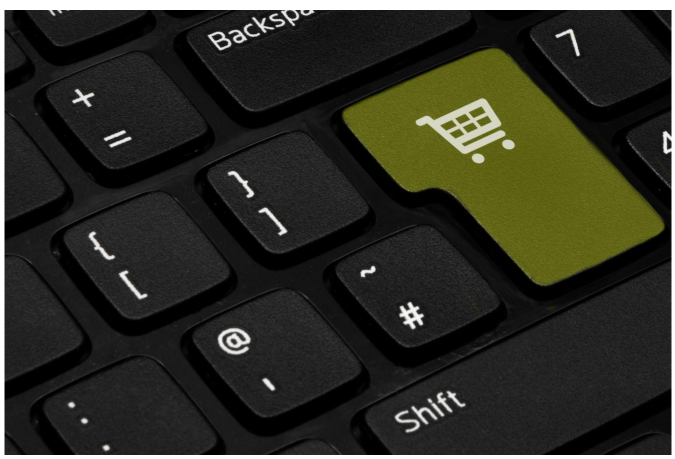
8.3 Pythagorean Trigonometric Identities

Learning Objectives

Learn to prove and use the Pythagorean identities for the six trigonometric functions to simplify expressions and write proofs.

Introduction

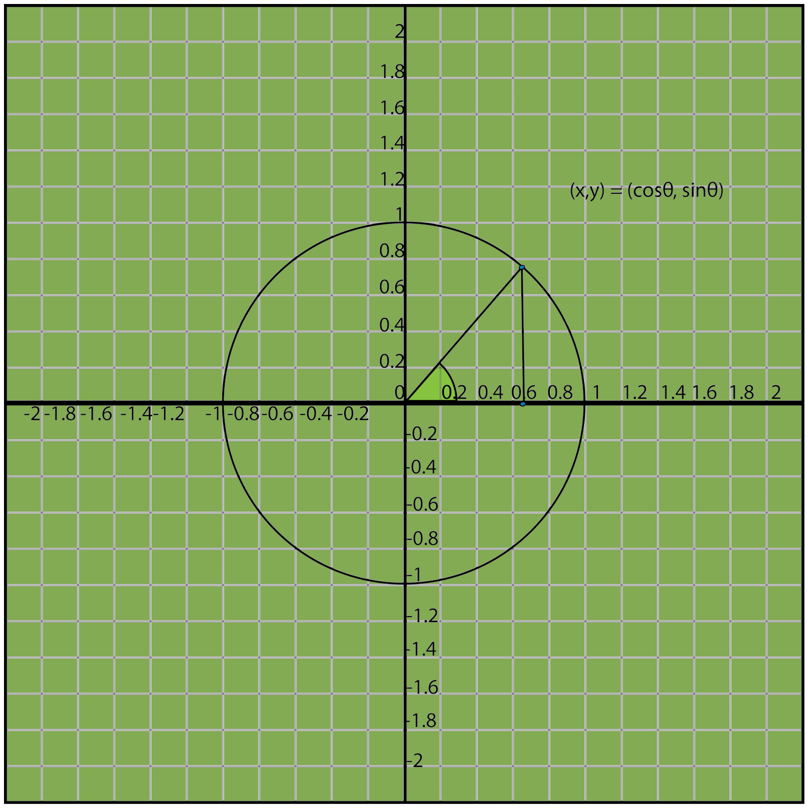
A data scientist works for an online retailer. The scientist is exploring patterns in data collected on purchasing patterns from the internet, and building mathematical functions that predict customer spending patterns.



To explore new graphs, the scientist graphed the function $y = \sin^2 x + \cos^2 x$, and the graph turned out to be a straight line. Confused, the scientist wants to know why such a complicated function has such a simple graph.

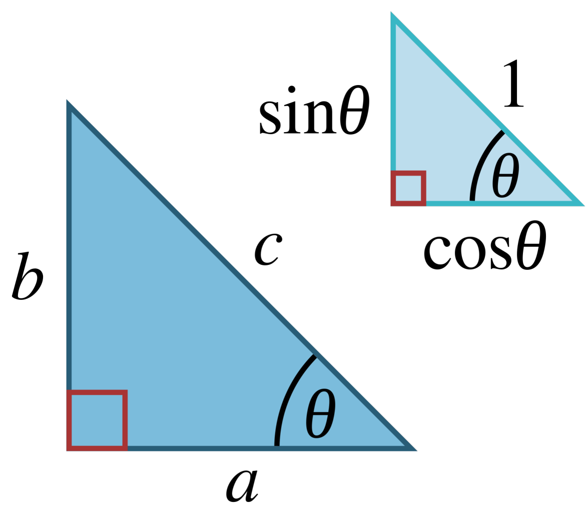
Pythagorean Trigonometric Identities

Pythagorean Trigonometric Identities leverage the power of the Pythagorean Theorem to simplify trigonometric expressions. Consider the x and y coordinates of a point on the unit circle.



Since $(x,y) = (\cos\theta, \sin\theta)$, and the radius of the unit circle is 1, the Pythagorean Theorem immediately yields the identity:

$$x^2 + y^2 = 1$$
$$\cos^2\theta + \sin^2\theta = 1$$
$$\sin^2\theta + \cos^2\theta = 1$$



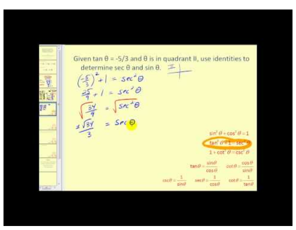
To establish the other two Pythagorean Trigonometric Identities, start with:

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1\\ \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} &= \frac{1}{\sin^2 x}\\ 1 + \cot^2 x &= \csc^2 x\end{aligned}$$

The proof for the other Pythagorean Trigonometric Identity, dealing with tangent and secant, will be shown in Example 3.

Pythagorean Trigonometric Identities

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1\\ 1 + \cot^2 x &= \csc^2 x\\ \tan^2 x + 1 &= \sec^2 x\end{aligned}$$



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Examples

Example 1

Simplify the following expression: $\frac{\sin x \cdot (\csc x - \sin x)}{1 - \sin x}$.

Solution:

$$\begin{aligned}\frac{\sin x \cdot (\csc x - \sin x)}{1 - \sin x} &= \frac{\sin x \cdot \csc x - \sin^2 x}{1 - \sin x}\\ &= \frac{\sin x \cdot \frac{1}{\sin x} - \sin^2 x}{1 - \sin x}\\ &= \frac{1 - \sin^2 x}{1 - \sin x}\\ &= \frac{(1 - \sin x)(1 + \sin x)}{1 - \sin x}\\ &= 1 + \sin x\end{aligned}$$

Note that factoring the Pythagorean Trigonometric Identity is powerful. This is a very common technique used to simplify.

Example 2

Prove the following trigonometric identity: $(\sec^2 x+\csc^2 x)-(\tan^2 x+\cot^2 x)=2$.

Solution:

Group the terms and apply a different form of the 2nd two Pythagorean Trigonometric Identities: $1+\cot^2 x=\csc^2 x$ and $\tan^2 x+1=\sec^2 x$.

$$\begin{aligned}(\sec^2 x+\csc^2 x)-(\tan^2 x+\cot^2 x)&=2\\(\sec^2 x-\tan^2 x)+(\csc^2 x-\cot^2 x)&=2\\1+1&=2\end{aligned}$$

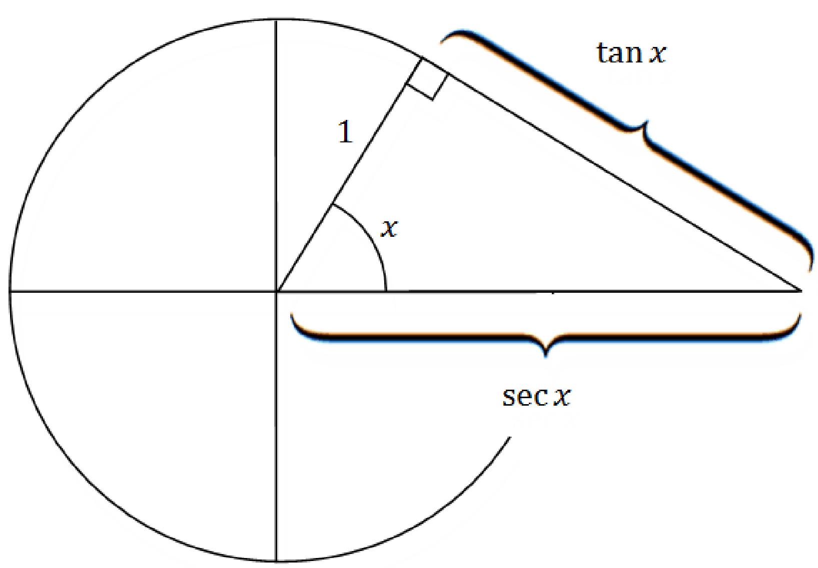
Example 3

Explain the fact that cofunctions are not always connected directly through a Pythagorean identity. For example, $\tan^2 x+\cot^2 x\neq 1$.

Solution:

Step 1: Visually, the right triangle connecting tangent and secant can also be observed in the unit circle. Tangent is named "tangent" because it refers to the distance of the line tangent from the point on the unit circle to the x axis: $\tan x=\frac{opp}{adj}$.

Step 2: Since the adjacent side is equal to 1 (the radius of the circle), $\tan x$ simply equals the opposite side. Similar logic can explain the placement of $\sec x$. So, $\tan^2 x+1=\sec^2 x$.



While the above connects tangent and secant through a right triangle, tangent and cotangent don't make up two legs of the same right triangle and thus don't add up to one for all values of x when they are squared.

Example 4

Return to the problem from the Introduction, where a data scientist works for an online retailer. The scientist graphed the function $y = \sin^2 x + \cos^2 x$ and the graph was a straight line. Confused, the scientist wants to know how such a complicated function has such a simple graph.

Solution:

The scientist must have forgotten the Pythagorean Trigonometric Identities! By him graphing $y = \sin^2 x + \cos^2 x$, he actually graphed $y = 1$ because $\sin^2 x + \cos^2 x = 1$. The simple graph the scientist saw was a horizontal line.

Example 5

Simplify the following expression:

$$(\sec^2 x)(1-\sin^2 x) - \left(\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}\right).$$

Solution:

$$\begin{aligned}
&(\sec^2 x)(1-\sin^2 x) - \left(\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}\right)\\
&= (\sec^2 x)(1-\sin^2 x) - \left(\frac{\sin x}{\frac{1}{\sin x}} + \frac{\cos x}{\frac{1}{\cos x}}\right)\\
&= \frac{1}{\cos^2 x} \cdot \cos^2 x - (\sin^2 x + \cos^2 x)\\
&= 1-1\\
&= 0
\end{aligned}$$

Example 6

Simplify the following expression:

$$(\cos t - \sin t)^2 + (\cos t + \sin t)^2.$$

Solution:

$$\begin{aligned}
&(\cos t - \sin t)^2 + (\cos t + \sin t)^2\\
&= \cos^2 t - 2\cos t \cdot \sin t + \sin^2 t + \cos^2 t + 2\cos t \cdot \sin t + \sin^2 t\\
&= 1 - 2\cos t \cdot \sin t + 1 + 2\cos t \cdot \sin t\\
&= 2
\end{aligned}$$

Summary

- The Pythagorean Identities allow for the use of the Pythagorean Theorem to simplify trigonometric expressions. They are:
  - $\sin^2 x + \cos^2 x = 1$
  - $1 + \cot^2 x = \csc^2 x$
  - $\tan^2 x + 1 = \sec^2 x$

Review

Prove each of the following:

1. $(1-\cos^2 x)(1+\cot^2 x)=1$
2. $\cos x(1-\sin^2 x)=\cos^3 x$
3. $\sin^2 x=(1-\cos x)(1+\cos x)$
4. $\sin x=\frac{\sin^2 x+\cos^2 x}{\csc x}$
5. $\sin^4 x-\cos^4 x=\sin^2 x-\cos^2 x$
6. $\sin^2 x\cos^3 x=(\sin^2 x-\sin^4 x)(\cos x)$

Simplify:

7. $\tan^3 x\csc^3 x$
8. $\frac{\csc^2 x-1}{\sec^2 x}$
9. $\frac{1-\sin^2 x}{1+\sin x}$
10. $\sqrt{1-\cos^2 x}$
11. $\frac{\sin^2 x-\sin^4 x}{\cos^2 x}$
12. $(1+\tan^2 x)(\sec^2 x)$
13. $\frac{\sin^2 x+\tan^2 x+\cos^2 x}{\sec x}$
14. $\frac{1+\tan^2 x}{\csc^2 x}$
15. $\frac{1-\sin^2 x}{\cos x}$

Review (Answers)

Please see the Appendix.

8.4 Sum and Difference Identities

Learning Objectives

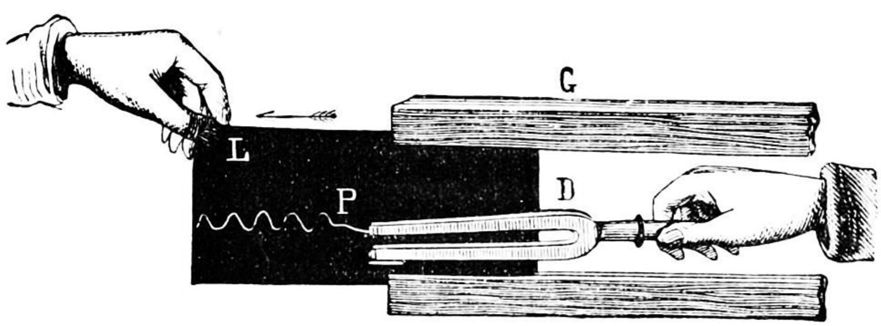
Learn about six identities: the sum and difference identities for sine, cosine, and tangent. You will use these identities along with previous identities for proofs and simplifying expressions.

Introduction

Sound waves and tones can be modeled by tuning forks. One sound might be modeled by the equation $s = 3\sin 2\pi t$, and another might be modeled by $s = 4\sin(t+3)$. It may be difficult to compare the tones without graphing. While trigonometric functions do not follow the properties of integers, they have identities that can be used to manipulate their variables.

In mathematics, the difference between exact and approximate values is always an issue. At this point with trigonometry, the only trigonometric functions known exactly are summarized in the unit circle. However, knowledge of these functions provides enough information to find the sine and cosine of 15° (the difference of 45° and 30°) and 75° (the sum of 45° and 30°).

Using the unit circle and a new set of identities, determine $\sin 15^\circ$ and $\sin 75^\circ$.

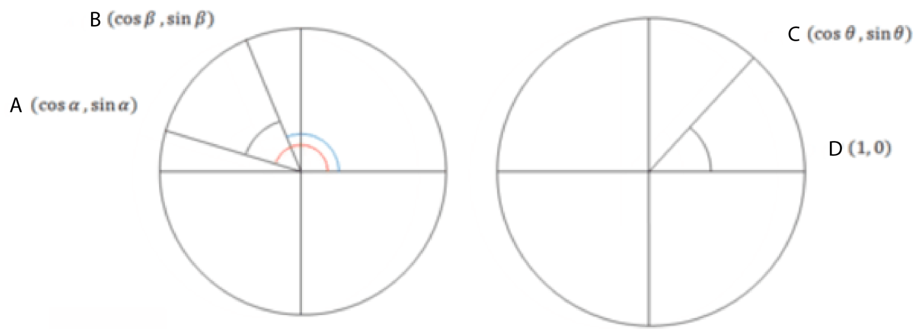


Sum and Difference Identities

There are some intuitive but incorrect formulas for sums and differences with respect to trigonometric functions. For example, $\sin(\theta+\beta) \neq \sin\theta + \sin\beta$ is one of the most common **incorrect** guesses as to the sum and difference identity.

First, look at the derivation of the cosine difference identity:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$



Start by drawing two arbitrary angles, α and β. In the image above, α is the angle in red and β is the angle in blue. The difference $\alpha - \beta$ is noted in black as θ. The reason why there are two pictures is because the image on the right has the same angle θ in a rotated position. This will be useful to work with because the length of $\overline{AB}$ will be the same as the length of $\overline{CD}$.

The length of $\overline{AB}$ by using the distance formula:

$$\overline{AB} = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$$

The length of $\overline{CD}$ by using the distance formula:

$$\overline{CD} = \sqrt{(\cos\theta - 1)^2 + (\sin\theta - 0)^2}$$

Since $\overline{AB} = \overline{CD}$,

$$\begin{aligned}\sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} &= \sqrt{(\cos\theta - 1)^2 + (\sin\theta - 0)^2}\\ (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 &= (\cos\theta - 1)^2 + (\sin\theta - 0)^2.\end{aligned}$$

Multiply through the squared terms and simplify, using algebra.

$$(\cos\alpha)^2 - 2\cos\alpha\cos\beta + (\cos\beta)^2 + (\sin\alpha)^2 - 2\sin\alpha\sin\beta + (\sin\beta)^2 = (\cos\theta)^2 - 2\cos\theta + 1 + (\sin\theta)^2$$

Use the Pythagorean Trigonometric Identity to further simplify.

$$\begin{aligned}2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= (\cos\theta)^2 - 2\cos\theta + 1 + (\sin\theta)^2\\ 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= 1 - 2\cos\theta + 1\\ -2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= -2\cos\theta\\ \cos\alpha\cos\beta + \sin\alpha\sin\beta &= \cos\theta\\ &= \cos(\alpha - \beta)\end{aligned}$$

The proof for the sine and tangent functions are left as an example and in the exercises. Cotangent, secant, and cosecant are excluded because it is better to use reciprocal identities.

Difference Identities

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$\tan(\alpha-\beta)=\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

The sum identities are nearly the same as the difference identities. However, notice the change in signs in the sum identities.

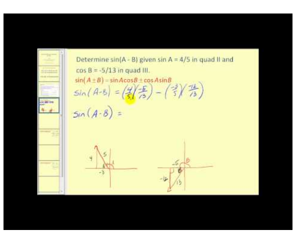
Sum Identities

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

Examples using the sum and difference formulas for sine and tangent functions can be seen in the following videos:



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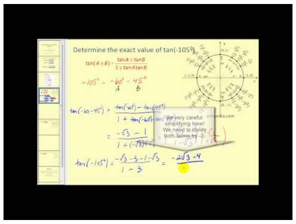

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Examples

Example 1

Prove the sum identity for the cosine function.

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta)$$

Solution:

Step 1: Start with the cosine of a difference and make a substitution. Then use the odd-even identity.

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)$$

Step 2: Let $\gamma = -\beta$

$$\begin{aligned} \cos\alpha\cos(-\gamma) + \sin\alpha\sin(-\gamma) &= \cos(\alpha+\gamma) \\ \cos\alpha\cos\gamma - \sin\alpha\sin\gamma &= \cos(\alpha+\gamma) \end{aligned}$$

Example 2

Find the exact value of $\tan 15^\circ$ without using a calculator.

Solution:

Step1:

$$\begin{aligned} \tan 15^\circ &= \tan(45^\circ - 30^\circ) \\ &= \frac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ} \\ &= \frac{1-\frac{\sqrt{3}}{3}}{1+1\cdot\frac{\sqrt{3}}{3}} \\ &= \frac{3-\sqrt{3}}{3+\sqrt{3}} \end{aligned}$$

Step 2: A final solution is rationalized. In this case, multiplying through by the conjugate of the denominator will eliminate the radical.

$$\begin{aligned}&=\frac{(3-\sqrt{3})\cdot(3-\sqrt{3})}{(3+\sqrt{3})\cdot(3-\sqrt{3})}\\&=\frac{9-6\sqrt{3}+3}{9-3}\\&=\frac{12-6\sqrt{3}}{6}\\&=2-\sqrt{3}\end{aligned}$$

Example 3

Evaluate the expression exactly without using a calculator.

$$\cos 50^\circ\cdot\cos 5^\circ+\sin 50^\circ\cdot\sin 5^\circ$$

Solution:

Apply cosine of a difference.

$$\begin{aligned}\cos 50^\circ\cos 5^\circ+\sin 50^\circ\sin 5^\circ&=\cos(50^\circ-5^\circ)\\&=\cos 45^\circ\\&=\frac{\sqrt{2}}{2}\end{aligned}$$

Example 4

Return to the problem in the Introduction, using the unit circle and a new set of identities to determine $\sin 15^\circ$ and $\sin 75^\circ$.

Solution:

In order to evaluate $\sin 15^\circ$ and $\sin 75^\circ$ exactly without a calculator, use the sine of a difference and the sine of a sum.

$$\begin{aligned}\sin(45^\circ-30^\circ)&=\sin 45^\circ\cos 30^\circ-\cos 45^\circ\sin 30^\circ\\&=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\&=\frac{\sqrt{6}-\sqrt{2}}{4}\\\sin(45^\circ+30^\circ)&=\sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ\\&=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\&=\frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Example 5

Prove the sine of a difference identity.

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

Solution:

Start with the cofunction identity, and then distribute and work out the cosine of a sum and cofunction identities.

$$\begin{aligned}\sin(\alpha-\beta) &= \cos\left(\frac{\pi}{2}-(\alpha-\beta)\right)\\ &= \cos\left(\left(\frac{\pi}{2}-\alpha\right)+\beta\right)\\ &= \cos\left(\frac{\pi}{2}-\alpha\right)\cos\beta-\sin\left(\frac{\pi}{2}-\alpha\right)\sin\beta\\ &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\end{aligned}$$

Example 6

Use a sum or difference identity to find an exact value of $\cot\left(\frac{5\pi}{12}\right)$.

Solution:

Start with the definition of cotangent as the inverse of tangent.

$$\begin{aligned}\cot\left(\frac{5\pi}{12}\right) &= \frac{1}{\tan\left(\frac{5\pi}{12}\right)}\\ &= \frac{1}{\tan\left(\frac{9\pi}{12}-\frac{4\pi}{12}\right)}\\ &= \frac{1}{\tan\left(\frac{3\pi}{4}-\frac{\pi}{3}\right)}\\ &= \frac{1+\tan\frac{3\pi}{4}\cdot\tan\frac{\pi}{3}}{\tan\frac{3\pi}{4}-\tan\frac{\pi}{3}}\\ &= \frac{1+(-1)\cdot\sqrt{3}}{(-1)-\sqrt{3}}\\ &= \frac{1-\sqrt{3}}{-1-\sqrt{3}}\cdot\frac{-1+\sqrt{3}}{-1+\sqrt{3}}\\ &= \frac{-1+2\sqrt{3}-3}{1-3}\\ &= \frac{-4+2\sqrt{3}}{-2}\\ &= 2-\sqrt{3}\end{aligned}$$

Example 7

Prove the following identity: $\frac{\sin(x-y)}{\sin(x+y)}=\frac{\tan x-\tan y}{\tan x+\tan y}$.

Solution:

$$\begin{aligned}
\frac{\sin(x-y)}{\sin(x+y)} &= \frac{\sin x\cos y-\cos x\sin y}{\sin x\cos y+\cos x\sin y}\\
&= \frac{\sin x\cos y-\cos x\sin y}{\sin x\cos y+\cos x\sin y}\cdot\frac{\left(\frac{1}{\cos x\cdot\cos y}\right)}{\left(\frac{1}{\cos x\cdot\cos y}\right)}\\
&= \frac{\left(\frac{\sin x\,\cancel{\cos y}}{\cos x\cdot\cancel{\cos y}}\right)-\left(\frac{\cancel{\cos x}\,\sin y}{\cancel{\cos x}\cdot\cos y}\right)}{\left(\frac{\sin x\,\cancel{\cos y}}{\cos x\cdot\cancel{\cos y}}\right)+\left(\frac{\cancel{\cos x}\,\sin y}{\cancel{\cos x}\cdot\cos y}\right)}\\
&= \frac{\tan x-\tan y}{\tan x+\tan y}
\end{aligned}$$

Summary

- Difference Identities

$$\begin{aligned}
\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\
\tan(\alpha-\beta) &= \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}
\end{aligned}$$

- Sum Identities

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\
\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}
\end{aligned}$$

Review

Find the exact value for each expression by using a sum or difference identity:

1. $\cos 75^\circ$

2. $\cos 105^\circ$

3. $\cos 165^\circ$

4. $\sin 105^\circ$

5. $\sec 105^\circ$

6. $\tan 75^\circ$

7. Prove the sine of a sum identity.

8. Prove the tangent of a sum identity.

9. Prove the tangent of a difference identity.

10. Evaluate without a calculator: $\cos 50^\circ \cos 10^\circ - \sin 50^\circ \sin 10^\circ$.
11. Evaluate without a calculator: $\sin 35^\circ \cos 5^\circ - \cos 35^\circ \sin 5^\circ$.
12. Evaluate without a calculator: $\sin 55^\circ \cos 5^\circ + \cos 55^\circ \sin 5^\circ$.
13. If $\cos\alpha\cos\beta = \sin\alpha\sin\beta$, then what does $\cos(\alpha+\beta)$ equal?
14. Prove that $\tan\left(x+\frac{\pi}{4}\right) = \frac{1+\tan x}{1-\tan x}$.
15. Prove that $\sin(x+\pi) = -\sin x$.

Review (Answers)

Please see the Appendix.

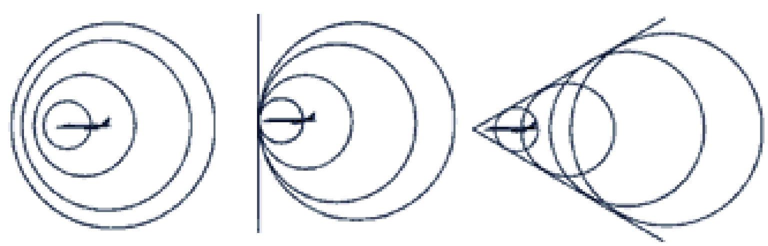
8.5 Double, Half, and Power-Reducing Identities

Learning Objectives

Learn to prove and use double-angle, half-angle, and power-reducing identities.

Introduction

A Mach number is the speed of a supersonic aircraft named for Austrian physicist Ernst Mach. The sonic boom cone represents the sound waves created by a sonic boom caused by aircraft flying faster than the speed of sound. The relationship between the cone's vertex angle, θ, and the Mach speed, M, can be represented by the formula $\sin\frac{\theta}{2} = \frac{1}{M}$.



One very useful goal of developing trigonometric identities is to create tools to be able to solve equations. Like angles, trigonometric functions can take many forms. Being able to change a trigonometric expression from form to form helps to complete that task.

The final set of identities are less geometrically intuitive than earlier identities. But practicing and working with these advanced identities will increase your toolbox for calculus. Each identity in this concept is aptly named. Double-angle identities can be used to find $\sin 80^\circ$ when $\sin 40^\circ$ is known. Half-angle identities are used to find $\sin 15^\circ$ when $\sin 30^\circ$ is known. Power-reducing identities can be used to find $(\sin 15^\circ)^2$ when the sine and cosine of 30° are known.

Double-Angle Identities

The double-angle identities are proved by applying the sum and difference identities.

For instance, the sum identity for the sine function is $\sin(x+y) = \sin x \cos y + \cos x \sin y$. Suppose the two angles are the same measure. Then the sum identity becomes

$$\begin{aligned}\sin(x+x) &= \sin x \cos x + \cos x \sin x\\ \sin(2x) &= 2\sin x \cos x.\end{aligned}$$

Similarly, the sum identity for the tangent function is $\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$. When the two angles are the same, then the sum identity becomes

$$\tan(x+x)=\frac{\tan x+\tan x}{1-\tan x\tan x}$$
$$\tan(2x)=\frac{2\tan x}{1-\tan^2 x}.$$

The double-angle proof for the cosine function is left as an exercise. Notice there are three versions of the double-angle identity for the cosine function. The other two are derived from using a Pythagorean Trigonometric Identity.

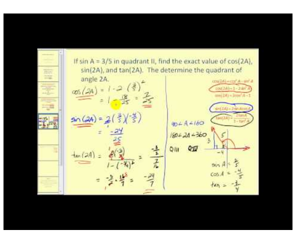
Double-Angle Identities

$$\sin 2x=2\sin x\cos x$$

$$\cos 2x=\cos^2 x-\sin^2 x$$
$$=1-2\sin^2 x$$
$$=2\cos^2 x-1$$

$$\tan 2x=\frac{2\tan x}{1-\tan^2 x}$$

Examples using the double-angle identities can be seen in the following video:



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Half-Angle Identities

The half-angle identities can be proved by applying the double-angle identities.

For instance, one of the double-angle identities for the cosine function is $\cos 2x=1-2\sin^2 x$. Suppose the angle is half as large, such that $\theta=\frac{x}{2}$. Then the double-angle identity becomes

$$\cos\left(2\cdot\frac{x}{2}\right)=1-2\sin^2\frac{x}{2}$$
$$\cos x=1-2\sin^2\frac{x}{2}.$$

Solve for $\sin\frac{x}{2}$.

$$2\sin^2\frac{x}{2}=1-\cos x$$
$$\sin^2\frac{x}{2}=\frac{1-\cos x}{2}$$
$$\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}$$

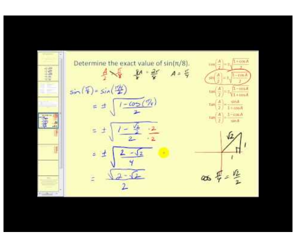
Similarly, use the double-angle identity for the cosine function $\cos 2x=2\cos^2 x-1$, and $\theta=\frac{x}{2}$ to solve for the half-angle identity for the cosine function. The half-angle proof for the tangent function is left as an exercise.

Half-Angle Identities

$$\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2}=\pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$



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Power-Reducing Identities

The power-reducing identities allow you to write a trigonometric function that is squared in terms of smaller powers. These identities come directly from the double-angle and half-angle identities. The proofs are left as an example and in the exercises.

Power-Reducing Identities

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$$

Examples

Example 1

Rewrite $\sin^4 x$ as an expression without powers greater than one.

Solution:

While $\sin x \cdot \sin x \cdot \sin x \cdot \sin x$ is one correct answer, applying another identity will produce a sum:

$$\begin{aligned}\sin^4 x &= (\sin^2 x)^2 \\ &= \left(\frac{1-\cos 2x}{2}\right)^2 \\ &= \frac{1-2\cos 2x+\cos^2 2x}{4} \\ &= \frac{1}{4}\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)\end{aligned}$$

Example 2

Write the following expression with only $\sin x$ and $\cos x$: $\sin 2x + \cos 3x$.

Solution:

$$\begin{aligned}\sin 2x + \cos 3x &= 2\sin x \cos x + \cos(2x + x)\\ &= 2\sin x\cos x + \cos 2x \cos x - \sin 2x \sin x\\ &= 2\sin x \cos x + (\cos^2 x - \sin^2 x)\cos x - (2\sin x \cos x)\sin x\\ &= 2\sin x\cos x + \cos^3 x - \sin^2 x\cos x - 2\sin^2 x\cos x\\ &= 2\sin x\cos x + \cos^3 x - 3\sin^2 x\cos x\end{aligned}$$

Example 3

Use the following half-angle identity to find an exact value of $\tan 22.5^\circ$ without using a calculator.

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

Solution:

$$\begin{aligned}\tan 22.5^\circ &= \tan\left(\frac{45^\circ}{2}\right)\\ &= \sqrt{\frac{1-\cos 45^\circ}{1+\cos 45^\circ}}\\ &= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}}\\ &= \sqrt{\frac{\frac{2}{2}-\frac{\sqrt{2}}{2}}{\frac{2}{2}+\frac{\sqrt{2}}{2}}}\\ &= \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}\\ &= \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2}}{2-\sqrt{2}}}\\ &= \sqrt{\frac{4-4\sqrt{2}+2}{4-2}}\\ &= \sqrt{\frac{6-4\sqrt{2}}{2}}\\ &= \sqrt{3-2\sqrt{2}}\end{aligned}$$

Example 4

Return to the problem in the Introduction: Determine $\sin^2 15^\circ$.

Solution:

Use the power-reducing identities to determine $\sin^2 15^\circ$.

$$\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2}\\ \sin^2 15^\circ &= \frac{1-\cos 30^\circ}{2}\\ &= \frac{1}{2}-\frac{\sqrt{3}}{4}\end{aligned}$$

Example 5

Prove the power-reducing identity for sine.

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

Solution:

Start with the double-angle identity for cosine.

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= (1-\sin^2 x)-\sin^2 x\\ &= 1-2\sin^2 x\end{aligned}$$

Note this expression is an equivalent expression to the double-angle identity, and is often considered an alternate form.

$$\begin{aligned}2\sin^2 x &= 1-\cos 2x\\ \sin^2 x &= \frac{1-\cos 2x}{2}\end{aligned}$$

Example 6

Simplify the following identity: $\sin^4 x - \cos^4 x$.

Solution:

$$\begin{aligned}\sin^4 x - \cos^4 x &= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)\\ &= -(\cos^2 x - \sin^2 x)\\ &= -\cos 2x\end{aligned}$$

Example 7

What is the period of the function below?

$$f(x) = \sin 2x \cdot \cos x + \cos 2x \cdot \sin x$$

Solution:

$$\begin{aligned} f(x) &= \sin 2x \cdot \cos x + \cos 2x \cdot \sin x \\ &= \sin(2x + x) \\ &= \sin 3x \end{aligned}$$

Since $b = 3$, this implies the period is $\frac{2\pi}{3}$.

Summary

- Double-Angle Identities:

$$\sin 2x = 2\sin x \cos x \qquad \cos 2x = \cos^2 x - \sin^2 x \qquad \tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

- Half-Angle Identities:

$$\sin \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}} \qquad \cos \frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}} \qquad \tan \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

- Power-Reducing Identities:

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \qquad \tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Review

Prove the following identities.

1. $\cos 2x = \cos^2 x - \sin^2 x$
2. $\cos 2x = 1 - 2\sin^2 x$
3. $\cos 2x = 2\cos^2 x - 1$
4. $\cos^2 x = \frac{1+\cos 2x}{2}$
5. $\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$
6. $\tan \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}$
7. $\csc 2x = \frac{1}{2}\csc x \sec x$
8. $\cot 2x = \frac{\cot^2 x - 1}{2\cot x}$
9. $\tan \frac{x}{2} = \frac{1-\cos x}{\sin x}$
10. Show that $\tan \frac{x}{2} = \frac{\sin x}{1+\cos x}$.
11. Rewrite $\cos^4 x$ as an expression without powers greater than one.

Find the value of each expression using half-angle identities.

12. $\csc 22.5^\circ$

13. $\tan 15^\circ$

14. $\tan 22.5^\circ$

15. $\sec 22.5^\circ$

Review (Answers)

Please see the Appendix.

8.6 Trigonometric Equations

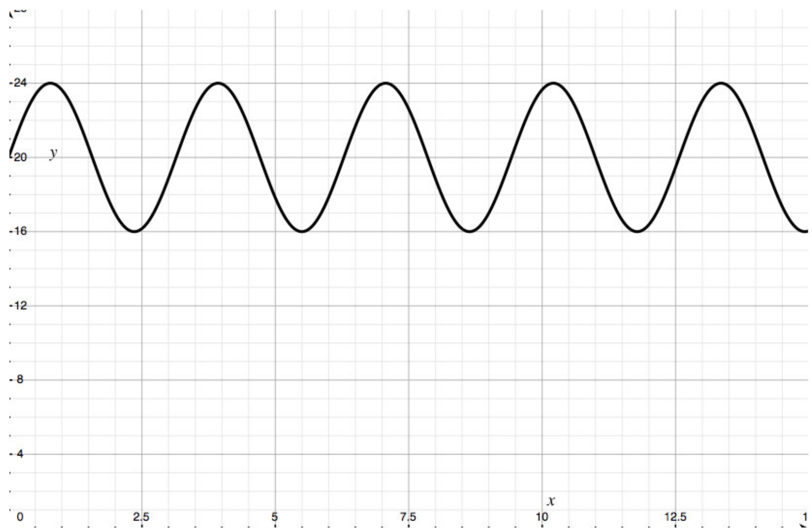
Learning Objectives

Learn to solve equations that contain trigonometric functions. You will also learn to identify when an equation is an identity and when it is has no solutions.

Introduction

A temporary employment agency manager gathered data on the number of movers needed during one year by local moving companies. The manager developed the model

$N(t) = 20 + 4\sin(2t)$, $t \geq 0$. The graph of the function is:



The manager needs to know the months during which 20 movers are needed. In the first 12 months, this happens several times. The equation to be solved is

$$20 = 20 + 4\sin(2t).$$

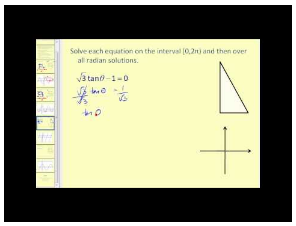
Solving this trigonometric equation is similar to solving a polynomial, rational, exponential, or logarithmic equation. Factoring and other algebraic techniques are used to isolate the variable. The biggest difference with trigonometric equations is the opportunity for an infinite number of solutions that must be described with a pattern. The equation $\cos x = 1$ has many solutions, including 0 and 2π. How can they all be described?

How to Solve a Trigonometric Equation

- Solve for the variable as you would a linear equation.
- Use inverse trigonometric functions to isolate the variable.

- Use Pythagorean Trigonometric, sum and difference, and other trigonometric identities to get a single trigonometric function, such as sine or cosine.
- Use sum and difference, double angle, and half angle identities to get a single angle.
- Use algebraic techniques, such as factoring or squaring, to further simplify the expressions.
- Use the unit circle to find the initial solution or to determine values within the specified interval. There may be more than one value.
- Provide an appropriate set of solutions.

When you solve an equation, the equation is an identity if the two sides of the equation are always equal, such as $\pi = \pi$. The equation has no solution if the two sides of an equation are never equal, such as $\sin x = 3$.



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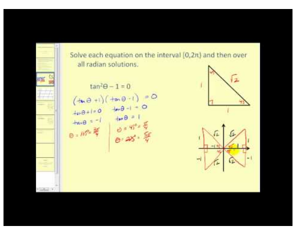
The following video provides examples of solving trigonometric equations that are factorable or in quadratic form:



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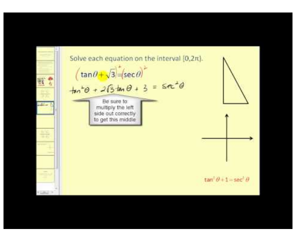
The following video provides examples of solving trigonometric equations that require the use of identities, are not factorable, or need to be squared to solve:



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URL: http://www.ck12.org/flx/render/embeddedobject/61364

Examples

Example 1

Solve the following equation algebraically and confirm graphically on the interval $[-2\pi, 2\pi]$:

$$\cos 2x = \sin x.$$

Solution:

Step 1: Use the double angle identity to get a single trigonometric function instead of a mixture of sine and cosine functions. Then use algebraic techniques to further simplify.

$$\begin{aligned}
\cos 2x &= \sin x \\
1 - 2\sin^2 x &= \sin x \\
0 &= 2\sin^2 x + \sin x - 1 \\
0 &= (2\sin x - 1)(\sin x + 1)
\end{aligned}$$

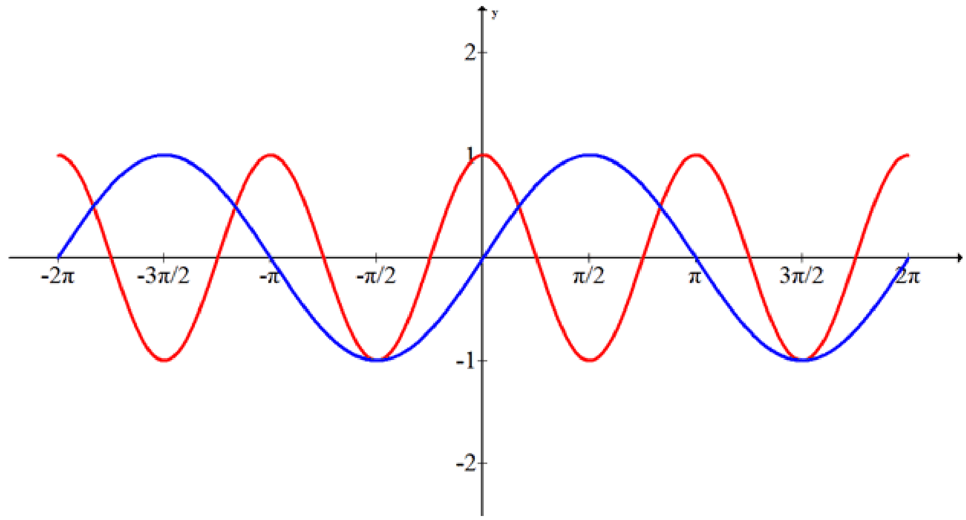
Step 2: Set the two factors equal to 0. Use the unit circle to determine values within the specified interval.

$$\begin{aligned}
0 &= 2\sin x - 1 \\
\frac{1}{2} &= \sin x \\
x &= \frac{\pi}{6}, \frac{5\pi}{6}
\end{aligned}$$

The coterminal angles for these angles are $-\frac{11\pi}{6}$ and $-\frac{7\pi}{6}$.

$$\begin{aligned}
0 &= \sin x + 1 \\
-1 &= \sin x \\
x &= -\frac{\pi}{2}, \frac{3\pi}{2}
\end{aligned}$$

Step 3: These are the six solutions to the initial equation in the given interval. The solutions appear as intersection points for the two graphs $f(x) = \cos 2x$ (in red and intersecting the y-axis at 1) and $g(x) = \sin x$ (in blue and intersecting the origin).



Example 2

Determine the general solution to the following equation:

$$\cot x - 1 = 0.$$

Solution:

Step 1: Solve for the variable as you would a linear equation.

$$\begin{aligned}\cot x - 1 &= 0\\ \cot x &= 1\end{aligned}$$

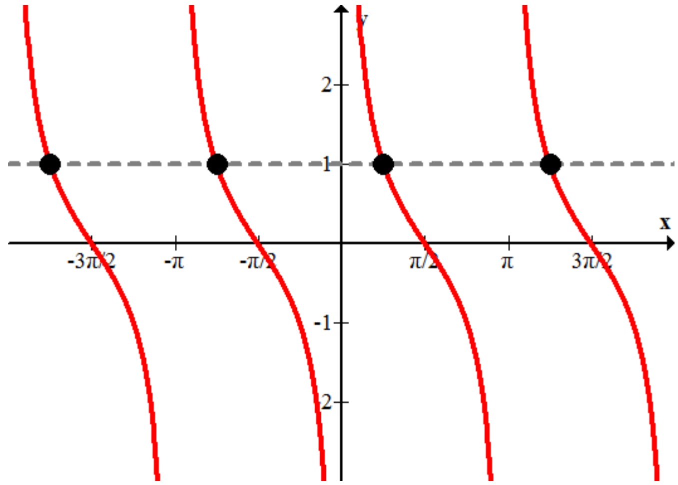
Step 2: Use the unit circle to find the initial solution. One solution is $x = \frac{\pi}{4}$.

Step 3: However, since this question asks for the general solution, find every possible solution. The cotangent has a period of π, which means if π is added or subtracted from $\frac{\pi}{4}$, then

$$x = \frac{\pi}{4} + n \cdot \pi$$

will also yield a height of 1. To capture all these other possible x values, use this notation:

$x = \frac{\pi}{4} + n \cdot \pi$, where n is a integer.



Example 3

Solve the following equation:

$$4\cos^2 x - 1 = 3 - 4\sin^2 x.$$

Solution:

Use algebraic techniques to simplify. Then use a Pythagorean Trigonometric Identity to further simplify.

$$\begin{aligned} 4\cos^2 x - 1 &= 3 - 4\sin^2 x \\ 4\cos^2 x + 4\sin^2 x &= 3 + 1 \\ 4(\cos^2 x + \sin^2 x) &= 4 \\ 4 &= 4 \end{aligned}$$

This equation is always true, which means the right side is always equal to the left side. This is an identity.

Example 4

Return to the problem from the Introduction. During which months over a 12-month interval will there be a demand for 20 movers?

$$20 = 20 + 4\sin(2t),\ t \in [0, 12]$$

Solution:

Use the unit circle to determine values within the specified interval.

$$\begin{aligned}
20 &= 20 + 4\sin(2t) \\
0 &= 4\sin(2t) \\
0 &= \sin 2t \\
2t &= 0 + n\pi, n \text{ is an integer} \\
t &= \frac{n\pi}{2}, n \text{ is an integer} \\
n &= 0, 1, 2, 3, 4, 5, 6, 7 \\
t &\approx 0,\ 1.57,\ 3.14,\ 4.71,\ 6.28,\ 7.85,\ 9.42,\ 11.00
\end{aligned}$$

The manager can expect that demand for movers in the 1st (month 0), 2nd, 4th, 5th, 7th, 8th, 10th, and 12th months.

Example 5

Solve the following equation on the interval $(2\pi, 4\pi)$:

$$2\sin x + 1 = 0.$$

Solution:

Step 1: Solve for the solutions within one period, and then find the solutions in the correct interval.

$$\begin{aligned}
2\sin x + 1 &= 0 \\
\sin x &= -\frac{1}{2} \\
x &= \frac{7\pi}{6}, \frac{11\pi}{6}
\end{aligned}$$

Step 2: Add 2π to each of these solutions to get solutions that are in the interval.

$$x = \frac{19\pi}{6}, \frac{23\pi}{6}$$

Example 6

Solve the following equation exactly:

$$2\cos^2 x + 3\cos x - 2 = 0.$$

Solution:

Step 1: Start by factoring:

$$\begin{aligned}
2\cos^2 x + 3\cos x - 2 &= 0 \\
(2\cos x - 1)(\cos x + 2) &= 0
\end{aligned}$$

Step 2: Set the two factors equal to 0. Use the unit circle to determine the initial solutions.

$$2\cos x - 1 = 0$$
$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}, -\frac{\pi}{3}$$

$$\cos x + 2 = 0$$
$$\cos x = -2$$

Since $\cos x \neq -2$, no solutions exist from this factor.

Step 3: These are the solutions within the interval $-\pi$ to π. Since this represents one full period of cosine, the rest of the solutions are just multiples of 2π added and subtracted to these two values.

$$x = \pm\frac{\pi}{3} + n \cdot 2\pi, \text{ where } n \text{ is an integer}$$

Example 7

Create an equation that includes the solutions $\frac{\pi}{4} + n \cdot 2\pi$, where n is an integer.

Solution:

There are an infinite number of possible equations that will work.

If we use the sine function, $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, an option that works is

$$\sin x = \frac{\sqrt{2}}{2}.$$

Note that this equation also has the solutions $\frac{3\pi}{4} + n \cdot 2\pi$.

Summary

- To solve a trigonometric equation, simplify algebraically to isolate the variable.
- Use trigonometric identities when possible.
- Use the unit circle to find the initial solution or to determine values within the specified interval.
- Trigonometric equations may have an infinite number of solutions that repeat in a certain pattern because they are periodic functions.

Review

Solve each equation on the interval $[0, 2\pi)$.

1. $3\cos^2\frac{x}{2} = 3$

2. $4\sin^2 x = 8\sin^2\frac{x}{2}$

Find approximate solutions to each equation on the interval $[0,2\pi)$.

3. $3\cos^2 x + 10\cos x + 2 = 0$
4. $\sin^2 x + 3\sin x = 5$
5. $\tan^2 x + \tan x = 3$
6. $\cot^2 x + 5\tan x + 14 = 0$
7. $\sin^2 x + \cos^2 x = 1$

Solve each equation on the interval $[0,360^\circ)$.

8. $2\sin\left(x - \frac{\pi}{2}\right) = 1$
9. $4\cos(x - \pi) = 4$

Solve each equation on the interval $[2\pi,4\pi)$.

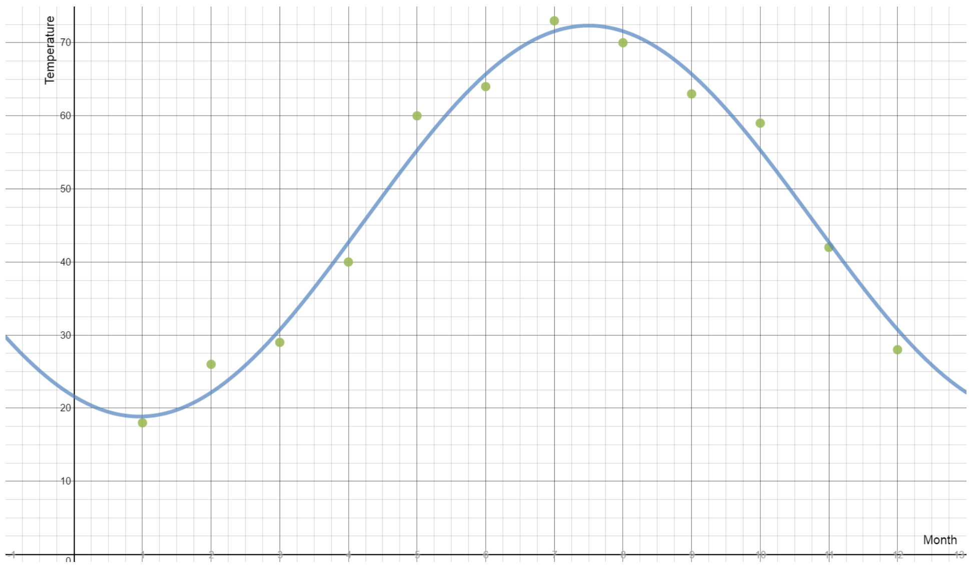
10. $\cos^2 x + 2\cos x + 1 = 0$
11. $3\sin x = 2\cos^2 x$
12. $\tan x \sin^2 x = \tan x$
13. $\sin^2 x + 1 = 2\sin x$
14. $\sec^2 x = 4$
15. $\sin^2 x - 4 = \cos^2 x - \cos 2x - 4$

Review (Answers)

Please see the Appendix.

8.7 Project: Analytic Trigonometry

The sine and cosine functions can be used to model fluctuations in temperature data throughout the year. For example, the average monthly temperatures for a city were used to create the following graph:



A trigonometric function that provides a good fit to model this data is $T(m) = 26.74\sin(0.48m - 2.03) + 45.6$.

Project: Create a Model of the Temperature Data to Plan a Family Vacation

Four members of a family are using a democratic approach to choose a vacation. Each family member wants to visit a different city. Your task is to create a model of the average monthly temperatures for all four cities on one graph, so the family can use this information to discuss the ideal location and time of year. The four cities chosen are Bar Harbor, Maine; Maui, Hawaii; Anchorage, Alaska; and Orlando, Florida.

Source: http://www.usclimatedata.com/

Bar Harbor - Maine												
	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Average high in °F:	31	35	42	53	65	74	79	78	71	59	48	37
Average low in °F:	14	17	25	35	44	54	59	59	52	42	33	21
Av. precipitation in inch:	4.92	4.33	5.39	4.76	4.65	4.13	3.5	3.31	4.45	5.31	6.46	5.47
Maui, Hawaii												
	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Average high in °F:	68	68	68	68	71	71	72	73	73	72	70	68
Average low in °F:	49	49	50	51	53	54	55	56	55	55	54	51
Av. precipitation in inch:	9.29	8.78	11.26	9.49	6.42	5.12	7.4	6.73	6.22	7.28	12.6	12.05
Anchorage - Alaska												
	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Average high in °F:	23	27	34	44	56	63	65	64	55	40	28	25
Average low in °F:	11	14	19	29	40	48	52	50	42	29	17	13
Av. precipitation in inch:	0.75	0.71	0.59	0.47	0.71	0.98	1.81	3.27	2.99	2.05	1.14	1.1
Orlando - Florida												
	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Average high in °F:	71	73	77	82	88	91	92	92	89	84	78	72
Average low in °F:	50	53	57	62	68	73	76	76	74	68	60	54
Av. precipitation in inch:	2.76	2.83	3.78	2.48	3.31	8.74	7.09	7.83	6.02	3.31	2.4	2.64

1. Graph the data.
2. Write a trigonometric equation for each city using the sine function that best models this situation.
3. Rewrite the equation using the cosine function using a cofunction identity.

4. Graph the function that approximates the data.
5. What features of the data are represented by a, b, h, and k in your equation? Explain the meaning of these values in context.
6. Set up an equation to solve for the times that the average monthly temperature reached a specific value within the range of the dataset.
7. Solve the equation. On how many dates in the domain of the dataset was this temperature reached?

8.8 Summary: Analytic Trigonometry

Chapter Summary

In this chapter, we learned about:

- **Trigonometric Identities**
  - Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

  - Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

  - Cofunction Identities

$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta \quad \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta \quad \cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$$
$$\sec\left(\frac{\pi}{2}-\theta\right) = \csc\theta \quad \csc\left(\frac{\pi}{2}-\theta\right) = \sec\theta \quad \tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$$

  - Even-Odd Trigonometric Identities

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$
$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$
$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

  - Pythagorean Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \cot^2 x = \csc^2 x$$
$$\tan^2 x + 1 = \sec^2 x$$

  - Difference Identities

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\tan(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

- Sum Identities

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

- Double Angle Identities

$$\sin 2x=2\sin x\cos x \qquad \cos 2x=\cos^2 x-\sin^2 x \qquad \tan 2x=\frac{2\tan x}{1-\tan^2 x}$$

- Half Angle Identities

$$\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}} \qquad \cos\frac{x}{2}=\pm\sqrt{\frac{1+\cos x}{2}} \qquad \tan\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

- Power Reducing Identities

$$\sin^2 x=\frac{1-\cos 2x}{2} \qquad \cos^2 x=\frac{1+\cos 2x}{2} \qquad \tan^2 x=\frac{1-\cos 2x}{1+\cos 2x}$$

- **How to Solve a Trigonometric Equation**
  - Solve for the variable as you would a linear equation.
  - Use inverse trigonometric functions to isolate the variable.
  - Use Pythagorean Trigonometric, sum and difference, and other trigonometric identities to get a single trigonometric function.
  - Use sum and difference, double angle, and half angle identities to get a single angle.
  - Use algebraic techniques, such as factoring or squaring, to further simplify the expressions.
  - Use the unit circle to find the initial solution or to determine values within the specified interval. There may be more than one value.
  - Provide an appropriate set of solutions.

Review

Try the following cumulative review problems to practice the concepts in this chapter:

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URL: http://www.ck12.org/flx/render/embeddedobject/195814

8.9 References

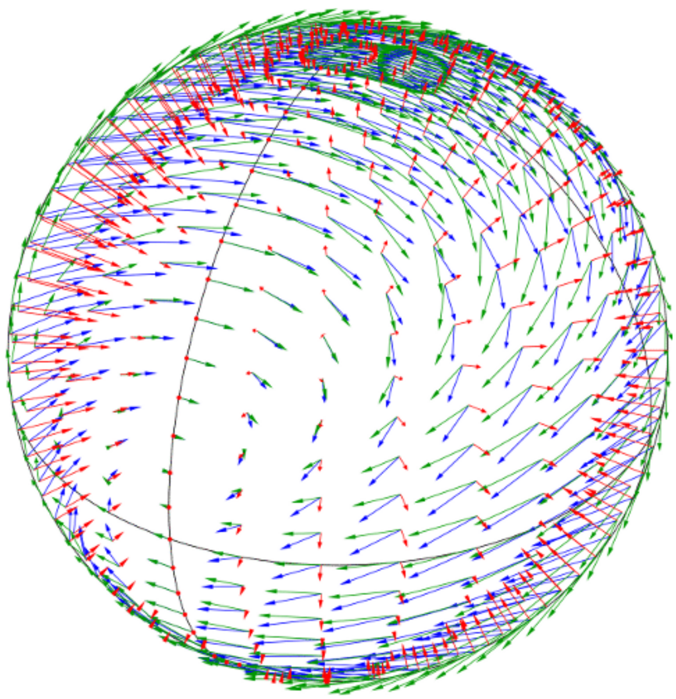
1. werner22brigitte. https://pixabay.com/en/chicago-illinois-usa-downtown-river-51326/ .
2. Amanda Mills. http://www.freestockphotos.biz/stockphoto/15411 .
3. patrickJMT. http://play.tojsiab.com/Njhlbk5SaEZPUmMz .
4. George Hodan. http://www.publicdomainpictures.net/view-image.php?image=164040&picture=online-winkel .
5. CK-12 Foundation. .
6. CK-12. Example 3 .
7. See page for author [Public domain], via Wikimedia Commons. https://commons.wikimedia.org/wiki/File%3APSM_V13_D055_Tuning_fork_and_sound_vibration.jpg .
8. . Angle Sum Illustration.
9. By US gov (US gov) [Public domain], via Wikimedia Commons. https://commons.wikimedia.org/wiki/File%3ASonic_boom_diagram.png .
10. CK-12 Foundation. .
11. CK-12. Example 1 .
12. . Example 2.
13. Paula Evans. Chicago Temperature Data .

CHAPTER 9 Vectors

Chapter Outline

9.1 Introduction: Vectors

Physicists, video game designers, meteorologists, and air traffic controllers all use vectors. Algebraically, vectors provide a whole new way to think about points, lines, and angles. Most importantly, they give you an opportunity to connect your knowledge of trigonometry to real-world applications. Typical quantities measured by vectors include velocity, force, and acceleration. Vector quantities include a magnitude like other measurements in units, such as those in kilograms and miles, but also include direction.



9.2 Two-Dimensional Vectors

Learning Objectives

Learn the definition of a vector, as well as how to find its components and to graph vectors on a coordinate plane.

Introduction

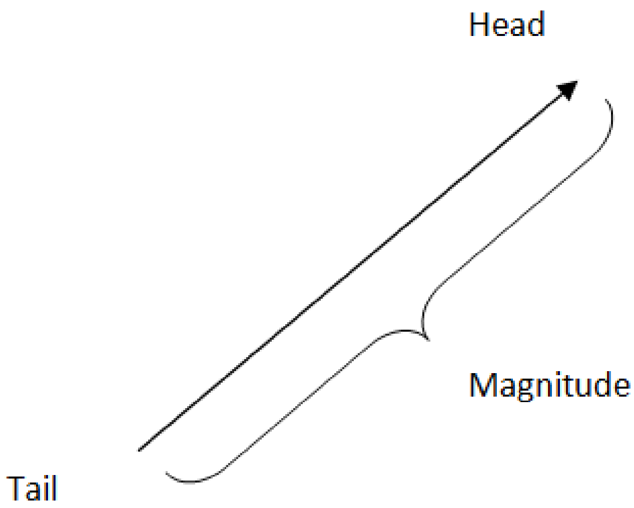
An airplane being pushed off course by wind and a swimmer's movement across a flowing river are both examples of vectors in action.

Points in the coordinate plane describe location. Vectors, on the other hand, have no location and indicate only magnitude and direction. Vectors can describe the strength of forces like gravity or speed and the direction of a ship at sea. Vectors are extremely useful in modeling complex situations in the real world.

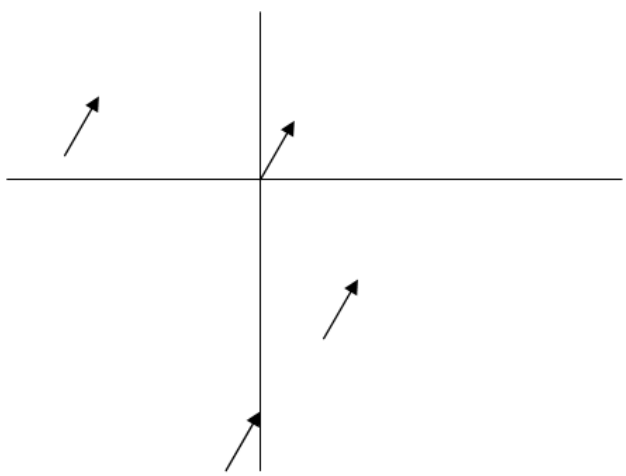
What are other differences between points and vectors?

Vectors

One way to define a **vector** is as a line segment with a direction. Vectors can be represented graphically using an arrow that points from the vector's initial point, called the **tail**, to its terminal point, called the **head**.



The two defining characteristics of a vector are its magnitude and its direction. The **magnitude** is shown graphically by the length of the arrow, and the direction is indicated by the angle in which the arrow is pointing. Notice how the vector below is shown multiple times on the same coordinate plane. This image emphasizes that the location on the coordinate plane does not matter and is not unique. Each representation of the vector has identical direction and magnitude.

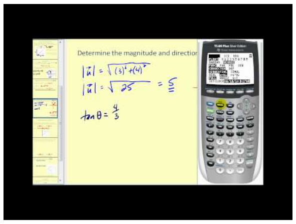


There are a few different ways to symbolize a vector, v:

$$v,\ \overrightarrow{v},\ \text{or}\ \vec{v}.$$

There are also a few ways to describe a specific vector. First, you can describe its angle and magnitude. Second, you can describe it as an ordered pair or in component form, $< x, y >$. Note that when discussing vectors, you should use brackets $<>$ instead of parentheses to help avoid confusion between a vector and a point. Vectors can be two-dimensional, three-dimensional, or n-dimensional, where n is an integer greater than 3.

The following video discusses vectors and vector vocabulary:



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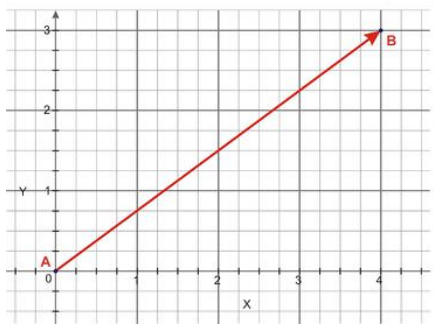
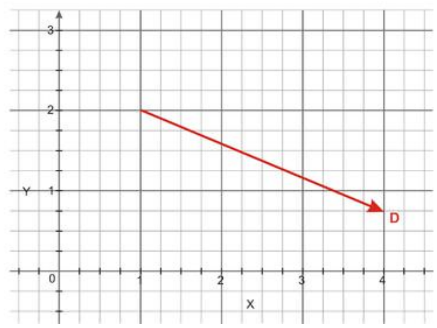
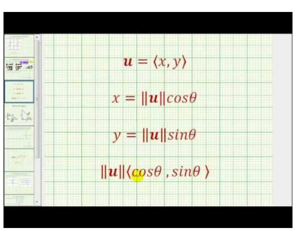
When a vector is angled in a certain direction, it is often useful to break the vector into individual components. The **component** of a vector depicts the influence of that vector in a given direction. A two-dimensional vector can be thought of as having an influence in two different directions: the horizontal direction (x-axis) and the vertical direction (y-axis).

For example, the vector on the left in the image below has components in both the x-direction, ***AB****x*, and the y-direction, ***AB****y*. Here, ***AB***x = 4 and ***AB****y* = 3 because point B is 4 units to the right and 3 units up from point A. Similarly, vector ***D*** has components ***D****x* = 3 and ***D****y* = -1.25. The negative sign in ***D****y* indicates that the y component of vector ***D*** is downward, in the negative y-direction. As with other numbers, we usually only include the negative sign explicitly.





The following video further explains how to find the component form of a vector given the vector's magnitude and direction:



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To determine the vector between two points, start with the terminal point and subtract the initial point. Suppose $A(u,v)$ and $B(x,y)$. Then:

$$\vec{AB} =$$
$$\vec{BA} = .$$

Vectors are said to be equal if they have the *same magnitude* and the *same direction.* The absolute value of a vector $|\vec{v}|$ is the same as the length or magnitude of the vector. Magnitude can be found by using the Pythagorean Theorem or the distance formula.

Play, Learn, and Explore with two-dimensional vectors using the example of a river ferry and currents: www.ck12.org/a/2137380 .

Examples

Example 1

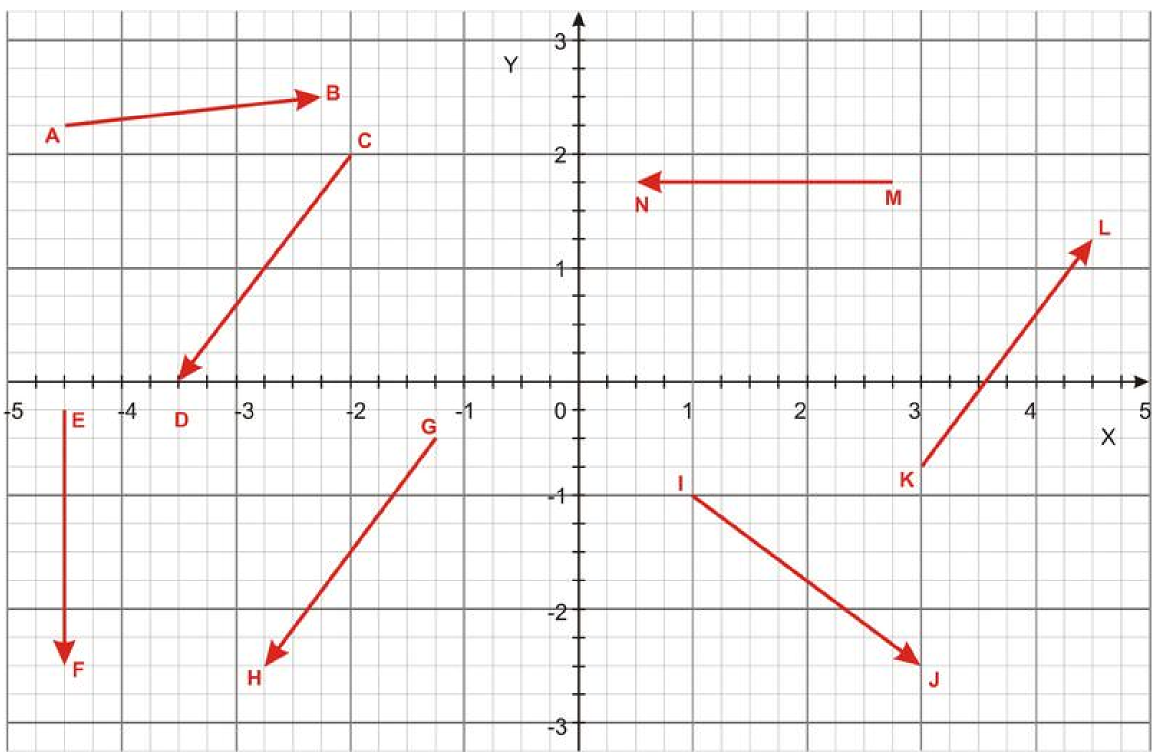
Consider the points $A(1,3), B(-4,-6), C(5,-13)$. Find the vectors in component form of $\overrightarrow{AB}, \overrightarrow{BA}, \overrightarrow{AC}, \overrightarrow{CB}$.

Solution:

$$\overrightarrow{AB} = < -5, -9 >$$
$$\overrightarrow{BA} = < 5, 9 >$$
$$\overrightarrow{AC} = < 4, -16 >$$
$$\overrightarrow{CB} = < -9, 7 >$$

Example 2

What are the x and y components of the vectors shown in the diagram?



Solution:

TABLE 9.1:

	AB	*CD*	*EF*	*GH*	*IJ*	*KL*	*MN*

TABLE 9.1: (continued)

	AB	***CD***	***EF***	***GH***	***IJ***	***KL***	***MN***
***x* component**	2.25	-1.5	0	-1.5	2.0	1.5	-2.25
***y* component**	0.25	-2.0	-2.25	-2.0	-1.5	2.0	0

In the diagram, each division is 0.25 units. All vectors that point toward the left have negative x components, and those that point downward have negative y components. Notice that for the horizontal vector, ***MN***, the y component is equal to 0. Likewise, for the vertical vector, ***EF***, the x component is equal to 0.

Example 3

1) Which of the vectors in Example 2 is equal to vector ***CD***?

Solution:

Vector ***GH* = *CD*.** Both vectors have the same length and the same direction or orientation.

2) Which vector is equal to ***-CD***?

Solution:

Vector ***KL* = *-CD***. Both vectors have the same length, and the two vectors point in opposite directions.

Example 4

Return to the problem from the Introduction: What are other differences between points and vectors?

Solution:

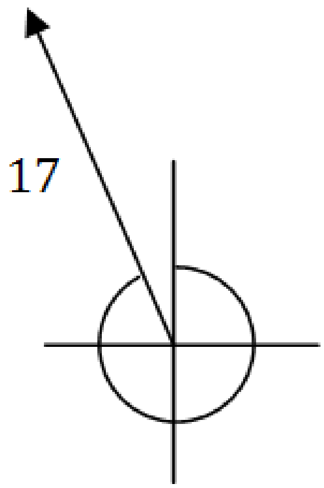
There are many differences between points and vectors. Points are specific locations, and vectors are made up of distance and angles. Parentheses are used for points, and <> are used for vectors. Without the starting point, the vector could start from anywhere.

Example 5

A marine traffic controller sees a ship traveling NNW at 17 knots (nautical mph) on radar. Describe the ship's movement using a vector.

Solution:

NNW is halfway between NW and N. When you describe ships at sea, it is best to use bearing that has 0° as due North and 270° as due West. This makes NW equal to 315° and NNW equal to 337.5°.



When the problem is depicted above, it becomes a basic trig question to find the x and $y$components of the vector. Note that the reference angle the vector makes with the negative portion of the x axis is 67.5°.

$$\sin 67.5^\circ = \frac{y}{17}, \cos 67.5^\circ = \frac{x}{17}$$
$$< x, y > \approx < -6.5, 15.7 >$$

Note: 270° clockwise is equivalent to 90° counterclockwise. This problem can also be worked using this angle measurement.

Example 6

A father and daughter are sledding. After going downhill, the father pulls his daughter up the hill using a rope attached to the sled. The sled sits on the ground. The hill has a 20° incline. The rope makes a 39° angle with the slope. Draw a force diagram showing how these forces act on the daughter's center of gravity:

a. The force of gravity.

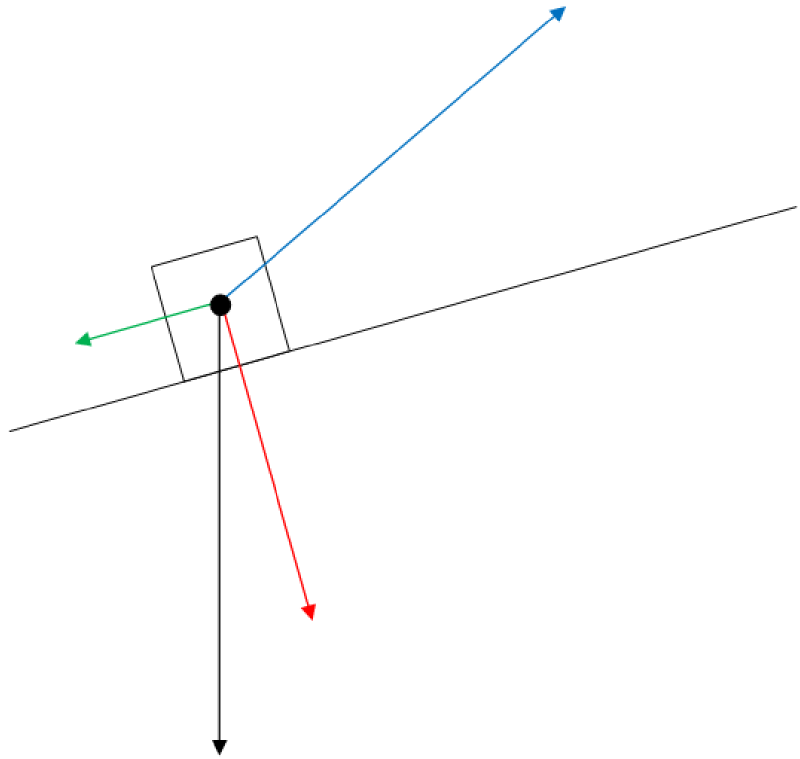
b. The force holding the daughter in the sled to the ground.

c. The force pulling the daughter backwards down the slope.

d. The force of the father pulling the daughter up the slope.

Solution:

The girl's center of gravity is represented by the black dot. The force of gravity is the black arrow straight down. The green arrow is the effect of gravity pulling the girl down the slope. The red arrow is the effect of gravity pulling the girl straight into the slope. The blue arrow represents the force the father is exerting as he pulls the girl up the hill.



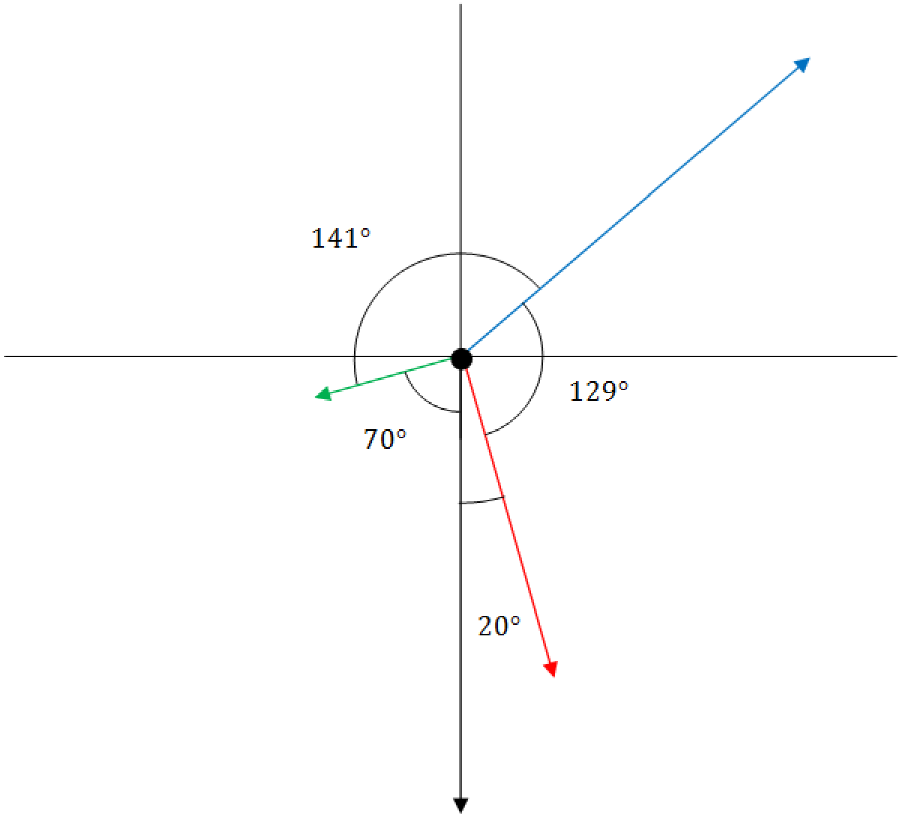
Notice that the father's force vector (blue) is longer than the force pulling the girl down the hill. This means that, over time, father and daughter will make progress and ascend the hill. Also note that the father is wasting some of his energy lifting rather than just pulling. If he could pull at an angle directly opposing the force pulling the girl down the hill, he would be using all his energy efficiently.

Example 7

Center the force diagram from the previous question onto the origin, and identify the angle between each consecutive force vector.

Solution:

The x and y axes are included as reference; note that the gravity vector overlaps with the negative y axis. To find each angle, you must use your knowledge of supplementary, complementary, and vertical angles and all the clues from the question. To check, see if all the angles sum to be 360°.



Summary

- A **vector** is a line segment defined by its direction and its magnitude.
- The **tail of a vector** is the initial point where the vector starts.
- The **head of a vector** is the terminal point where it ends.
- **Magnitude** refers to the length of the vector, and is associated with measurements such as the strength of a force or the speed of an object.
- The **component** of a vector depicts the influence of that vector in a given direction.

Review

1. Describe what a vector is and give a real-life example of something that a vector could model.

Consider the points $A(3,5), B(-2,-4), C(1,-12), D(-5,7)$. Find the vectors in component form of:

2. $\overrightarrow{AB}$

3. $\overrightarrow{BA}$

4. $\overrightarrow{AC}$

5. $\overrightarrow{CB}$

6. $\overrightarrow{AD}$

7. $\overrightarrow{DA}$

For each of the following vectors, draw the vector on a coordinate plane starting at the origin, and find its magnitude:

8. $< 3,7 >$

9. $< -3,4 >$

10. $< -5, 10 >$

11. $< 6, -8 >$

12. Can the x or y component of a vector ever have a greater magnitude than the vector itself?

13. If two vectors have magnitudes that are not equal, can the sums of their magnitudes ever be zero?

14. A ship is traveling SSW at 13 knots. Describe this ship's movement in a vector.

15. A vector that describes a ship's movement is $< 5\sqrt{2}, 5\sqrt{2} >$. In what direction is the ship traveling, and what is its speed in knots?

16. If a boat is being motored perpendicularly at 35 km/hr across a stream that is flowing at 25 km/hr, how can the direction and speed it travels be clearly shown using vectors?

Review (Answers)

Please see the Appendix.

9.3 Operations with Vectors

Learning Objectives

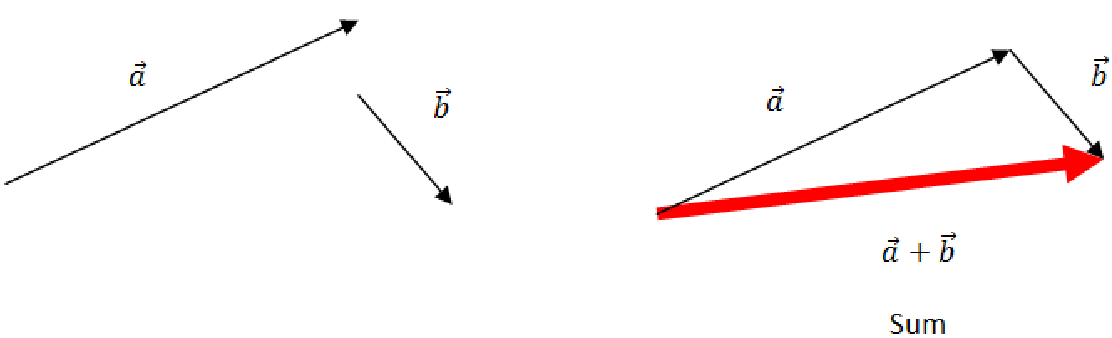
Learn to calculate resultant vectors by adding or subtracting the corresponding components or multiplying by a scalar.

Introduction

When two or more forces are acting on the same object, they combine to create a new force. A bird flying due south at 10 mph in a headwind of 2 mph makes headway at a rate of only 8 mph. These forces directly oppose each other. In real life, most forces are not parallel. What will happen when the headwind has a slight crosswind as well, blowing northeast at 2 mph? How far will the bird get in one hour?

Operations with Vectors

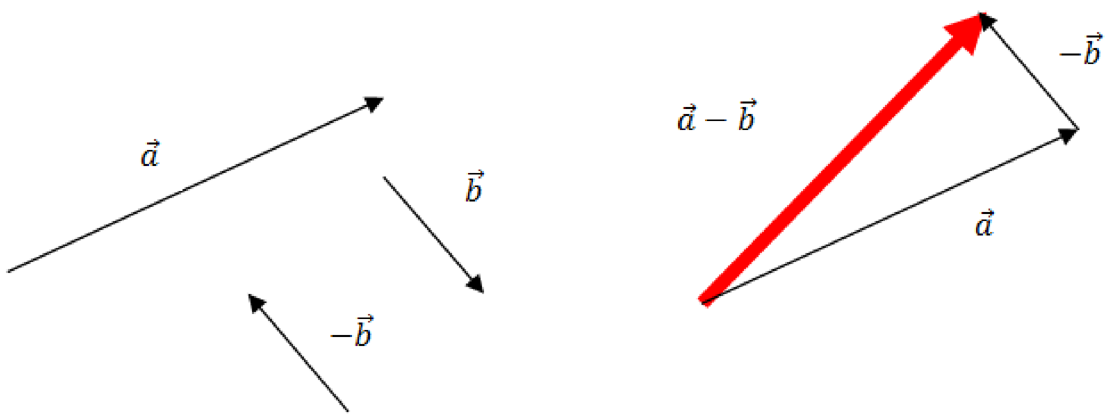
When adding vectors, place the tail of one vector at the head of the other. This is called the **tail-to-head rule.** The vector formed by joining the tail of the 1st with the head of the 2nd is called the **resultant vector**. The order in which we add vectors is irrelevant, because we will obtain the same result regardless of the order.



To determine the sum of two vectors, add the corresponding components of the vectors. Suppose $\vec{c}$ is the sum of $\vec{a}$ and $\vec{b}$:

$$\begin{aligned}\vec{a} &=< 5,12 >\\ \vec{b} &=< 3,8 >\\ \vec{c} &=< (5+3),(12+8) >=< 8,20 > .\end{aligned}$$

Vector subtraction reverses the direction of the 2nd vector: $\overrightarrow{a} - \overrightarrow{b} = \overrightarrow{a} + (-\overrightarrow{b})$. Subtracting two vectors is equivalent to putting the head of the vectors together.



To determine the difference of two vectors, reverse the direction of the vector we would like to subtract, and then add the two vectors. Suppose $\vec{t}$ is the difference of $\vec{r}$ and vector $\vec{s}$:

$$\begin{aligned}A = (1,3),\ \overrightarrow{v} &=< 4,8 >,\ \overrightarrow{u} =< -1,-5 >\\ A+\overrightarrow{v}+\overrightarrow{u} &= (4,6).\end{aligned}$$

Adding vectors can be done in either order (just as with regular numbers). Subtracting vectors must be done in a specific order or else the vector will be negative (just as with regular numbers). In either case, use geometric reasoning and the Law of Cosines with the parallelogram that is formed to find the magnitude of the resultant vector.

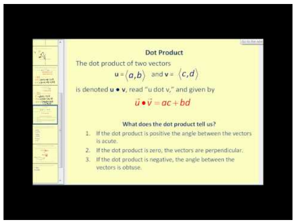
Scalar multiplication means to multiply a vector's components by a scalar number.

$$\begin{aligned}\vec{a} &=< x,y >\\ k\cdot\vec{a} &= k\cdot < x,y >=\end{aligned}$$

Scalar multiplication changes the magnitude of the vector, but not its direction.

Suppose $\overrightarrow{v} =< 3,4 >$. Then $2\overrightarrow{v} =< 6,8 >$.

The following video further explains basic vector operations, such as vector addition, vector subtraction, and scalar multiplication:



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URL: http://www.ck12.org/flx/render/embeddedobject/61368

Examples

Example 1

Given the following vectors, compute the sum:

$\overrightarrow{w} =< 1,3 >, \ \overrightarrow{v} =< 4,8 >, \ \overrightarrow{u} =< -1,-5 >$

$\overrightarrow{w} + \overrightarrow{v} + \overrightarrow{u} = ?$

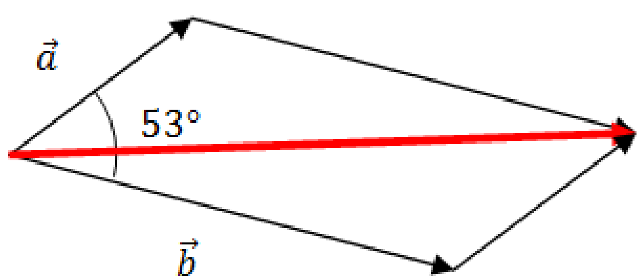
Solution:

$$\overrightarrow{w} =< 1,3 >, \ \overrightarrow{v} =< 4,8 >, \ \overrightarrow{u} =< -1,-5 >$$
$$\overrightarrow{w} + \overrightarrow{v} + \overrightarrow{u} =< 4,6 >$$

Example 2

Two vectors, $\overrightarrow{a}$ and $\overrightarrow{b}$, have magnitudes of 5 and 9 respectively. The angle between the vectors is 53°. Determine the magnitude of the resultant vector, $|\overrightarrow{a} + \overrightarrow{b}|$.

Solution:



To find the magnitude of the resulting vector (x), note the triangle on the bottom that has sides 9 and 5, with included angle 127°. Since you know two sides of the triangle and its included angle, use the Law of Cosines to calculate the 3rd side of the triangle.

$$x^2 = 9^2 + 5^2 - 2 \cdot 9 \cdot 5 \cdot \cos 127^\circ$$
$$x \approx 12.66$$

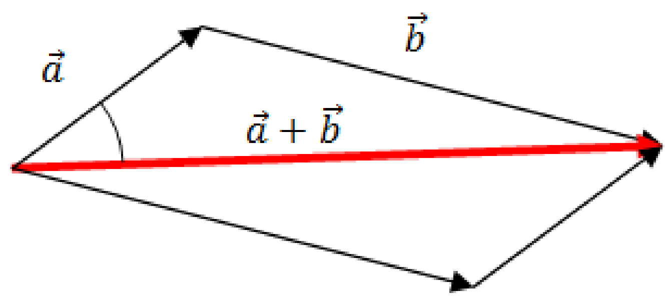
Example 3

Using the picture from Example 2, what is the angle that the sum $\overrightarrow{a} + \overrightarrow{b}$ makes with $\overrightarrow{a}$?

Solution:

Start by drawing a picture and labeling what you know:

$$|\overrightarrow{a}| = 5,\ |\overrightarrow{b}| = 9,\ |\overrightarrow{a} + \overrightarrow{b}| \approx 12.66.$$



Since you know three sides of the triangle and you need to find one angle, use the Law of Cosines.

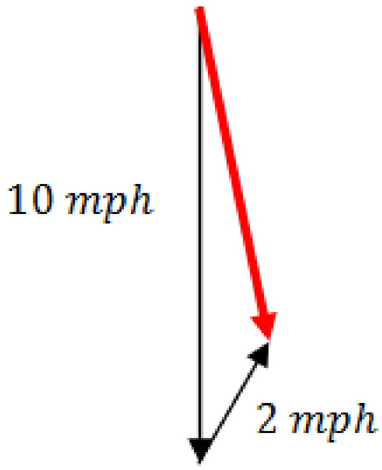
$$9^2 = 12.66^2 + 5^2 - 2 \cdot 12.66 \cdot 5 \cdot \cos \theta$$
$$\theta = 34.6^\circ$$

Example 4

Return to the problem from the Introduction, in which a bird is flying due south at 10 mph, with a cross headwind of 2 mph heading northeast.

Solution:

The force diagram looks like this:



The angle between the bird's vector and the wind vector is 45°, which means this is a perfect situation for the Law of Cosines. Let $x =$ the red vector.

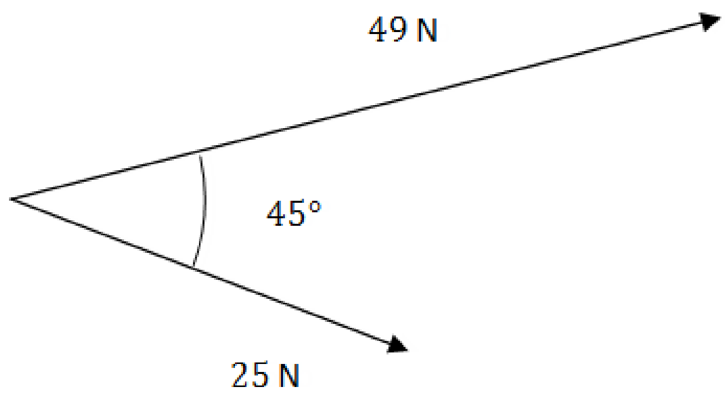
$$x^2 = 10^2 + 2^2 - 2 \cdot 10 \cdot 2 \cdot \cos 45^\circ$$
$$x \approx 8.7$$

The bird is blown slightly off track and travels only about 8.7 mph.

Example 5

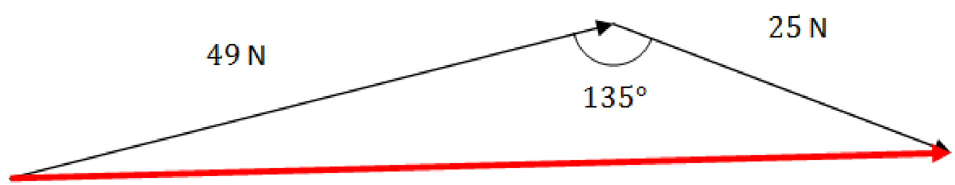
Elaine started a dog-walking business. She walks two dogs at a time. The dogs, Elvis and Ruby, each pull her in a different direction at a 45° angle, with different forces.

Elvis pulls at a force of 25 N, and Ruby pulls at a force of 49 N. How hard does Elaine need to pull with a constant force so she can stay in place? (Note: N stands for Newtons, which is the standard unit of force.)



Solution:

Even though the two vectors are centered at Elaine, the forces are added, which means you need to use the tail-to-head rule to add the vectors together. Finding the angle between each component vector requires logical use of supplement angles.



Evaluate the Law of Cosines with the given information.

$$x^2 = 49^2 + 25^2 - 2 \cdot 49 \cdot 25 \cdot \cos 135^\circ$$
$$x \approx 68.98\ N$$

In order for Elaine to stay in place, she will need to counteract this force with an equivalent force of her own in the exact opposite direction.

Example 6

Consider vector $\overrightarrow{v} =< 2,5 >$ and vector $\overrightarrow{u} =< -1,9 >$. Determine the component form of $3\overrightarrow{v} - 2\overrightarrow{u}$.

Solution:

$$\begin{aligned} 3 \cdot \overrightarrow{v} - 2 \cdot \overrightarrow{u} &= 3 \cdot < 2,5 > -2 \cdot < -1,9 > \\ &=< 6,15 > - < -2,18 > \\ &=< 8,-3 > \end{aligned}$$

Example 7

An airplane is flying at a bearing of 270° at 400 mph. A wind is blowing due south at 30 mph. Does this crosswind affect the plane's speed?

Solution:

Since the crosswind is perpendicular to the plane, it pushes the plane south as the plane tries to go directly east. As a result, the plane still has an airspeed of 400 mph, but the groundspeed (true speed) needs to be calculated.

$$400^2 + 30^2 = x^2$$
$$x \approx 401$$

Summary

- A **resultant vector** is the vector that is produced when two or more vectors are summed or subtracted. It is also what is produced when a single vector is scaled by a constant.
- To determine the **sum of two vectors**, add the corresponding components of the vectors.
- To determine the **difference of two vectors**, reverse the direction of the vector that we would like to subtract, and then add the two vectors.
- **Scalar multiplication** means to multiply a vector's components by a scalar number.
- Calculations involving vectors can be solved using formulas from trigonometry such as the Law of Sines and the Law of Cosines.

Review

Consider vector $\overrightarrow{v} =< 1,3 >$ and vector $\overrightarrow{u} =< -2,4 >$.

1. Determine the component form of $5\overrightarrow{v} - 2\overrightarrow{u}$.
2. Determine the component form of $-2\overrightarrow{v} + 4\overrightarrow{u}$.
3. Determine the component form of $6\overrightarrow{v} + \overrightarrow{u}$.
4. Determine the component form of $3\overrightarrow{v} - 6\overrightarrow{u}$.
5. Find the magnitude of the resultant vector from number 1.
6. Find the magnitude of the resultant vector from number 2.
7. Find the magnitude of the resultant vector from number 3.
8. Find the magnitude of the resultant vector from number 4.
9. The vector $< 3,4 >$ starts at the origin. What is the direction of the vector?
10. The vector $< -1,2 >$ starts at the origin. What is the direction of the vector?
11. The vector $< 3,-4 >$ starts at the origin. What is the direction of the vector?
12. A bird flies due south at 8 mph with a cross headwind blowing due east at 15 mph. How far does the bird get in one hour?
13. In what direction is the bird in the previous problem actually moving?
14. A football is thrown at 50 mph due north. A wind is blowing due east at 8 mph. What is the actual speed of the football?
15. In what direction is the football in the previous problem actually moving?

Review (Answers)

Please see the Appendix.

9.4 Unit Vectors

Learning Objectives

Learn about unit vectors and how to convert vectors into linear combinations of standard unit vectors and component vectors.

Introduction

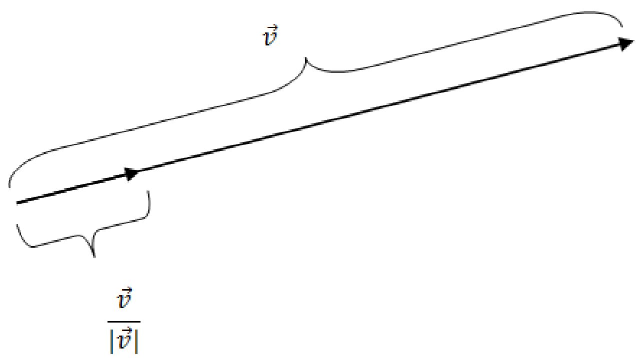
Sometimes, working with horizontal and vertical components of a vector can be significantly easier than working with just an angle and a magnitude. This is especially true when combining several forces together.

Consider four siblings fighting over a box of candy in a four-way tug of war. Lanie pulls with 8 lbs of force at an angle of 41°. Connie pulls with 10 lbs of force at an angle of 100°. Cynthia pulls with 12 lbs of force at an angle of 200°. How much force and in what direction does poor little Terry have to pull the candy so it doesn't move?

Unit Vectors

A **unit vector** is a vector of length 1. Sometimes you might wish to scale a vector you already have so it has a length of 1. If the length was 5, you'd scale the vector by a factor of $\frac{1}{5}$, so the resulting vector has magnitude of 1. A unit vector of vector $\vec{v}$ is notated as

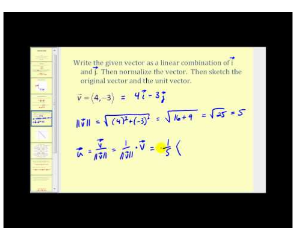
$$\frac{\overrightarrow{v}}{|\overrightarrow{v}|}.$$



Two standard unit vectors make up all other vectors in the coordinate plane. They are $\overrightarrow{i}$, which is the vector $< 1,0 >$, and $\overrightarrow{j}$, which is the vector $< 0,1 >$. These two unit vectors are perpendicular to each other. A linear combination of $\overrightarrow{i}$ and $\overrightarrow{j}$ will allow you to uniquely describe any other vector in the coordinate plane. For instance, the vector $< 5,3 >$ is the same as $5\overrightarrow{i}+3\overrightarrow{j}$.

Working with vectors written as an angle and magnitude requires extremely precise geometric reasoning and excellent pictures. One advantage of rewriting the vectors in component form is that much of this work is simplified.

The video below shows how to determine a unit vector given a vector. It also explains how to determine the component form of a vector in standard position that intersects the unit circle.



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URL: http://www.ck12.org/flx/render/embeddedobject/61370

Play, Learn, and Explore Unit Vectors and Components: www.ck12.org/a/2164219 .

Examples

Example 1

A plane has a bearing of 60° and is going 350 mph. Find the component form of the velocity of the airplane.

Solution:

A bearing of 60° is the same as a 30° angle on the unit circle, which corresponds to the point $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$. When written as a vector, $< \frac{\sqrt{3}}{2},\frac{1}{2} >$ is a unit vector because it has magnitude of 1.

Since the plane is going 350 mph, scale the vector by a factor of 350.

$$350\cdot < \frac{\sqrt{3}}{2}, \frac{1}{2} > = < 175\sqrt{3}, 175 >$$

Thus, the velocity of the airplane in component form is $< 175\sqrt{3}, 175 >$.

Example 2

Consider the plane flying in Example 1. If there is wind blowing with a bearing of 300° at 45 mph, what is the component form of the total velocity of the airplane?

Solution:

A bearing of 300° is the same as a 150° angle on the unit circle, which corresponds to the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Since the wind is blowing at 45 mph, scale the vector by a factor of 45.

$$45\cdot < -\frac{\sqrt{3}}{2}, \frac{1}{2} > = < -\frac{45\sqrt{3}}{2}, \frac{45}{2} >$$

Since both the wind vector and the velocity vector of the airplane are written in component form, you can simply sum them to find the component vector of the total velocity of the airplane.

$$< 175\sqrt{3}, 175 > + < -\frac{45\sqrt{3}}{2}, \frac{45}{2} > = < \frac{305\sqrt{3}}{2}, \frac{395}{2} >$$

Example 3

Consider the plane and wind in Examples 1 and 2. Find the actual ground speed and direction of the plane (as a bearing).

Solution:

You already know the component vector of the total velocity of the airplane. You should remember that these components represent an x distance and a y distance, and the question asks for the hypotenuse.

$$\left(\frac{305\sqrt{3}}{2}\right)^2 + \left(\frac{395}{2}\right)^2 = c^2$$
$$329.8 \approx c$$

The airplane is traveling at about 329.8 mph.

Since you know the x and y components, you can use tangent to find the angle. Then convert this angle into bearing.

$$\tan\theta = \frac{\left(\frac{395}{2}\right)}{\left(\frac{305\sqrt{3}}{2}\right)}$$
$$\theta \approx 36.8^\circ$$

An angle of 36.8° on the unit circle is equivalent to a bearing of 53.2°.

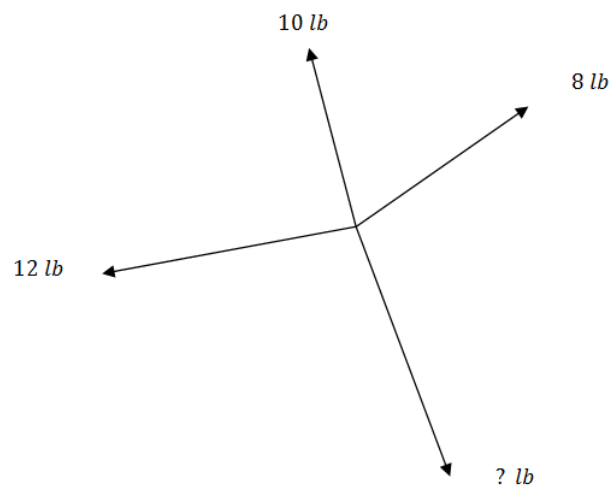
Note: You can do the entire problem in bearing just by switching sine and cosine, but it is best to truly understand what you are doing every step of the way. This will probably involve always going back to the unit circle.

Example 4

Recall the problem from the Introduction: Four siblings are fighting over a box of candy in a four-way tug of war. Lanie pulls with 8 lbs of force at an angle of 41° from the positive x-axis. Connie pulls with 10 lbs of force at an angle of 100° from the positive x-axis. Cynthia pulls with 12 lbs of force at an angle of 200° from the positive x-axis. How much force and in what direction does poor little Terry have to pull the candy so it doesn't move?

Solution:

The problem can be illustrated by the following diagram:



To add the three vectors together would require several iterations of the Law of Cosines. Instead, write each vector in component form and set equal to a 0 vector, indicating that the candy does not move.

$$\overrightarrow{L}+\overrightarrow{CON}+\overrightarrow{CYN}+\overrightarrow{T}=<0,0>$$

$$<8\cdot\cos 41^\circ, 8\cdot\sin 41^\circ>+<10\cdot\cos 100^\circ, 10\cdot\sin 100^\circ>$$
$$+<12\cdot\cos 200^\circ, 12\cdot\sin 200^\circ>+\overrightarrow{T}=<0,0>$$

Use a calculator to add all the x components and bring them to the right side of the equation and the y components, and then subtract from the right side to get

$$\overrightarrow{T}\approx<6.98,-10.99>.$$

Turning this component vector into an angle and magnitude yields how hard and in what direction Terry will have to pull. He'll have to pull with about 13 lbs of force at an angle of 302.4°.

Example 5

$\overrightarrow{v}=<2,-5>, \overrightarrow{u}=<-3,2>, \overrightarrow{t}=<-4,-3>, \overrightarrow{r}=<5,y>$

$B = (4, -5), P = (-3, 8)$

1) Solve for y in vector $\vec{r}$ to make $\vec{r}$ perpendicular to $\vec{t}$.

Solution:

$\vec{t}$ has slope $\frac{3}{4}$, which means that $\vec{r}$ must have slope $-\frac{4}{3}$. A vector's slope is found by putting the y component over the x component, just as with $\frac{rise}{run}$.

$$\frac{y}{5} = -\frac{4}{3}$$
$$y = -\frac{20}{3}$$

2) Find the unit vectors in the same direction as $\vec{u}$ and $\vec{t}$.

Solution:

To find a unit vector, divide each vector by its magnitude.

$$\frac{\vec{u}}{|\vec{u}|} = < \frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}} >, \frac{\vec{t}}{|\vec{t}|} = < \frac{-4}{5}, \frac{-3}{5} >$$

3) Find the point 10 units away from B in the direction of P.

Solution:

The vector $\overrightarrow{BP}$ is $< -7, 13 >$. First, take the unit vector and scale it so it has a magnitude of 10.

$$\frac{BP}{|BP|} = < \frac{-7}{\sqrt{218}}, \frac{13}{\sqrt{218}} >$$
$$10 \cdot \frac{BP}{|BP|} = < \frac{-70}{\sqrt{218}}, \frac{130}{\sqrt{218}} >$$

You end up with a vector that is 10 units long in the right direction. The question asked for a point from B, which means you need to add this vector to point B.

$$(4, -5) + < \frac{-70}{\sqrt{218}}, \frac{130}{\sqrt{218}} > \approx (-0.74, 3.8)$$

Summary

- A **unit vector** is a vector of magnitude 1.
- The **standard unit vectors** are $\vec{i}$, which is the vector $< 1, 0 >$, and $\vec{j}$, which is the vector $< 0, 1 >$.
- A **linear combination** of vectors $\vec{u}$ and $\vec{v}$ means a multiple of one plus a multiple of the other.

Review

Use the following defined vectors and points to answer 1-8: $\vec{v} = < 1, -3 >$, $\vec{u} = < 2, 5 >$, $\vec{t} = < 9, -1 >$, $\vec{r} = < 2, y >$

$A = (-3, 2), B = (5, -2)$

1. Solve for y in vector $\vec{r}$ to make $\vec{r}$ perpendicular to $\vec{t}$.

2. Find the unit vector in the same direction as $\vec{u}$.

3. Find the unit vector in the same direction as $\vec{t}$.

4. Find the unit vector in the same direction as $\vec{v}$.

5. Find the unit vector in the same direction as $\vec{r}$.

6. Find the point exactly 3 units away from A in the direction of B.

7. Find the point exactly 6 units away from B in the direction of A.

8. Find the point exactly 5 units away from A in the direction of B.

9. Jack and Jill went up a hill to fetch a pail of water. When they got to the top of the hill, they were very thirsty, so they each pulled on the bucket. Jill pulled at 30° with 20 lbs of force. Jack pulled at 45° with 28 lbs of force. What is the resulting vector for the bucket?

10. A plane is flying on a bearing of 60° at 400 mph. Find the component form of the velocity of the plane. What does the component form tell you?

11. A baseball is thrown at a 70° angle with the horizontal, with an initial speed of 30 mph. Find the component form of the initial velocity.

12. A plane is flying on a bearing of 200° at 450 mph. Find the component form of the velocity of the plane.

13. A plane is flying on a bearing of 260° at 430 mph. At the same time, a wind is blowing at a bearing of 30° at 60 mph. What is the component form of the velocity of the plane?

14. Use the information from the previous problem to find the actual ground speed and direction of the plane.

15. Wind is blowing at a magnitude of 40 mph with an angle of 25°, with respect to the east. What is the velocity of the wind blowing to the north? What is the velocity of the wind blowing to the east?

Review (Answers)

Please see the Appendix.

9.5 Dot Products

Learning Objectives

Learn to compute the dot product between two vectors and interpret its meaning.

Introduction

A ramp allows you to use less force than a straight lift to move an object to the same height. Will realizes this is why movers use ramps to load and unload trucks. He has three options to get a heavy chest of drawers into his new apartment:

Option 1: Will can simply lift the dresser through the window into his apartment. To do this, he'll need to overcome the force of gravity and use a 1,400 Newton (N) force applied at a 90° angle to move the dresser 1.5 meters at a 90° angle.

Option 2: Will can use a 243 N force applied at a 30° angle to the horizontal, and move the chest of drawers 5.7 meters up a 10° ramp into the front door of the apartment. This method requires less force than a straight lift, because the force to overcome gravity depends on the sine of the angle of the ramp being used to move an object.

Option 3: Will can exert 73 N of force at a 30° angle to the horizontal, and move the chest of drawers 19 meters up the slope of a hill with a 3° grade and into the back door.

If Will wants to do the least work possible, which is the best option for moving his heavy chest of drawers?

Dot Products

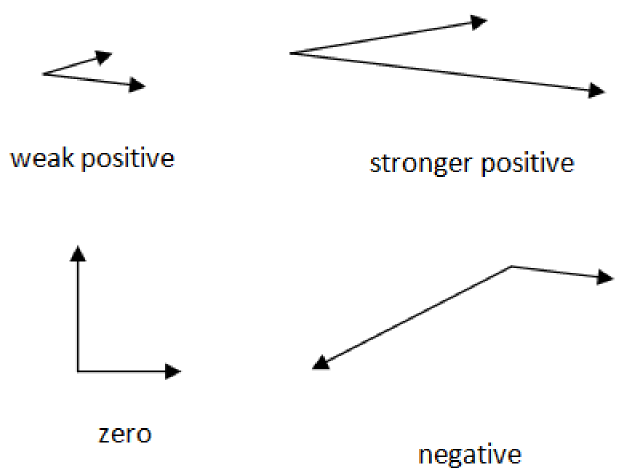
While two vectors cannot be strictly multiplied like numbers can, there are two different ways to find the product between two vectors. The cross product between two vectors results in a new vector perpendicular to the other two vectors. The 2nd type of product is the **dot product** between two vectors, which results in a scalar number. This number represents *how much of one vector goes in the direction of the other*. In one sense, it indicates how much the two vectors agree with each other.

The dot product is defined as

$$u \cdot v = < u_1, u_2 > \cdot < v_1, v_2 > = u_1 v_1 + u_2 v_2.$$

This procedure states that you multiply the corresponding components and then sum the resulting products. It can work with vectors that are more than two dimensions in the same way, as long as the vectors involved have the same number of dimensions.

Before trying this procedure with specific numbers, look at the following pairs of vectors and relative estimates of their dot product:



Notice how vectors going in generally the same direction have a positive dot product. Think of two forces acting on a single object. A positive dot product implies that these forces are working together at least a little bit. Another way of saying this is the angle between the vectors is less than 90°.

There are many important properties related to the dot product that you will prove in the examples and practice problems. The two most important are:

1) What happens when a vector has a dot product with itself.

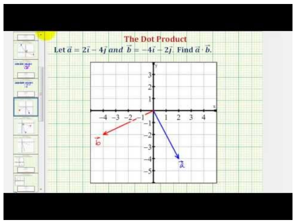
2) What is the dot product of two vectors that are perpendicular to each other.

- $v \cdot v = |v|^2$
- v and u are perpendicular if and only if $v \cdot u = 0$.

The dot product can help you determine the angle between two vectors using the formula below. Notice that in the numerator, the dot product is required because each term is a vector. In the denominator, only regular multiplication is required because the magnitude of a vector is just a regular number indicating length.

$$\cos \theta = \frac{u \cdot v}{|u||v|}$$

The following video discusses the meaning of the dot product, and provides several examples of how to determine the dot product of vectors in two dimensions:



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/195018

Play, Learn, and Explore the Dot Product: www.ck12.org/a/2328699 .

Examples

Example 1

Show the commutative property holds for the dot product between two vectors. In other words, show that $u \cdot v = v \cdot u$.

Solution:

This proof is for two-dimensional vectors, although it holds for *any*-dimensional vectors. Start with the vectors in component form.

$$u = < u_1, u_2 >$$
$$v = < v_1, v_2 >$$

Apply the definition of dot product and rearrange the terms. Then apply the commutative property.

$$\begin{aligned} u \cdot v &= < u_1, u_2 > \cdot < v_1, v_2 > \\ &= u_1 v_1 + u_2 v_2 \\ &= v_1 u_1 + v_2 u_2 \\ &= < v_1, v_2 > \cdot < u_1, u_2 > \end{aligned}$$
$$\& = v \cdot u$$

Example 2

Find the dot product between the following vectors: $< 3, 1 > \cdot < 5, -4 >$.

Solution:

$$\begin{aligned} < 3, 1 > \cdot < 5, -4 > &= 3 \cdot 5 + 1 \cdot (-4) \\ &= 15 - 4 \\ &= 11 \end{aligned}$$

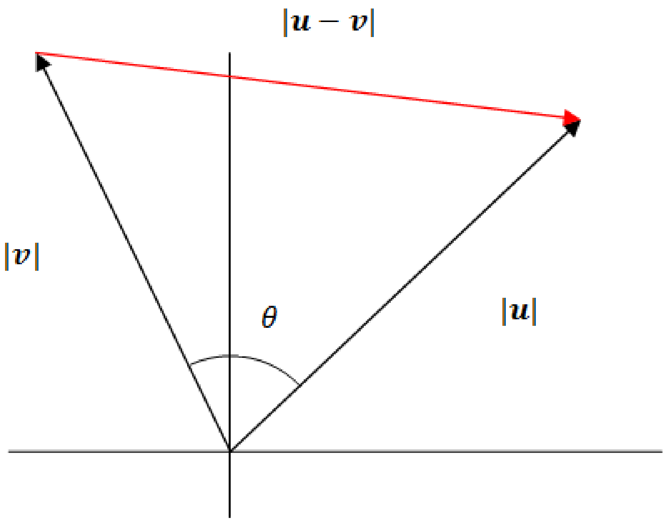
Example 3

Prove the angle between two vectors formula:

$$\cos\theta = \frac{u \cdot v}{|u||v|}.$$

Solution:

Use the Law of Cosines.



$$\begin{aligned}|u-v|^2 &= |v|^2+|u|^2-2|v||u|\cos\theta\\(u-v)\cdot(u-v) &=\\u\cdot u-2u\cdot v+v\cdot v &=\\|u|^2-2u\cdot v+|v|^2 &=\\-2u\cdot v &= -2|v||u|\cos\theta\\\frac{u\cdot v}{|u||v|} &= \cos\theta\end{aligned}$$

Example 4

Return to the question from the Introduction: Which is the best option for moving Will's heavy chest of drawers if he wants to do the least work possible?

Solution:

Option 1: For this move, θ is 0 because there is no difference in angle between the force vector and the movement vector.

$$\begin{aligned}\text{Work} &= \vec{f}\cdot\vec{d} = \left\|\vec{f}\right\| \left\|\vec{d}\right\| \cos\theta\\&= 1{,}400\cdot 1.5\cdot\cos(0)\\&= 2{,}100\ J\end{aligned}$$

If Will uses the 1st option, he will have to do 2,100 Joules of work.

Option 2: The angle between the force he's applying and the surface of the ramp is

$$30^\circ - 10^\circ = 20^\circ.$$

$$\begin{aligned}\text{Work} &= \left\|\vec{f}\right\| \left\|\vec{d}\right\| \cos\theta \\ &= 243 \cdot 5.7 \cdot \cos(20) \\ &= 1{,}302\ J\end{aligned}$$

If he moves the dresser up the ramp and into the front door, he will do 1,302 Joules of work.

Option 3: The angle between the force and the hill is 27°.

$$\begin{aligned}\text{Work} &= \left\|\vec{f}\right\| \left\|\vec{d}\right\| \cos\theta \\ &= 73 \cdot 19 \cdot \cos(27) \\ &= 1{,}236\ J\end{aligned}$$

If Will wants to do the least work possible, he should choose option 3. He should go up the hill and into the back door.

Example 5

Show the distributive property holds under the dot product**.**

$$u \cdot (v + w) = uv + uw$$

Solution:

This proof will work with two-dimensional vectors, although the property does hold in general.

$$u = < u_1, u_2 >, v = < v_1, v_2 >, w = < w_1, w_2 >$$

$$\begin{aligned}u \cdot (v + w) &= u \cdot (< v_1, v_2 > + < w_1, w_2 >) \\ &= u \cdot < v_1 + w_1, v_2 + w_2 > \\ &= < u_1, u_2 > \cdot < v_1 + w_1, v_2 + w_2 > \\ &= u_1(v_1 + w_1) + u_2(v_2 + w_2) \\ &= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 \\ &= u_1v_1 + u_2v_2 + u_1w_1 + u_2w_2 \\ &= u \cdot v + v \cdot w\end{aligned}$$

Example 6

Find the dot product between the following vectors:

$$(4i-2j)\cdot(3i-8j).$$

Solution:

The standard unit vectors can be written as component vectors.

$$<4,-2>\cdot<3,-8>=12+(-2)(-8)=12+16=28$$

Example 7

What is the angle between $v=<3,5>$ and $u=<2,8>$?

Solution:

$$\begin{aligned}
\frac{u\cdot v}{|u||v|} &= \cos\theta \\
\frac{<3,5>\cdot<2,8>}{\sqrt{34}\cdot\sqrt{68}} &= \cos\theta \\
\frac{6+40}{\sqrt{34}\cdot\sqrt{68}} &= \cos\theta \\
\cos^{-1}\left(\frac{46}{\sqrt{34}\cdot\sqrt{68}}\right) &= \theta \\
&\approx \theta
\end{aligned}$$

Summary

- The **dot product** is also known as the **inner product** and **scalar product**. It produces a scalar number that can be interpreted to tell how much one vector goes in the direction of the other.
- Dot product formula: $u\cdot v=<u_1,u_2>\cdot<v_1,v_2>=u_1v_1+u_2v_2$

Review

Find the dot product for each of the following pairs of vectors:

1. $<2,6>\cdot<-3,5>$

2. $<5,-1>\cdot<4,4>$

3. $<-3,-4>\cdot<2,2>$

4. $<3,1>\cdot<6,3>$

5. $<-1,4>\cdot<2,9>$

Find the angle between each pair of vectors below.

6. $<2,6>\cdot<-3,5>$

7. $<5,-1>\cdot<4,4>$

8. $<-3,-4>\cdot<2,2>$

9. $<3,1>\cdot<6,3>$

10. $< -1, 4 > \cdot < 2, 9 >$

11. What is $v \cdot v$?

12. How can you use the dot product to find the magnitude of a vector?

13. What is $0 \cdot v$?

14. Show that $(cu) \cdot v = u \cdot (cv)$, where c is a constant.

15. Show that $< 2, 3 >$ is perpendicular to $< 1.5, -1 >$.

16. If Julie applies a force of 5 N at an angle of 30° to the horizontal, and moves an object 12 m along a 3° slope, how much work does she do?

17. If Michelle applies a force of 8 N at an angle of 20° to the horizontal, and moves an object 18 m along a 12° slope, how much work does she do?

18. If John applies a force of 10 N at an angle of 10° to the horizontal, and moves an object 25 m along a flat surface, how much work does he do?

Review (Answers)

Please see the Appendix.

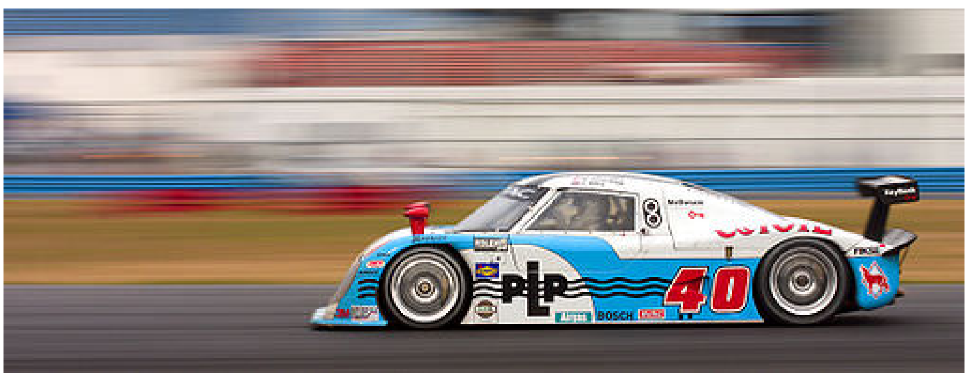
9.6 Scalar and Vector Projections

Learning Objectives

Learn how to use dot products and unit vectors in scalar projections. You will also learn how to project one vector onto another and apply this technique as it relates to force.

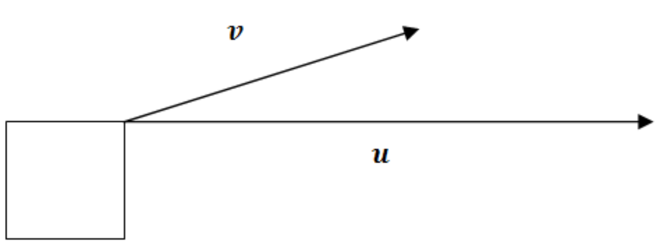
Introduction

For an engineer, a pilot, a race car driver, and even a chef, it can be useful to determine the result of different amounts and directions of force applied to a particular action. This information can help decide on the best use of added strength, altitude, speed, or heat to achieve the optimum result.



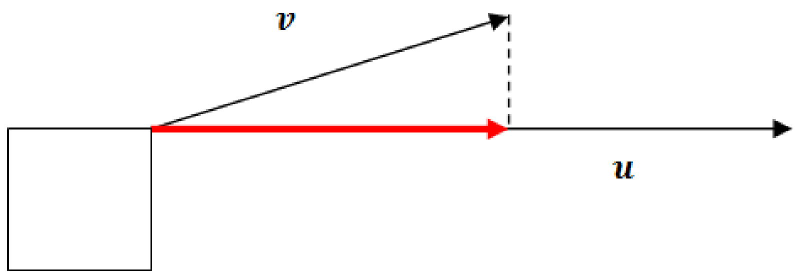
Projecting one vector onto another explicitly answers the question, "How much of one vector goes in the direction of the other vector?" The dot product is useful because it produces a scalar quantity that helps to answer this question. In this section, you will produce an actual vector, not just a scalar.

Why is vector projection useful when considering pulling a box in the direction of v, instead of horizontally in the direction of u?



Scalar and Vector Projections

Consider the question from above.



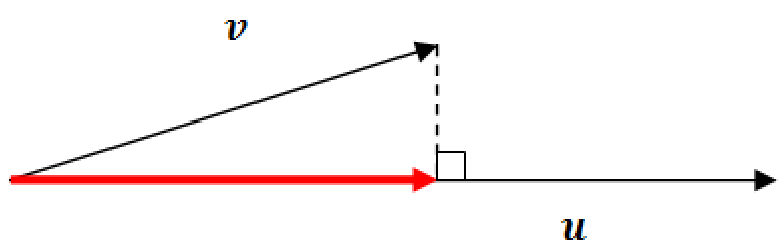
The definition of **vector projection** for the indicated red vector is the called $proj_u v$. When you read $proj_u v$, you should say "the vector projection of v onto u." This implies that the new vector is going in the direction of u. The vector projection is the vector produced when one vector is resolved into two component vectors, one that is parallel to the 2nd vector and one that is perpendicular to the 2nd vector. The parallel vector is the vector projection. Conceptually, this means that if someone is pulling the box at an angle and strength of vector v, then some of their energy would be wasted pulling the box up, and some of their energy would actually contribute to pulling the box horizontally.

The vector projection formula can be written two ways, shown below. The version on the left is most simplified, but the version on the right makes the most sense conceptually:

$$proj_u v = \left(\frac{v \cdot u}{|u|^2}\right) u = \left(\frac{v \cdot u}{|u|}\right) \frac{u}{|u|}$$

The proof of the vector projection formula is as follows:

Given two vectors u, v, what is $proj_u v$?



First note that the projected vector in red will go in the direction of u. This means it will be a product of the unit vector $\frac{u}{|u|}$ and the length of the red vector (the scalar projection). To find the scalar projection, note the right triangle, the unknown angle θ between the two vectors, and the cosine ratio.

$$\cos\theta = \frac{\text{scalar projection}}{|v|}$$

Recall that $\cos\theta = \frac{u \cdot v}{|u||v|}$. Now, just substitute and simplify to find the length of the scalar projection.

$$\begin{aligned}
\cos\theta &= \frac{\text{scalar projection}}{|v|} \\
\frac{u \cdot v}{|u||v|} &= \frac{\text{scalar projection}}{|v|} \\
\frac{u \cdot v}{|u|} &= \text{scalar projection}
\end{aligned}$$

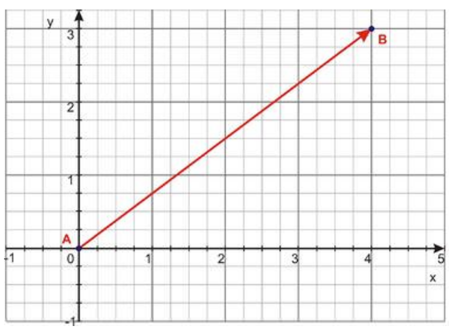
Now you have the length of the vector projection and the direction you want it to go:

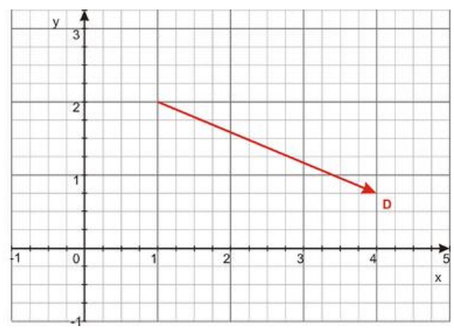
$$proj_u v = \left(\frac{u \cdot v}{|u|}\right)\frac{u}{|u|}.$$

The definition of **scalar projection** is the length of the vector projection. Recall that the dot product of a vector is a scalar quantity describing only the magnitude of a particular vector. A scalar projection is given by the dot product of a vector with a unit vector for that direction.

When the scalar projection is positive, it means that the angle between the two vectors is less than 90°. When the scalar projection is negative, it means that the two vectors are heading in opposite directions.

For example, the component forms for the vectors shown below are $AB = \langle 4,3 \rangle$ and $D = \langle 3,-1.25 \rangle$.





The scalar projection of vector $\boldsymbol{AB}$ onto $\hat{i}$ is given by

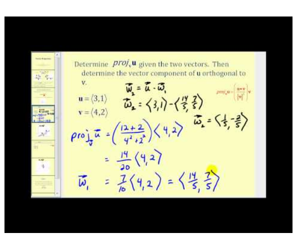
$$\overrightarrow{AB} \cdot \hat{i} = (4 \cdot 1) + (3 \cdot 0) = 4.$$

The scalar projection of vector $\boldsymbol{AB}$ onto $\hat{j}$ is given by

$$\overrightarrow{AB} \cdot \hat{j} = (4 \cdot 0) + (3 \cdot 1) = 3.$$

The scalar projections of $\boldsymbol{AB}$ onto the x and y directions are nonzero numbers because the vector is located in the xy -plane.

The following video explains how to determine the projection of one vector onto another vector:



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/61375

Play, Learn, and Explore Vector Projections: www.ck12.org/a/2121574 .

Examples

Example 1

Find the scalar projection of vector $v =< 3,4 >$ onto vector $u =< 5,-12 >$.

Solution:

As noted earlier, the scalar projection is the magnitude of the vector projection. This was shown to be $\left(\frac{u\cdot v}{|u|}\right)$, where u is the vector being projected onto.

$$\frac{u\cdot v}{|u|} = \frac{< 5,-12 > \cdot < 3,4 >}{13} = \frac{15-48}{13} = -\frac{33}{13}$$

Example 2

Find the vector projection of vector $v =< 3,4 >$ onto vector $u =< 5,-12 >$.

Solution:

Since the scalar projection has already been found in Example 1, you should multiply the scalar by the "onto" unit vector.

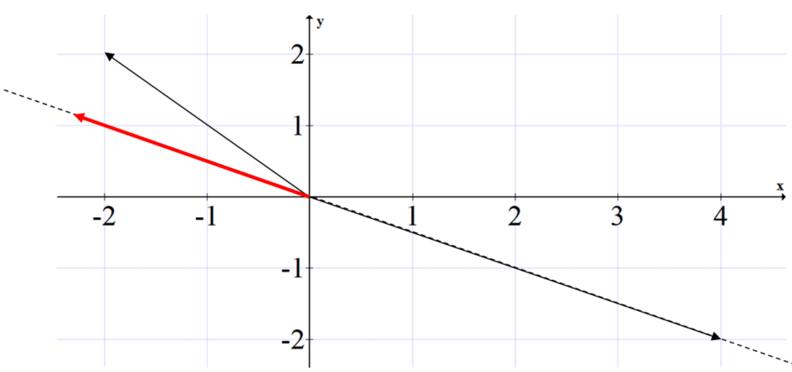
$$-\frac{33}{13}\left\langle \frac{5}{13},-\frac{12}{13}\right\rangle = \left\langle -\frac{165}{169},\frac{396}{169}\right\rangle$$

Example 3

Sketch the vector $< -2,-2 >$ and $< 4,-2 >$. Explain using a sketch why a negative scalar projection of $< -2,-2 >$ onto $< 4,-2 >$ makes sense.

Solution:

First, plot the two vectors and extend the "onto" vector. When the vector projection occurs, the vector $< -2,2 >$ goes in the opposite direction of the vector $< 4,-2 >$. This will create a vector projection going in the opposite direction of $< 4,-2 >$.

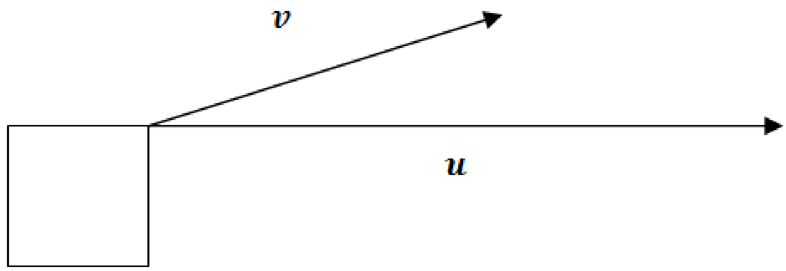


Example 4

Recall the problem from the Introduction: Why is vector projection useful when considering pulling a box in the direction of v, instead of horizontally in the direction of u?

Solution:

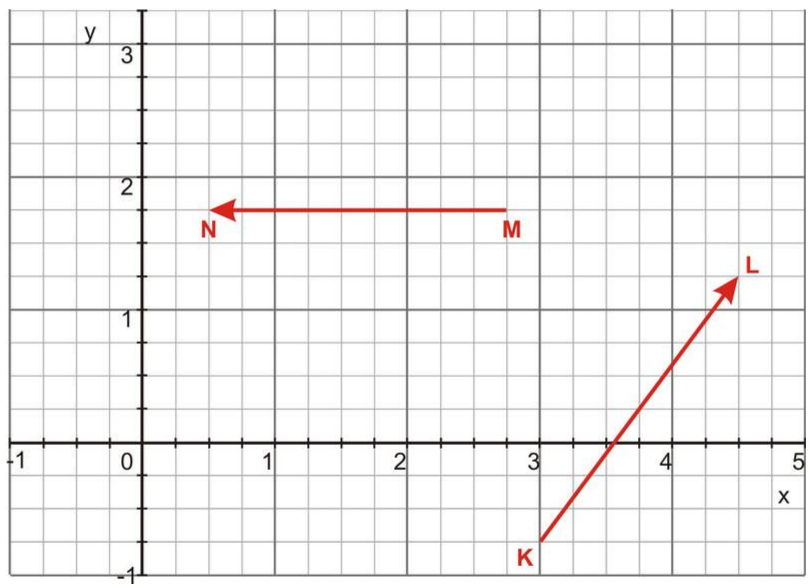
Vector projection is useful in physics applications involving force and work.



When the box is pulled by vector v, some of the force is wasted pulling up against gravity. In real life, this may be useful because of friction, but for now this energy is inefficiently wasted in the horizontal movement of the box.

Example 5

Determine the vector projection of vector $\overrightarrow{MN}$ onto the vector $\overrightarrow{KL}$.



Solution:

The vector projection of one vector onto a 2nd vector is the dot product of the two vectors and the unit vector defining the direction of the 2nd vector. In this case, $\left(\frac{\overrightarrow{MN} \cdot \overrightarrow{KL}}{|\overrightarrow{KL}|}\right)\overrightarrow{KL}$.

First, identify the components of the two vectors by using the information given on the graph. In this case, $\overrightarrow{MN} = \langle -2.25, 0 \rangle$ and $\overrightarrow{KL} = \langle 1.5, 2 \rangle$.

Next, determine the dot product of the two vectors.

$$\begin{aligned}\overrightarrow{MN} \cdot \overrightarrow{KL} &= (MN)_x\ (KL)_x\ +\ (MN)_y\ (KL)_y \\ &= (2.25)(1.5) + (2)(0) \\ &= 3.375\end{aligned}$$

Calculate the length of vector $\overrightarrow{KL}$.

$$\begin{aligned}|\overrightarrow{KL}| &= \sqrt{(KL)_x^2+(KL)_y^2}\\ &= \sqrt{(1.5)^2+(2)^2}\\ &= \sqrt{2.25+4}\\ &= \sqrt{6.25}\\ &= 2.5\end{aligned}$$

Then determine the unit vector in the direction of $\overrightarrow{KL}$. Remember that a unit vector is equal to the ratio of the vector and its magnitude.

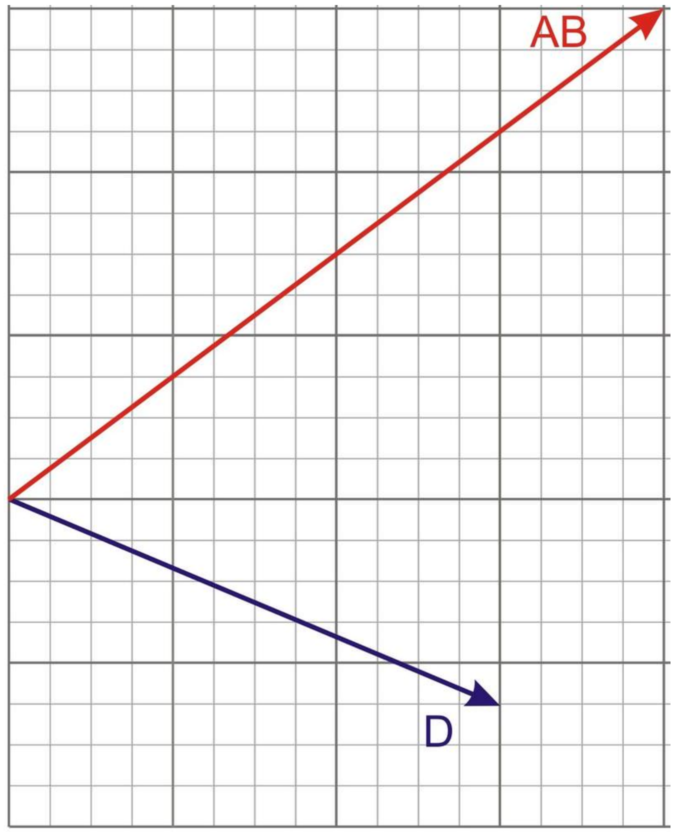
$$\begin{aligned}\frac{\overrightarrow{KL}}{|\overrightarrow{KL}|} &= \frac{\langle 1.5,2\rangle}{2.5}\\ &= \left\langle \frac{1.5}{2.5},\frac{2}{2.5}\right\rangle\\ &= \langle 0.6,0.8\rangle .\end{aligned}$$

Lastly, multiply the dot product of the two vectors by this unit vector:

$$(\overrightarrow{MN}\cdot\overrightarrow{KL})\frac{\overrightarrow{KL}}{|\overrightarrow{KL}|} = (3.375)\langle 0.6,0.8\rangle = \langle 2.025,2.7\rangle .$$

Example 6

The diagram below shows both vectors ***AB*** and ***D*** together on the same grid. Determine the scalar projection of vector ***AB*** onto the direction of vector ***D***.



Solution:

To find the scalar projection onto the direction of another vector, we need to know the unit vector in the direction of vector $\boldsymbol{D}$.

First, the components of $\overrightarrow{AB}$ are $\langle 4,3\rangle$ and $\overrightarrow{D}$ are $\langle 3,-1.25\rangle$.

Now we can use the dot product to calculate the scalar projection of $\boldsymbol{AB}$ onto the direction of vector $\boldsymbol{D}$.

$$\overrightarrow{AB}\cdot\overrightarrow{D}=(4\cdot 3)+(3\cdot -1.25)=8.25.$$

The magnitude of $\overrightarrow{D}$ is

$$\begin{aligned}|D| &= \sqrt{(D_x)^2+(D_y)^2}\\ &= \sqrt{3^2+(-1.25)^2}\\ &= \sqrt{9+1.5625}\\ &= \sqrt{10.5625}\\ &= 3.25.\end{aligned}$$

Thus, the scalar projection of $\boldsymbol{AB}$ onto the direction of vector $\boldsymbol{D}$ is

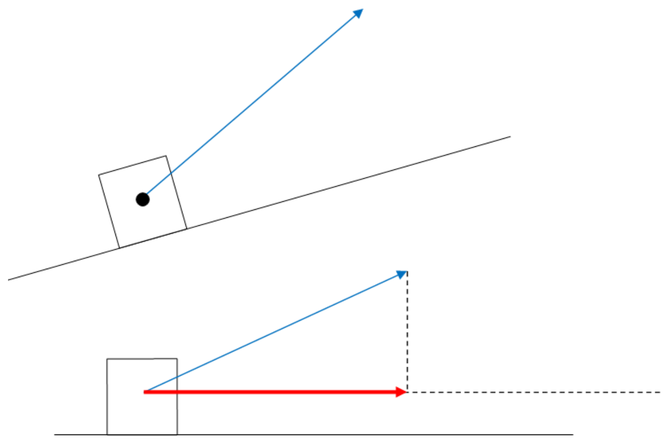
$$\frac{\overrightarrow{AB}\cdot\overrightarrow{D}}{|\overrightarrow{D}|}=\frac{8.25}{3.25}\approx 2.54.$$

Example 7

A father is pulling his daughter on a sled that has a rope attached to it. They are going up a hill with a 20° incline. The sled sits on the ground, and the father pulls the rope with a force of 100 lbs as he walks. The rope makes a 39° angle with the slope. What is the effective force the father exerts as he pulls his daughter and the sled up the hill?

Solution:

The box below represents the girl and the sled. The blue arrow indicates the father's 100-lb force. Notice that the question asks for simply the amount of force, which means scalar projection. Since this is not dependent on the slope of this hill, we can rotate our perspective and still get the same scalar projection.



The components of the father's force vector is $100 < \cos 39^{\circ}, \sin 39^{\circ} >$, and the "onto" vector is any vector horizontally to the right. Since we are looking only for the length of the horizontal component and you already have the angle between the two vectors, the scalar projection is

$$100 \cdot \cos 39^{\circ} \approx 77.1 \text{ lb.}$$

Summary

- The **vector projection** of a vector onto a given direction has a magnitude equal to the scalar projection.
- The formula for the projection vector is given by $proj_u v = \left(\frac{u \cdot v}{|u|}\right) \frac{u}{|u|}$.
- A vector $\overrightarrow{v}$ is multiplied by a scalar s. Its components are given by $\overrightarrow{sv} = \langle sv_x, sv_y \rangle$.
- A **scalar projection** is the length of the vector projection. It is given by the dot product of a vector with a unit vector for that direction.

Review

1. Calculate the result of a scalar multiple of 6 on the vector $\langle 1, 19 \rangle$.
2. Calculate the result of a scalar multiple of (-8) on the vector $\langle 7, 8 \rangle$.
3. Calculate the result of a scalar multiple of 15 on the vector $\langle -7, 3 \rangle$.
4. Calculate the result of a scalar multiple of (-11) on the vector $\langle -9, 20 \rangle$.
5. Calculate the result of a scalar multiple of 16 on the vector $\langle 9, 10 \rangle$.

6. Given vector A = $\langle -4, 6 \rangle$ and vector B = $\langle 9, 15 \rangle$, what is the projection of A onto B?
7. What is the projection of $\langle 2, 10 \rangle$ onto $\langle 1, 3 \rangle$?
8. Given vector C = $\langle 1, 2 \rangle$ and vector D = $\langle 1, 11 \rangle$, what is the projection of C onto D?

9. Given vector E = $\langle -5, 3\rangle$ and vector F = $\langle -1, 18\rangle$, what is the projection of E onto F?
10. What is the projection of $\langle -2, 1\rangle$ onto $\langle -5, 6\rangle$?
11. Given vector H = $\langle 8i + 11j\rangle$ and vector I = $\langle -2i + 15j\rangle$, what is the projection of H onto I?
12. What is the projection of $\langle -1, 6\rangle$ onto $\langle 8, 16\rangle$?
13. What is the projection of $\langle -1, 8\rangle$ onto $\langle 7, 8\rangle$?
14. Given vector J = $\langle -4i + 8j\rangle$ and vector K = $\langle -8i + 10j\rangle$, what is the projection of J onto K?
15. Given vector L = $\langle -4, 3\rangle$ and vector M = $\langle 7, 17\rangle$, what is the projection of L onto M?
16. A box is on the side of a hill inclined at 30°. The weight of the box is 40 lbs. What is the magnitude of the force required to keep the box from sliding down the hill?
17. Sarah is on a sled on the side of a hill inclined at 60°. The weight of Sarah and the sled is 125 lbs. What is the magnitude of the force required for Sam to keep Sarah from sliding down the hill?
18. A 1,780-lb car is parked on a street that makes an angle of 15° with the horizontal. Find the magnitude of the force required to keep the car from rolling down the hill.
19. A 1,900-lb car is parked on a street that makes an angle of 10° with the horizontal. Find the magnitude of the force required to keep the car from rolling down the hill.
20. A 30-lb force that makes an angle of 32° with an inclined plane is pulling a box up the plane. The inclined plane makes a 20° angle with the horizontal. What is the magnitude of the effective force pulling the box up the plane?

Review (Answers)

Please see the Appendix.

9.7 Vector Equation of a Line

Learning Objectives

Learn how to identify a line in three dimensions as the intersection of two planes. You will also explore the process of finding the vector equation of a line given two points on that line.

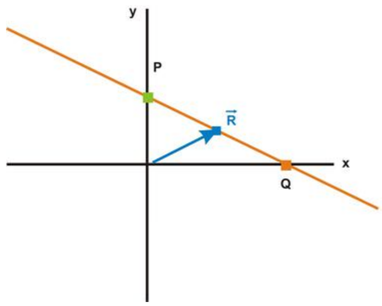
Introduction

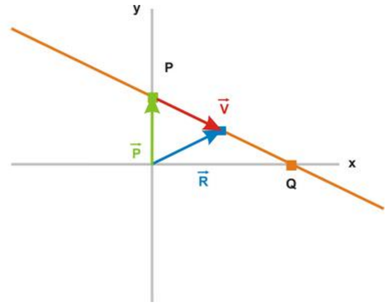
Air traffic control is tracking two planes in the vicinity of the airport. At a given moment, one plane is at a location 45 km east and 120 km north of the airport. The second plane is located 63 km east and 96 km south of the airport. The first plane is flying directly toward the airport, while the second plane is continuing at a constant altitude, with a heading defined by the vector $\overrightarrow{h_2} = \langle 3,4 \rangle$, to land eventually at another airport northwest of our air traffic controllers. Do the paths of these two aircrafts cross?

Vector Equation of a Line

In a two-dimensional plane, a line can be represented by the equation $y = mx + b$, where m is the slope of the line and b is the y-intercept.

One way to identify points on a line can be found using the vector addition method we discussed earlier in this chapter. To find the position vector, $\overrightarrow{r}$, for any point along a line, we can add the position vector of a point on the line that we already know, and add to that a vector, $\overrightarrow{v}$, which lies on the line as shown in the diagram below.



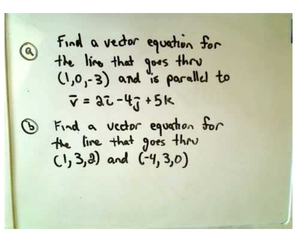


The position vector $\overrightarrow{r}$ for a point between P and Q is given by $\overrightarrow{r} = \overrightarrow{p} + \overrightarrow{v}$.

All other points on this line can be reached by traveling along the line from point P. Therefore, the position vector for any point on the line is given by $\overrightarrow{r} = \overrightarrow{p} + t\overrightarrow{v}$, where t is a real number.

If we know the locations of two points on a line, we can determine the equation of the line. All points on a line fulfill the equation $\overrightarrow{r} = \overrightarrow{p} + t\overrightarrow{v}$, where k is a scalar that varies from -∞ to ∞. If we already know the position vectors for two points on the line, $\overrightarrow{p}$ and $\overrightarrow{q}$, we can use the method of vector subtraction to determine the equation of the vector, $\overrightarrow{v} = \overrightarrow{p} - \overrightarrow{q}$. Therefore, $\overrightarrow{r} = \overrightarrow{p} + t(\overrightarrow{p} - \overrightarrow{q})$,where t varies from -∞ to ∞.

The following video provides the formula for finding the vector equation of a line, and demonstrates how to use the formula with two examples:



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/55016

Examples

Example 1

Write the equation of $y = -\frac{5}{3}x + 5$ as a vector equation.

Solution:

Start by choosing two points on the line, say (3,0) and (6,-5). Then, $\overrightarrow{p} = \langle 3,0 \rangle$ and $\overrightarrow{q} = \langle 6,-5 \rangle$. So,

$$\begin{aligned}\overrightarrow{p} - \overrightarrow{q} &= \langle 3,0 \rangle - \langle 6,-5 \rangle \\ &= \langle 3-6, 0-(-5) \rangle \\ &= \langle -3,5 \rangle .\end{aligned}$$

Finally, the vector equation of the line is

$$\begin{aligned}\overrightarrow{r} &= \langle 3,0 \rangle + k \langle -3,5 \rangle \\ &= \langle 3-3k, 5k \rangle .\end{aligned}$$

Example 2

Determine the equation for the line defined by the points *P* = (6, 7) and *Q* = (3, 2). Then find the position vector for a point, *R,* halfway between these two points.

Solution:

The vector of the line connecting the two points is given by

$$\begin{aligned}\overrightarrow{v} &= \overrightarrow{p} - \overrightarrow{q} \\ &= \langle (6-3), (7-2) \rangle \\ &= \langle 3,5 \rangle .\end{aligned}$$

The equation of the line then becomes

$$\overrightarrow{r} = \langle 6,7 \rangle + t \langle 3,5 \rangle .$$

Since the vector $\overrightarrow{v}$ points from *P*, the value of *t* for a particular point gives us some information about the location of that point. If $0 < 1$, the point lies on the line between *P* and *Q*. If $t < 0$, the point is not between *P* and *Q* and is closer to *P* than *Q*. If $t > 1$, the point is not between *P* and *Q* and is closer to *Q*.

The point halfway between the two points has $t = \frac{1}{2}$.

$$\begin{aligned}\overrightarrow{R} &= \langle 6,7 \rangle + \frac{1}{2} \langle 3,5 \rangle \\ &= \langle (6+1.5), (7+2.5) \rangle \\ &= \langle 7.5, 9.5 \rangle\end{aligned}$$

Example 3

Do the two vectors $\overrightarrow{D} = \langle 1,-1 \rangle + d \langle 1,-1 \rangle$ and $\overrightarrow{F} = \langle 2,4 \rangle + f \langle 2,1 \rangle$ intersect?

Solution:

If the two vectors intersect, there must be a point identified by position vector $\overrightarrow{p}$ that satisfies the equations of both lines. In other words, we must be able to find values for *d* and *f* such that

$$\overrightarrow{D} = \overrightarrow{F}$$

or

$$\langle 1,-1\rangle + d\langle 1,-1\rangle = \langle 2,4\rangle + f\langle 2,1\rangle .$$

Each component of vectors $\overrightarrow{D}$ and $\overrightarrow{F}$ must independently be equal if $\overrightarrow{D} = \overrightarrow{F}$.

$$\begin{aligned} 1+d &= 2+2f \\ -1-d &= 4+f \end{aligned}$$

By solving this system using elimination, $d = -3$ and $f = -2$.

Thus,

$$\begin{aligned} \langle 1,-1\rangle + d\langle 1,-1\rangle &= \langle 1,-1\rangle - 3\langle 1,-1\rangle \\ &= \langle 1,-1\rangle + \langle -3,3\rangle \\ &= \langle -2,2\rangle \\ \langle 2,4\rangle + f\langle 2,1\rangle &= \langle 2,4\rangle - 2\langle 2,1\rangle \\ &= \langle 2,4\rangle + \langle -4,-2\rangle \\ &= \langle -2,2\rangle . \end{aligned}$$

Both equations are equally satisfied, so the two lines do intersect and the point of intersection is (-2, 2).

Example 4

Recall the problem from the Introduction: One plane is at a location 45 km east and 120 km north of the airport. The second plane is located 63 km east and 96 km south of the airport. The first plane is flying directly toward the airport, while the second plane is continuing at a constant altitude, with a heading defined by the vector $\overrightarrow{h_2} = \langle 3,4\rangle$ to land eventually at another airport northwest of our air traffic controllers. Do the paths of the two aircraft cross?

Solution:

The first thing we need to do is to determine the position vectors of the two planes. Define our airport as the origin of coordinates, and define $\hat{x}$ = east and $\hat{y}$ = north. Call the position of the first plane *P*, and the position of the second plane *Q*. This gives $\overrightarrow{p} = \langle 45,120\rangle$ and $\overrightarrow{q} = \langle 63,-96\rangle$.

The first plane is heading directly toward the airport. The vector from the position of this plane to the origin is given by

$$\begin{aligned} \overrightarrow{v} = \overrightarrow{origin} - \overrightarrow{p} &= \langle (0-45),(0-120)\rangle \\ &= \langle -45,-120\rangle . \end{aligned}$$

The equation of the line representing the first plane's motion then becomes

$$\overrightarrow{r_1} = \langle 45,120\rangle + t_1\langle -45,-120\rangle .$$

The second plane continues at a constant altitude with a heading $\overrightarrow{h_2} = \langle 3,4\rangle$. The equation of the line representing this plane is

$$\overrightarrow{r_2} = \langle 63, -96 \rangle + t_2 \langle 3, 4 \rangle .$$

If the paths of the two planes intersect, the position of that intersection must have coordinates that satisfy both equations. In other words, $\vec{v_1} = \vec{v_2}$.

$$\begin{aligned} 45 - 45t_1 &= 63 + 3t_2 \\ 120 - 120t_1 &= -96 + 4t_2 \end{aligned}$$

By solving this system using elimination, $t_1 = 4$ and $t_2 = -66$.

Thus,

$$\begin{aligned} \langle 45, 120 \rangle + t_1 \langle -45, -120 \rangle &= \langle 45, 120 \rangle 4 \langle -45, -120 \rangle \\ &= \langle 45, 120 \rangle + \langle -180, -480 \rangle \\ &= \langle -135, -360 \rangle \\ \langle 63, -96 \rangle + t_2 \langle 3, 4 \rangle &= \langle 63, -96 \rangle - 66 \langle 3, 4 \rangle \\ &= \langle 63, -96 \rangle + \langle -198, -264 \rangle \\ &= \langle -135, -360 \rangle . \end{aligned}$$

Both equations are equally satisfied, so the two paths of the planes do intersect, and the point of intersection is (-135, 360). Therefore, the air traffic controller should take the next step to consider speed and altitude as well to determine if they simply have the same path or will be in danger of crashing.

Example 5

In physics, the motion of an object traveling at a constant speed is described by the equation $\overrightarrow{s_t} = \overrightarrow{s_i} + \overrightarrow{v} t$, where s_i is the initial position, s_t is the position at some later time t, and v is the velocity of the object. Write the vector equation that returns the set of position vectors $\overrightarrow{s}$ for an object having an initial position $\overrightarrow{s_i} = \langle 2, 3 \rangle$ and a velocity of $\overrightarrow{v} = \langle 1, 1 \rangle$, and determine the object's location at t = 10 s.

Solution:

$$\overrightarrow{s_t} = \overrightarrow{s_i} + \overrightarrow{v} t = \langle 2, 3 \rangle + t \langle 1, 1 \rangle = \langle 2 + t, 3 + t \rangle$$

At t = 10 s,

$$\overrightarrow{s_{10}} = \langle 2 + 10, 3 + 10 \rangle = \langle 12, 13 \rangle .$$

Example 6

An object has a position of $\overrightarrow{s_i} = \langle 3, 3 \rangle$ at t = 0 and a velocity of $\overrightarrow{v} = \langle 10, 7 \rangle$. Use the vector equation $\overrightarrow{s} = \overrightarrow{s_i} + \overrightarrow{v} t$ to determine the distance traveled by the object between t = 3 s and t = 5 s. (Distance is measured in meters.)

Solution:

The vector equation describing the motion of the object is

$$\begin{aligned}\overrightarrow{s} &= \overrightarrow{s_i} + \overrightarrow{v}t \\ &= \langle 3,3\rangle + t\,\langle 10,7\rangle \\ &= \langle 3+10t, 3+7t\rangle\,.\end{aligned}$$

The object's position at $t = 3$ s is obtained from the vector equation:

$$\begin{aligned}\overrightarrow{s_3} &= \langle 3,3\rangle + (3)\,\langle 10,7\rangle \\ &= \langle 3+10(3), 3+7(3)\rangle \\ &= \langle 33,24\rangle\,.\end{aligned}$$

The object's position at $t = 5$ s is obtained from the vector equation:

$$\begin{aligned}\overrightarrow{s_5} &= \langle 3,3\rangle + (5)\,\langle 10,7\rangle \\ &= \langle 3+10(5), 3+7(5),\rangle \\ &= \langle 53,38\rangle\,.\end{aligned}$$

The distance traveled between these two points is the magnitude of the vector starting at (33, 24) and ending at (53, 38).

$$\begin{aligned}\overrightarrow{\triangle s} &= \overrightarrow{s_5} - \overrightarrow{s_3} \\ &= \langle 53,38\rangle - \langle 33,24\rangle \\ &= \langle 20,14\rangle\end{aligned}$$

Now we can use the Pythagorean Theorem to determine the magnitude of the vector.

$$\begin{aligned}|\overrightarrow{\triangle s}| &= \sqrt{a^2+b^2} \\ &= \sqrt{(20)^2+(14)^2} \\ &\approx 24.41 \text{ meters}\end{aligned}$$

Example 7

Determine the vector equation of the straight line defined by the points (2, 2) and (1, 3).

Solution:

These two points have position vectors $\overrightarrow{p} = \langle 2,2 \rangle$ and $\overrightarrow{q} = \langle 1,3 \rangle$. The vector of the line connecting the two points is given by

$$\overrightarrow{v} = \overrightarrow{p} - \overrightarrow{q} = \langle (2-1),(2-3) \rangle = \langle 1,-1 \rangle .$$

The equation of the line is

$$\overrightarrow{r} = \langle 2,2 \rangle + t \langle 1,-1 \rangle .$$

Summary

- The **vector form of the equation of a line** is:
  $\vec{r} = \vec{r}_0 + t\vec{v} =< x_0, y_0, z_0 > +t.$

Review

Write the vector equation of the line defined by the the following points:

1. $(2,-2,5)$ and $(1,5,4)$
2. $(2,-9,5)$ and $(8,4,-6)$
3. $(15,3,-3)$ and $(4,-3,9)$
4. $(-1,-1,7)$ and $(3,11,8)$
5. $(1,-3,2)$ and $(-5,3,-1)$
6. $(25,17,42)$ and $(-16,12,23)$

Determine if the two vectors are skew lines or if they intersect each other.

7. $\overrightarrow{D} = \langle 2,3,1 \rangle + d \langle 5,3,6 \rangle$ and $\overrightarrow{F} = \langle -5,-3,-5 \rangle + f \langle 15,3,-3 \rangle$
8. $\overrightarrow{D} = \langle 3,4,7 \rangle + d \langle 3,3,2 \rangle$ and $\overrightarrow{F} = \langle -2,11,7 \rangle + f \langle -2,11,7 \rangle$
9. $\overrightarrow{D} = \langle 15,3,-3 \rangle + d \langle 3,11,8 \rangle$ and $\overrightarrow{F} = \langle 5,3,6 \rangle + f \langle 1,-4,6 \rangle$
10. $\overrightarrow{D} = \langle 13,-1,6 \rangle + d \langle -5,4,12 \rangle$ and $\overrightarrow{F} = \langle 6,9,0 \rangle + f \langle 21,0,14 \rangle$

Identify the position vector for the midpoint of each line below. (You already found the vector equations in problems 1 through 4.)

11. $(2,-2,5)$ and $(1,5,4)$
12. $(2,-9,5)$ and $(8,4,-6)$
13. $(15,3,-3)$ and $(4,-3,9)$
14. $(-1,-1,7)$ and $(3,11,8)$

Use the vector equation to find the position of the object at t = 3s and t = 5s. (Distance is measured in meters.)

15. An object has a position of $\overrightarrow{s_i} = \langle 3,-5,1 \rangle$ at t = 0 and a velocity of $\overrightarrow{v} = \langle 2,7,-3 \rangle$.
16. An object has a position of $\overrightarrow{s_i} = \langle 1,-4,-8 \rangle$ at t = 0 and a velocity of $\overrightarrow{v} = \langle 9,-7,3 \rangle$.
17. An object has a position of $\overrightarrow{s_i} = \langle -1,2,-3 \rangle$ at t = 0 and a velocity of $\overrightarrow{v} = \langle 3,-1,-2 \rangle$.

Review (Answers)

Please see the Appendix.

Vocabulary

9.8 Project: Vectors

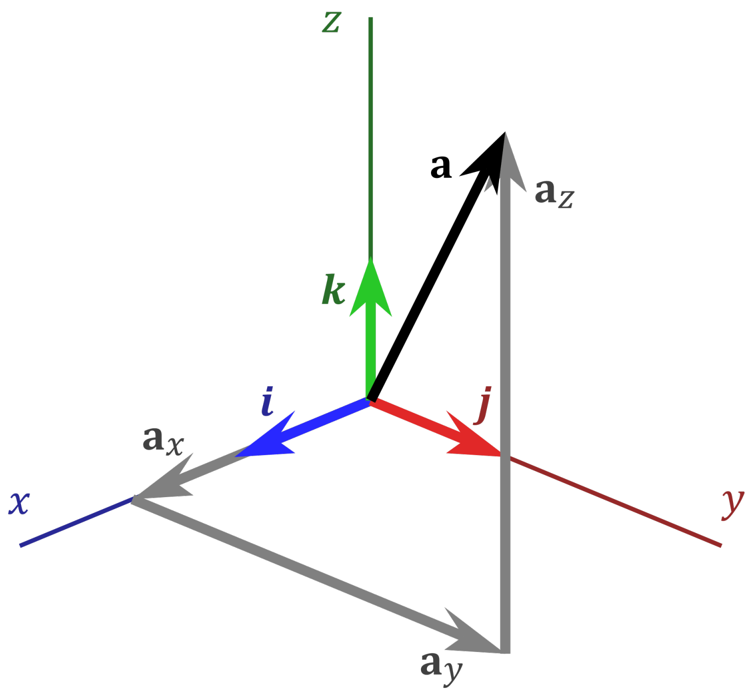
Vector Project

In 2015, about 140.43 billion gallons of gasoline were consumed in the United States, a daily average of about 384.74 million gallons (or about 9.16 million barrels per day). (Source: U.S. Energy Information Administration, https://www.eia.gov/tools/faqs/faq.cfm?id=23&t=10)

1. Crude oil is sold in barrels of 42 gallons each. How many barrels are used each year?
2. If one barrel of oil begins rolling down a hill at a 10° angle, and the only force acting on it is gravity with a force of 500 N, what is the work done by the force of gravity after the barrel has rolled 150 meters?
3. How much force will a man use if he is dragging a barrel at constant speed at a 25° angle with the ground?
4. A warehouse is storing 150 barrels of the crude oil, with each weighing about 300 pounds, when an earthquake occurs. The barrels tip over against the wall at an angle of 49°. What is the magnitude of the horizontal force against the wall? If the wall was built to withstand 30,000 lbs of force, what is the container angle that will cause the wall to collapse?

9.9 Summary: Vectors

Physicists, video game designers, meteorologists, and air traffic controllers all use vectors. Vector quantities have a direction and a magnitude. In this chapter, you learned that you can add and subtract vectors just as you can numbers. You saw that vectors also have other algebraic properties and essentially create an entirely new algebraic structure. There are two ways the product of vectors can be taken, and you studied one way called the dot product. Lastly, you learned to calculate how much of one vector goes in the direction of another vector.



Chapter Summary

In this chapter we learned:

- A vector is a set of instructions indicating direction and magnitude.
- A vector is determined by two points: the initial point, called the tail, and its terminal point, called the head.
- A force diagram is a collection of vectors where each represents a force, like gravity or wind, acting on an object.
- Magnitude refers to the length of the vector and is associated with the strength of the force or the speed of the object.
- Bearing is measured with 0° as due North, 90° as East, 180° as South, and 270° as West.
- The magnitude of the position vector is given by

$$|\overrightarrow{P}| = \sqrt{P_x^2 + P_y^2}$$

- A resultant vector is the vector that is produced when two or more vectors are summed or subtracted. It is also what is produced when a single vector is scaled by a constant.

- To scale a vector, multiply the vector's magnitude and direction by the common scale factor.
- A unit vector is a vector of magnitude 1.
- Component form means in the form $< x, y >$.
- The standard unit vectors are $\overrightarrow{i}$, which is the vector $< 1, 0 >$, and $\overrightarrow{j}$, which is the vector $< 0, 1 >$.
- The dot product is also known as inner product and scalar product. It is one of two kinds of products taken between vectors. It produces a number that can be interpreted to tell how much one vector goes in the direction of the other.
- The dot product formula is $u \cdot v =< u_1, u_2 > \cdot < v_1, v_2 >= u_1v_1 + u_2v_2$.
- The vector projection of a vector onto a given direction has a magnitude equal to the scalar projection.
- A scalar projection is given by the dot product of a vector, with a unit vector for that direction.
- Scalar projection is the length of the vector projection.
- The vector projection formula is $proj_u v = \left(\frac{u \cdot v}{|u|}\right) \frac{u}{|u|}$.
- The vector form of the equation of a line is $\vec{r} = \vec{r_0} + t\vec{v} =< x_0, y_0 > +t$.

Review

Try the following cumulative review problems to practice the concepts studied in this chapter:

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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/194743

9.10 References

1. Andrew D. Hwang. https://i.stack.imgur.com/WsIcy.png .
2. Richard Eriksson. https://www.flickr.com/photos/sillygwailo/814701516/in/photolist-2eZynS-fwLHZi-4d8QYh-6W7Bsm-4E3D6S-6W7B7U-8bZ29H-p5WV71-9ZPt6v-7WEzs8-9ZSkTN-9ZPuEZ-9n8jDW-9ZQ6JM-9ZQ6mZ-2wjPtK-9ZSjWG-6W3y3x-9ZPuDB-9ZSjnb-9DJwXe-9ZSkY3-9ZSk77-9ZPtAc-6hxGE5-fQpBzr-9ZPtw6-fMS1f5-4upeWD-8BdJrH-a6t854-97euqa-8hunwV-oWVar-8hwVDY-9ZSk8w-9ZPtnk-9ZSkHS-g9VEHJ-9ZPtxr-9ZSkG9-dqo7wA-57t2bW-9ZSjLU-4wBxR8-8WMLtF-9ZSnKL-9ZSk9s-9ZQ6CX-9ZPu9R .
3. Unknown. http://maxpixel.freegreatpicture.com/Bird-Soaring-Flight-Flying-Wild-Raptor-Bald-Eagle-1259676 .
4. Madison Scott-Clary. https://www.flickr.com/photos/ranna/11719031374/in/photolist-iRz6K5-dNXS9X-e2rxDM-e2xqcf-bJxdt-SX79rC-e2r4cr-e2xbCd-e2wm2s-e2qyc8-NsS9HP-o9V6Sv-cyLejh-nckQxX-e2rge6-e2wbVj-cyLSGN-SfyJh1-95ax7z-cyKM5W-e2wi2A-cQrjq5-b4m7gk-34QYCm-SfyJTS-HqaKm9-e2rdQe-97rcfB-qBmS5E-RCVCXU-4BcQuR-p4TtLF-ec3xQi-iRyA8D-e2wWw9-cyKY33-7Ypru8-ncmYiz-e2xoaq-SfymCE-e2rxoc-cyLcvL-hDb5UJ-e2wkam-AmG1-cyL9Tf-nNp7mV-cyLuMA-8QB2bD-jvX6GC .
5. josh james. https://www.flickr.com/photos/joshwept/5689555989/in/photolist-9ELsLV-hQHaku-aRPrjD-66Vo6T-bYfM9E-3mCL78-3mCKte-3mHcSY-3mD2ZX-rBNeAw-gjfx4o-3mHyYd-3mHxK7-3mD2dV-3mHssf-3mHpdL-3mCNDV-3mHoso-3mCULM-3mHg9L-aRPr6M-3mCN9g-3mHe21-kEo9bv-3mCQbX-3mHke3-3mHnCo-aRPrri-3mCYQR-rDZahU-3mHeBS-aRPrd8-3mCPtr-3mHm1w-3mD3Qt-cWi4TL-3mCLAr-3mHhTf-3mHjsN-8tiAF5-3mCY4e-94WSNd-5iCPtm-3mD5uz-3mHiFW-3mHrDA-efp5rZ-3mHvHJ-8tK58b-fPsDJr .
6. Chika Watanabe. https://www.flickr.com/photos/chikawatanabe/3252005289/ .
7. S Foster. https://commons.wikimedia.org/wiki/File:Prototype_Pontiac_race_car.jpg .
8. Demetrius Lester. https://commons.wikimedia.org/wiki/File:Bagram_control_tower_-d.jpg .

Chapter 10 Systems and Matrices

Chapter Outline

10.1 Introduction: Systems and Matrices

Relationships between numbers can be helpful in making predictions or determining future behavior of phenomena. Often the relationships are impacted by multiple factors or variables. You have solved systems of equations in your algebra studies using methods such as substitution and elimination to find important points of intersection between two equations. There are numerous applications of systems of equations: for example, comparing the nutritional information of recipes, determining how much different cell phone providers charge for their services, or deciding which vacation package is the best deal. As systems become more complex, matrices, or arrays of numbers, can be used to solve or manipulate multiple layers of data.

10.2 Systems of Two Equations and Two Unknowns

Learning Objectives

Learn to solve a system of two equations and two unknowns using the elimination method.

Introduction

The cost of two cell phone plans can be written as a **system of equations** based on the number of minutes used and the base monthly rate.

As a consumer, you would find it useful to know when the two plans cost the same, and when one plan is less expensive.

Plan A costs \$40 per month plus \$0.10 for each minute of talk time.

Plan B costs \$25 per month plus \$0.50 for each minute of talk time.

Plan B has a lower starting cost, but it costs more per minute. Therefore, it may not be the most cost-effective plan for someone who likes to spend a lot of time on the phone.

When do the two plans cost the same amount?

Solving Systems of Equations with Two Unknowns

There are many ways to solve a system of equations, including substitution, elimination, and graphing. Here we will focus on solving using elimination, because the knowledge and skills we use with this method will transfer directly to our work with matrices. When we solve systems of linear equations, there are three solution options:

1. If a system is **independent**, then the linear equations intersect at one point, so there is one solution to the system.
2. If a system is **dependent**, then the linear equations intersect at every point on the lines, so there are infinitely many solutions to the system.
3. If a system is **inconsistent**, then the linear equations are parallel, so there are no solutions to the system.

When using elimination to solve a system, first count the number of variables that are missing and the number of equations. The number of variables needs to be the same or fewer than the number of equations. For instance, a system with two equations and two variables can be solved, but one equation with two variables cannot.

Here's the procedure for solving a system using the elimination method:

- **Step 1**: Write both equations with two variables in standard form, $Ax+By=C$. This form helps to align the variables.
- **Step 2**: Determine which variable you want to eliminate.
- **Step 3**: Scale each equation as necessary by multiplying through by constants.
- **Step 4**: Add the equations together.
- **Step 5**:
  - If the system is independent, solve for the variable. Then substitute the result into one of the linear equations to determine the value for the second variable.
  - If the system is dependent, Step 4 results in $0=0$.
  - If the system is inconsistent, Step 4 results in $0=k$, where k is a nonzero number.

Here is a system of two equations and two variables in standard form: $5x+12y=72$ and $3x-2y=18$. Notice there is an x column and a y column on the lefthand side, and a constant column on the righthand side. If not, rewrite the equations as shown. Also notice that if you add the system as written, no variable will be eliminated.

Equation 1: $5x+12y=72$

Equation 2: $3x-2y=18$

Strategically choose to eliminate y by scaling the second equation by 6, so that the coefficient of y will match at 12 and -12.

$$\begin{aligned} 5x+12y&=72 \\ 18x-12y&=108 \end{aligned}$$

Add the two equations:

$$\begin{aligned} 23x&=180 \\ x&=\frac{180}{23} \end{aligned}$$

The value for x could be substituted into either of the original equations, and the result could be solved for y; however, since the value is a fraction, it may be easier to repeat the elimination process to solve for y. This time you will take the first two equations and eliminate x by making the coefficients of x to be 15 and -15. Scale the first equation by a factor of 3, and scale the second equation by a factor of -5.

Equation 1: $15x+36y=216$

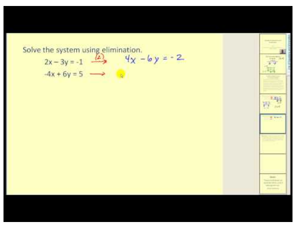
Equation 2: $-15x+10y=-90$

Adding the two equations:

$$0x+46y=126$$
$$y=\frac{126}{46}=\frac{63}{23}$$

The point $\left(\frac{180}{23},\frac{63}{23}\right)$ is where these two lines intersect.

The following video has further examples demonstrating how to solve a system of equations using the elimination method:



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/61405

Examples

Example 1

Recall the problem from the Introduction in which you were asked when the two cell phone plans cost the same amount.

Plan A costs $40 per month plus $0.10 for each minute of talk time.

Plan B costs $25 per month plus $0.50 for each minute of talk time.

Solution:

To find out when the two plans cost the same, represent each plan with an equation and solve the system of equations. Let y represent cost, and x represent number of minutes.

$$y=0.10x+40$$
$$y=0.50x+25$$

First put these equations in standard form.

$$x-10y=-400$$
$$x-2y=-50$$

Then scale the second equation by -1, add the equations together, and solve for y.

$$-8y = -350$$
$$y = 43.75$$

To solve for x, scale the second equation by -5, add the equations together, and solve for x.

$$-4x = -150$$
$$x = 37.5$$

The equivalent costs of plan A and plan B will occur at 37.5 minutes of talk time with a cost of \$43.75.

Example 2

Solve the following system of equations:

$$6x - 7y = 8$$
$$15x - 14y = 21$$

Solution:

Scaling the first equation by -2 will allow the y term to be eliminated when the equations are summed.

$$-12x + 14y = -16$$
$$15x - 14y = 21$$

The sum is

$$3x = 5$$
$$x = \frac{5}{3}.$$

Substitute x into the first equation to solve for y.

$$6 \cdot \frac{5}{3} - 7y = 8$$
$$10 - 7y = 8$$
$$-7y = -2$$
$$y = \frac{2}{7}$$

The point $\left(\frac{5}{3}, \frac{2}{7}\right)$ is where these two lines intersect.

Example 3

Solve the following system using elimination:

$$
\begin{aligned}
5x - y &= 22 \\
-2x + 7y &= 19
\end{aligned}
$$

Solution:

Start by scaling the first equation by 7, and notice that the y coefficient will immediately be eliminated when the equations are summed.

$$
\begin{aligned}
35x - 7y &= 154 \\
-2x + 7y &= 19
\end{aligned}
$$

Add; solve for $x = \frac{173}{33}$. Instead of substituting, practice eliminating x by scaling the first equation by 2 and the second equation by 5.

$$
\begin{aligned}
10x - 2y &= 44 \\
-10x + 35y &= 95
\end{aligned}
$$

Add; solve for y.

Final Answer: $\left(\frac{173}{33}, \frac{139}{33}\right)$

Example 4

Solve the following system of equations:

$$
\begin{aligned}
5 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} &= 11 \\
\frac{1}{x} + \frac{1}{y} &= 4
\end{aligned}
$$

Solution:

The strategy of elimination still applies. You can eliminate the $\frac{1}{y}$ term if the second equation is scaled by a factor of -2.

$$
\begin{aligned}
5 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} &= 11 \\
-2 \cdot \frac{1}{x} - 2 \cdot \frac{1}{y} &= -8
\end{aligned}
$$

Add the equations together and solve for x.

$$
\begin{aligned}
3 \cdot \frac{1}{x} + 0 \cdot \frac{1}{y} &= 3 \\
3 \cdot \frac{1}{x} &= 3 \\
\frac{1}{x} &= 1 \\
x &= 1
\end{aligned}
$$

Substitute into the second equation and solve for y.

$$\begin{aligned}\frac{1}{1}+\frac{1}{y}&=4\\1+\frac{1}{y}&=4\\\frac{1}{y}&=3\\y&=\frac{1}{3}\end{aligned}$$

The point $\left(1,\frac{1}{3}\right)$ is the point of intersection between these two curves.

Example 5

Solve the following system using elimination:

$$\begin{aligned}11\cdot\frac{1}{x}-5\cdot\frac{1}{y}&=-38\\9\cdot\frac{1}{x}+2\cdot\frac{1}{y}&=-25.\end{aligned}$$

Solution:

To eliminate $\frac{1}{y}$, scale the first equation by 2 and the second equation by 5.

To eliminate $\frac{1}{x}$, scale the first equation by -9 and the second equation by 11.

Final Answer: $\left(-\frac{1}{3},1\right)$

Summary

- A system of linear equations can be solved a number of ways, including substitution, elimination, and graphing.
- An **independent** system is when the linear equations intersect at one point, so there is one solution to the system.
- A **dependent** system is when the linear equations intersect at every point on the lines, so there are infinitely many solutions to the system.
- An **inconsistent** system is when the linear equations are parallel, so there are no solutions to the system.
- Elimination Method:
  - **Step 1**: Write both equations with two variables in standard form, $Ax+By=C$.
  - **Step 2**: Determine which variable you want to eliminate.
-
  - **Step 3**: Scale each equation as necessary by multiplying through by constants.
  - **Step 4**: Add the equations together.
  - **Step 5**:
    * If the system is independent, solve for the first variable. Then substitute the result into one of the linear equations to determine the value for the second variable.
    * If the system is dependent, Step 4 results in $0=0$.
    * If the system is inconsistent, Step 4 results in $0=k$, where k is a nonzero number.

Review

Solve each system of equations using the elimination method.

1. $x+y=-4; -x+2y=13$
2. $\frac{3}{2}x-\frac{1}{2}y=\frac{1}{2}; -4x+2y=4$
3. $6x+15y=1; 2x-y=19$
4. $x-\frac{2y}{3}=\frac{-2}{3}; 5x-2y=10$
5. $-9x-24y=-243; \frac{1}{2}x+y=\frac{21}{2}$
6. $5x+\frac{28}{3}y=\frac{176}{3}; y+x=10$
7. $2x-3y=50; 7x+8y=-10$
8. $2x+3y=1; 2y=-3x+14$
9. $2x+\frac{3}{5}y=3; \frac{3}{2}x-y=-5$
10. $5x=9-2y; 3y=2x-3$
11. How do you know if a system of equations has no solution?
12. If a system of equations has no solution, what does this imply about the relationship of the curves on the graph?
13. Give an example of a system of two equations with two unknowns with an infinite number of solutions. Explain how you know the system has an infinite number of solutions.
14. Solve:

$$12\cdot\frac{1}{x}-18\cdot\frac{1}{y}=4$$
$$8\cdot\frac{1}{x}+9\cdot\frac{1}{y}=5$$

15. Solve:

$$14\cdot\frac{1}{x}-5\cdot\frac{1}{y}=-3$$
$$7\cdot\frac{1}{x}+2\cdot\frac{1}{y}=3$$

16. A 150-yard pipe is cut to provide drainage for two fields. If the length of one piece is three yards less that twice the length of the second piece, what are the lengths of the two pieces?
17. Mr. Stein invested a total of \$100,000 in two companies for a year. Company A's stock showed a 13% annual gain, while Company B showed a 3% loss for the year. Mr. Stein made an 8% return on his investment over the year. How much money did he invest in each company?
18. Jack and James each buy some small fish for their new aquariums. Jack buys 10 clownfish and 7 goldfish for \$28.25. James buys 5 clownfish and 6 goldfish for \$17.25. How much does each type of fish cost?
19. The sum of two numbers is 35. The larger number is one less than three times the smaller number. What are the two numbers?
20. Rachel offers to go to the coffee shop to buy cappuccinos and lattes for her coworkers. She buys a total of nine drinks for \$35.75. If cappuccinos cost \$3.75 each, and the lattes cost \$4.25 each, how many of each drink did she buy?

Review (Answers)

Please see the Appendix.

10.3 Solving Linear Systems in Three Variables

Learning Objectives

Learn to solve linear systems in three variables (x, y, and z) algebraically.

Introduction

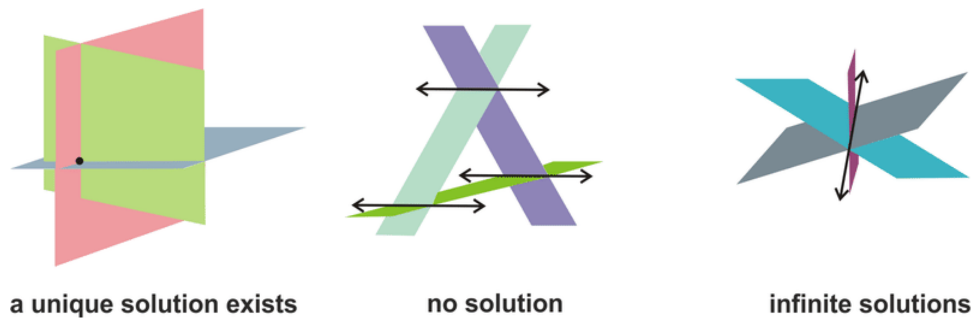
You want to make a platter for a summer picnic.

Three pounds of squash plus five pounds of green beans plus one pound of melon cost \$20. Three pounds of squash plus two pounds of green beans plus two pounds of melon cost \$21. Four pounds of squash plus three pounds of green beans plus three pounds of melon cost \$30. How much does each fruit cost?

Linear Systems in Three Variables

An equation in three variables, such as $2x-3y+4z=10$, is an equation of a plane in three dimensions. In other words, this equation expresses the relationship between the three coordinates of each point on a plane. The solution to a system of three equations in three variables is a point in space that satisfies all three equations. When we add a third dimension, we use the variable, z, for the third coordinate. For example, the point (3, -2, 5) would be $x=3, y=-2$, and $z=5$. A solution can be verified by substituting the x, y, and z values into the equations to see if they are valid.

A system of three equations in three variables consists of three planes in space. These planes could intersect with each other or not as shown in the diagrams below.



- In the first diagram, the three planes intersect at a single point. Thus, a unique solution exists and can be found.
- The second diagram illustrates one way that three planes can exist. It is also possible to have three parallel planes. Moreover, there could be two parallel planes and a third plane that intersects them. In all of these cases, there is no point that is in all three planes. Therefore, there are no solutions to these systems.
- The third diagram shows three planes intersecting in a line. Every point on this line is a solution to the system. Therefore, the system has infinite solutions.

Play, Learn, and Explore Intersecting Planes: www.ck12.org/a/2256724 .

To solve a system of three variables in three equations, we will be using the elimination method. This time we will take two equations at a time to eliminate one variable. Then we will use the resulting equations in two variables to eliminate a second variable and solve for the third. This process is simply an extension of the linear combination procedure used to solve systems with two equations in two variables.

Determining If a Point Is a Solution

How can you determine whether the point (6, -2, 5) is a solution to the system below?

$$\begin{aligned} x - y + z &= 13 \\ 2x + 5y - 3z &= -13 \\ 4x - y - 6z &= -4 \end{aligned}$$

In order for the point to be a solution to the system, it must satisfy each of the three equations.

First equation: $(6) - (-2) + (5) = 6 + 2 + 5 = 13$

Second equation: $2(6) + 5(-2) - 3(5) = 12 - 10 - 15 = -13$

Third equation: $4(6) - (-2) - 6(5) = 24 + 2 - 30 = -4$

The point, (6, -2, 5), satisfies all three equations. Therefore, it is a solution to the system.

Solving Systems

Solve the system using linear combinations:

$$\begin{aligned} 2x+4y-3z &= -7 \\ 3x-y+z &= 20 \\ x+2y-z &= -2 \end{aligned}$$

We can start by taking two equations at a time and eliminating the same variable. We can use the first two equations to eliminate z. Then we can use the second and third equations to eliminate z.

$$\begin{aligned} 2x+4y-3z &= -7 & \Rightarrow \quad 2x+4y-\cancel{3z} &= -7 \\ 3(3x-y+z &= 20) & \underline{9x-3y+\cancel{3z} = 60} \\ & & 11x+y &= 53 \end{aligned}$$

Result from equations 1 and 2: $11x+y=53$

$$\begin{aligned} 3x-y+\cancel{z} &= 20 \\ \underline{x+2y-\cancel{z}} &\underline{= -2} \\ 4x+y &= 18 \end{aligned}$$

Result from equations 2 and 3: $4x+y=18$

Now we have reduced our system to two equations in two variables. We can eliminate y most easily next, and solve for x.

$$\begin{aligned} 11x+y &= 53 & \Rightarrow \quad 11x+\cancel{y} &= 53 \\ -1(4x+y &= 18) & \underline{-4x-\cancel{y} = -18} \\ & & 7x &= 35 \\ & & x &= 5 \end{aligned}$$

Now use this value to find y:

$$\begin{aligned} 4(5)+y &= 18 \\ 20+y &= 18 \\ y &= -2 \end{aligned}$$

Finally, we can go back to one of the original three equations and use our x and y values to find z.

$$\begin{aligned} 2(5)+4(-2)-3z &= -7 \\ 10-8-3z &= -7 \\ 2-3z &= -7 \\ -3z &= -9 \\ z &= 3 \end{aligned}$$

Therefore, the solution is (5, -2, 3).

Don't forget to check your answer by substituting the point into each equation.

Equation 1: $2(5)+4(-2)-3(3)=10-8-9=-7$

Equation 2: $3(5)-(-2)+(3)=15+2+3=20$

Equation 3: $(5)+2(-2)-(3)=5-4-3=-2$

Solve this new system using linear combinations:

$$\begin{aligned} x+y+z&=5\\ 5x+5y+5z&=20\\ 2x+3y-z&=8 \end{aligned}$$

We can start by combining equations 1 and 2 together by multiplying the first equation by -5.

$$\begin{aligned} -5(x+y+z=5) \quad &\Rightarrow \quad -5x-5y-5z=-25\\ 5x+5y+5z=20 \quad &\qquad \underline{5x+5y+5z=20}\\ &\qquad\qquad\qquad 0=-5 \end{aligned}$$

Since the result is a false equation, there is no solution to the system. If you end up with 0 = 0, then there will be infinitely many solutions.

Play, Learn, and Explore 3 Variable Systems: www.ck12.org/a/2114551 .

Examples

Example 1

In the Introduction, you were asked to find how much each fruit cost.

The system of linear equations represented by this situation is:

$$\begin{aligned} 2s+5g+m&=20\\ 3s+2g+2m&=21\\ 4s+3g+3m&=30 \end{aligned}$$

Solution:

Let's start by multiplying the first equation, $2s+5g+m=20$, by 2. We obtain

$4s+10g+2m=40$.

Now we can subtract the third equation from this equation:

$$\begin{aligned} \cancel{4s}+10g+2m&=40\\ \underline{\cancel{4s}+3g+3m}&\underline{=30}\\ 7g-m&=10 \end{aligned}$$

Now let's multiply the second equation by 4 and the third equation by 3. We obtain:

$$\begin{aligned} 12s + 8g + 8m &= 84 \\ 12s + 9g + 9m &= 90 \end{aligned}$$

Now we can subtract the new third equation from the new second equation:

$$\begin{aligned} \cancel{12s} + 8g + 8m &= 84 \\ \underline{\cancel{12s} + 9g + 9m} &\underline{= 90} \\ -g - m &= -6 \end{aligned}$$

If we multiply this result by 7 and add it to the result we obtain above, we obtain:

$$\begin{aligned} \cancel{7g} - m &= 10 \\ \underline{\cancel{-7g} - 7m} &\underline{= -42} \\ -8m &= -32 \end{aligned}$$

This implies $m = 4$. We can substitute this value of m into the previous equation to obtain the value of g:

$$\begin{aligned} 7g - (4) &= 10 \\ 7g &= 14 \\ g &= 2 \end{aligned}$$

Finally, we substitute these values for m and g into one of our original equations to determine the value of s:

$$\begin{aligned} 2s + 5(2) + (4) &= 20 \\ 2s + 14 &= 20 \\ 2s &= 6 \\ s &= 3 \end{aligned}$$

Therefore, squash cost \$3 per pound, green beans cost \$2 per pound, and melon costs \$4 per pound.

Example 2

Is the point (-3, 2, 1) a solution to the system below?

$$\begin{aligned} x + y + z &= 0 \\ 4x + 5y + z &= -1 \\ 3x + 2y - 4z &= -8 \end{aligned}$$

Solution:

Check to see if the point satisfies all three equations.

Equation 1: $(-3)+(2)+(1)=-3+2+1=0$

Equation 2: $4(-3)+5(2)+(1)=-12+10+1=-1$

Equation 3: $3(-3)+2(2)-4(1)=-9+4-4=-9\neq-8$

Since the third equation is not satisfied by the point, the point is not a solution to the system.

Example 3

Solve the following system using linear combinations:

$$\begin{aligned}5x-3y+z&=-1\\x+6y-4z&=-17\\8x-y+5z&=12\end{aligned}$$

Solution:

Combine the first and second equations to eliminate z. Then combine the first and third equations to eliminate z.

$$\begin{aligned}4(5x-3y+z=-1)\quad&\Rightarrow\quad 20x-12y+\cancel{4z}=-4\\x+6y-4z=-17\quad&\qquad\quad \underline{x+6y-\cancel{4z}=-17}\\&\qquad\qquad 21x-6y=-21\end{aligned}$$

Result from equations 1 and 2: $21x-6y=-21$

$$\begin{aligned}-5(5x-3y+z=-1)\quad&\Rightarrow\quad -25x+15y-\cancel{5z}=5\\8x-y+5z=12\quad&\qquad\quad \underline{8x-y+\cancel{5z}=12}\\&\qquad\qquad -17x+14y=17\end{aligned}$$

Result from equations 1 and 3: $-17x+14y=17$

Now we have reduced our system to two equations in two variables. We can eliminate y most easily next, and solve for x.

$$\begin{aligned}7(21x-6y=-21)\quad&\Rightarrow\quad 147x-\cancel{42y}=-147\\3(-17x+14y=17)\quad&\qquad\quad \underline{-51x+\cancel{42y}=51}\\&\qquad\qquad 96x=-96\\&\qquad\qquad x=-1\end{aligned}$$

Now find y:

$$\begin{aligned}21(-1)-6y&=-21\\-21-6y&=-21\\-6y&=0\\y&=0\end{aligned}$$

Finally, we can go back to one of the original three equations and use our y and z values to find x.

$$\begin{aligned} 5(-1)-3(0)+z &= -1 \\ -5+z &= -1 \\ z &= 4 \end{aligned}$$

Therefore, the solution is (-1, 0, 4).

Don't forget to check your answer by substituting the point into each equation.

Equation 1: $5(-1)-3(0)+(4)=-5+4=-1$

Equation 2: $(-1)+6(0)-4(4)=-1-16=-17$

Equation 3: $8(-1)-(0)+5(4)=-8+20=12$

Example 4

Solve the following system using elimination:

$$\begin{aligned} 2x+y-z &= 3 \\ x-2y+z &= 5 \\ 6x+3y-3z &= 6 \end{aligned}$$

Solution:

Combine equations 1 and 2 to eliminate z. Then combine equations 2 and 3 to eliminate z.

$$\begin{aligned} 2x+y-\cancel{z} &= 3 \\ \underline{x-2y+\cancel{z}} &\underline{= 5} \\ 3x-y &= 8 \end{aligned}$$

Result from equations 1 and 2: $3x-y=8$

$$\begin{aligned} 3(x-2y+z=5) \quad &\Rightarrow \quad 3x-6y+\cancel{3z}=15 \\ 6x+3y-3z=6 \quad &\qquad \underline{6x+3y-\cancel{3z}=6} \\ &\qquad\qquad 9x-3y=21 \end{aligned}$$

Result from equations 2 and 3: $9x-3y=21$

Now we have reduced our system to two equations in two variables. Now we can combine these two equations and attempt to eliminate another variable.

$$\begin{aligned} -3(3x-y=8) \quad &\Rightarrow \quad -9x+3y=-24 \\ 9x-3y=21 \quad &\qquad \underline{9x-3y=21} \\ &\qquad\qquad 0=-3 \end{aligned}$$

Since the result is a false equation, there is no solution to the system.

Summary

- A minimum of three equations is necessary to solve a system with three variables.
- The elimination method can be used to solve a system of three variables in three equations.
- There can be three possible results when solving systems of three linear equations: one solution, infinitely many solutions, and no solution.

Review

1. Is the point (2, -3, 5) the solution to the system below?

$$\begin{aligned} 2x+5y-z &= -16 \\ 5x-y-3z &= -2 \\ 3x+2y+4z &= 20 \end{aligned}$$

2. Is the point (-1, 3, 8) the solution to the system below?

$$\begin{aligned} 8x+10y-z &= 14 \\ 11x+4y-3z &= -23 \\ 2x+3y+z &= 10 \end{aligned}$$

3. Is the point (0, 3, 5) the solution to the system below?

$$\begin{aligned} 5x-3y+2z &= 1 \\ 7x+2y-z &= 1 \\ x+4y-3z &= -3 \end{aligned}$$

4. Is the point (1, -1, 1) the solution to the system below?

$$\begin{aligned} x-2y+2z &= 5 \\ 6x+y-4z &= 1 \\ 4x-3y+z &= 8 \end{aligned}$$

Solve the systems below in three variables using linear combinations.

5.

$$\begin{aligned} 3x-2y+z &= 0 \\ 4x+y-3z &= -9 \\ 9x-2y+2z &= 20 \end{aligned}$$

6.

$$\begin{aligned}11x+15y+5z&=1\\3x+4y+z&=\text{-}2\\7x+13y+3z&=3\end{aligned}$$

7.

$$\begin{aligned}2x+y+7z&=5\\3x-2y-z&=\text{-}1\\4x-y+3z&=5\end{aligned}$$

8.

$$\begin{aligned}x+3y-4z&=\text{-}3\\2x+5y-3z&=3\\\text{-}x-3y+z&=\text{-}3\end{aligned}$$

9.

$$\begin{aligned}3x-2y-5z&=\text{-}8\\3x+2y+5z&=\text{-}8\\6x+4y-10z&=\text{-}16\end{aligned}$$

10.

$$\begin{aligned}x+2y-z&=\text{-}1\\2x+4y+z&=10\\3x-y+8z&=6\end{aligned}$$

11.

$$\begin{aligned}x+y-z&=\text{-}3\\2x-y-z&=6\\4x+y+z&=0\end{aligned}$$

12.

$$\begin{aligned} 4x+y+3z &= 8 \\ 8x+2y+6z &= 15 \\ 3x-3y-z &= 5 \end{aligned}$$

13.

$$\begin{aligned} 2x+3y-z &= -1 \\ x+2y+3z &= -4 \\ -x+y-2z &= 3 \end{aligned}$$

14.

$$\begin{aligned} x-3y+4z &= 14 \\ -x+2y-5z &= -13 \\ 2x+5y-3z &= -5 \end{aligned}$$

15.

$$\begin{aligned} x+y+z &= 3 \\ x+y-z &= 3 \\ 2x+2y+z &= 6 \end{aligned}$$

16. When Kaitlyn went to the store with $10, she had some choices about what to buy. She could get one apple, one onion, and one basket of blueberries for $9. She could get two apples and two onions for $10. She could also get two onions and one basket of blueberries for $10. Write and solve a system of linear equations.

17. The sum of the digits of a three-digit number is 15. The tens digits is four less than the hundreds digit, and the units digit is five less than the sum of the other two digits. What is the number?

18. John has $42,000 to invest. He invests a portion of this money into a savings account paying 5% interest, a time deposit paying 7%, and a bond paying 9%. He earned $2,600 in annual interest from these three investments. If the earned interest from the savings account was $200 less than the sum of earned interest from the other two investments, how much did John invest in each type of investment?

19. A cashier has 25 coins consisting of nickels, dimes, and quarters with a value of $3.95. If the number of quarters is two less than twice the number of nickels, how many of each type of coin does the cashier have?

20. A theater sells tickets at $6 for adults, $4 for students, and $3 for children under 12 years old. A total of 260 tickets were sold for one showing, with a total revenue of $1,200. If the number of adult tickets sold was three times the number of student tickets sold, how many of each type of ticket were sold for the showing?

Review (Answers)

Please see the Appendix.

10.4 Matrices to Represent Data

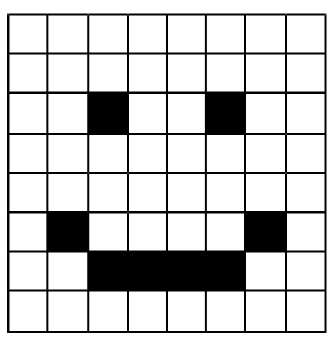
Learning Objectives

Learn what a matrix is and how to use one to represent data.

Introduction

A **matrix** is a rectangular array of numbers representing data in a variety of forms. Computers work very heavily with matrices because operations with matrices are efficient with memory. Matrices can represent statistical data with numbers, but also graphical data with pictures.

How might you use a matrix to write the image below as something a computer could recognize and work with?



Introduction to Matrices

A matrix is a means of storing information effectively and efficiently. The rows and columns each mean something very specific, and the location of a number is just as important as its value. The following are all examples of matrices:

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \begin{bmatrix}2 & 3 & 4 & 5\\0 & 4 & 5 & 0\\0 & 0 & 2 & -9\\0 & 0 & 0 & 10\end{bmatrix}$$

The entries in a matrix can be written out using brackets like [ ], but they can also be described individually using a set of two subscript indices, i and j, which stand for the row number and the column number. Alternatively, the matrix can be named with just a capital letter like A.

$$A = [a_{ij}] = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}$$

Square matrices have the same number of rows as columns. The **order of a matrix**, or the **dimensions of a matrix**, describes the number of rows and the number of columns in the matrix. The matrix below is said to have order 2×3 because it has two rows and three columns. A 1×1 matrix is just a regular number.

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$$

The **identity matrix** of order $n \times n$ has zeros everywhere, except along the main diagonal where it has ones. The identity matrix of any order has special properties. The following are examples of identity matrices of order 1, 2, and 3, respectively:

$$[1], \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

A **symmetric matrix** is a special type of square matrix that has reflection symmetry across the main diagonal. The identity matrix is an example of a symmetric matrix.

When you turn the rows of a matrix into the columns of a new matrix, the two matrices are **transpositions** of one another. The superscript T stands for transpose. Sometimes using the transpose of a matrix is more useful than using the matrix itself.

$$A = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$$

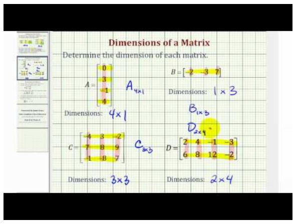
$$A^T = \begin{bmatrix}1 & 4\\2 & 5\\3 & 6\end{bmatrix}$$

A **triangular matrix** is a matrix that has a triangle of zeros within the matrix. A **lower triangular matrix** is a square matrix where every entry below the diagonal is zero. An **upper triangular matrix** is a square matrix where every entry above the diagonal is zero. Shown below is a lower triangular matrix. When you work with solving matrices, look for triangular matrices because they are much easier to solve.

$$\begin{bmatrix}2 & 3 & 4 & 5\\0 & 4 & 5 & 0\\0 & 0 & 2 & -9\\0 & 0 & 0 & 10\end{bmatrix}$$

A **diagonal matrix** is both upper and lower triangular, which means all the entries except those along the diagonal are zero. The identity matrix is a special case of a diagonal matrix.

The following video further explains how to determine the dimensions of a matrix, and why it is important to be able to determine the dimensions of a matrix:



MEDIA

Click image to the left or use the URL below.

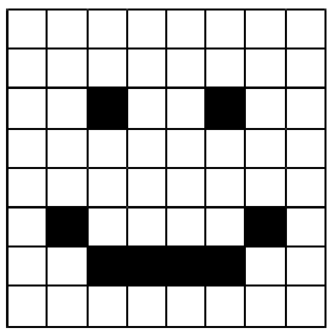
URL: http://www.ck12.org/flx/render/embeddedobject/61418

Play, Learn, and Explore Matrices: www.ck12.org/a/1824089 .

Examples

Example 1

In the Introduction, you were asked how you might use a matrix to write the image below as something a computer could recognize and work with.

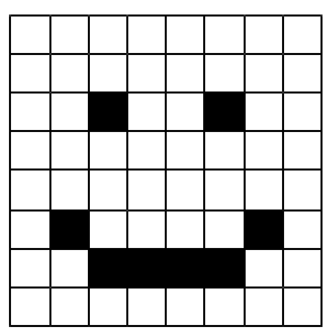


Solution:

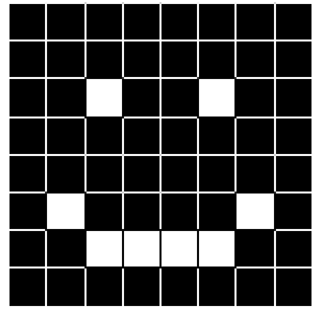
By writing every hollow square as a 0 and every blank square as a 1, a computer could read the picture.

When you use computers to manipulate images, the computer manipulates just the numbers. In this case, if you swap zeros and ones, you get the negative image.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



Real photos and computer images have matrices that are much larger and include more numbers than just zero and one to account for more colors.

Example 2

Kate runs three bakeries, and each bakery sells bagels and muffins.

The rows represent the bakeries, and the columns represent bagels (left) and muffins (right) sold.

$$K = \begin{bmatrix} 144 & 192 \\ 115 & 127 \\ 27 & 34 \end{bmatrix}$$

Answer the questions below about Kate's sales.

1) What does 127 represent?

Solution:

It represents the number of muffins Kate sold in her 2nd location. You know this because it is in the muffin column and the 2nd row.

2) How many muffins did Kate sell in total?

Solution:

The total muffins sold is equal to the sum of the righthand column: $192 + 127 + 34 = 353$.

3) How many bagels did Kate sell at her 1st location?

Solution:

Kate sold 144 bagels at her 1st location.

4) Which location is doing poorly?

Solution:

The 3rd location is doing much worse than the other two locations.

Example 3

Identify the order of the following matrices:

$$A = \begin{bmatrix} 1 & 3 & 4 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 21 & 45 & 1 \\ 34 & 1 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 25 & 235 \\ 562 & 562 \\ 4 & 413 \\ 454 & 33 \\ 1 & 141 \end{bmatrix}$$

Solution:

A is 1×4, B is 2×3, C is 5×2. Note that $4 \times 1, 3 \times 2, 2 \times 5$ are not the same orders and would be incorrect.

Example 4

Write out the 5×4 matrix whose entries are $a_{ij} = \frac{i+j}{j}$.

Solution:

$$\begin{bmatrix} 2 & \frac{3}{2} & \frac{4}{3} & \frac{5}{4} & \frac{6}{5} \\ 3 & 2 & \frac{5}{3} & \frac{3}{2} & \frac{7}{5} \\ 4 & \frac{5}{2} & 2 & \frac{7}{4} & \frac{8}{5} \\ 5 & 3 & \frac{7}{3} & 2 & \frac{9}{5} \end{bmatrix}$$

Example 5

Create a 3×3 matrix for each of the following:

1) Diagonal Matrix

Solution:

Possible answer:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

2) Lower Triangular Matrix

Solution:

Possible answer:

$$\begin{bmatrix} 4 & 1 & 1 \\ 0 & 3 & 14 \\ 0 & 0 & 5 \end{bmatrix}$$

3) Symmetric Matrix

Solution:

Possible answer:

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 14 \\ 1 & 14 & 5 \end{bmatrix}$$

4) Identity Matrix

Solution:

Possible Answer:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: While the identity matrix is technically a correct answer for all four parts of this problem, it does not highlight the differences between each definition.

Summary

- A **matrix** is a rectangular array of numbers representing data.
- **Square matrices** have the same number of rows as columns.
- The **order** or **dimension of a matrix** describes the number of rows and the number of columns in the matrix.
- A **symmetric matrix** is a special type of square matrix that has reflection symmetry across the main diagonal. The identity matrix is an example of a symmetric matrix.
- The **identity matrix** of order $n \times n$ has zeros everywhere, except along the main diagonal where it has ones. Just like the number one has an important property with numbers, the identity matrix of any order has special properties as well.

Review

State the order of each of the following matrices:

1.

$$A = \begin{bmatrix} 4 & 2 & 4 & 7 \\ 5 & 2 & 1 & 0 \end{bmatrix}$$

2.

$$B = \begin{bmatrix} 0 & 1 \\ 34 & 1 \end{bmatrix}$$

3.

$$C = \begin{bmatrix} 2 & 62 \\ 14 & 3 \\ 4 & 3 \\ 1 & 11 \end{bmatrix}$$

4.

$$D = \begin{bmatrix} 12 & 0 & 2 \\ 0 & 3 & 3 \\ 4 & 0 & 1 \\ 1 & 4 & 0 \end{bmatrix}$$

5. $E = \begin{bmatrix} 1 & 11 \end{bmatrix}$
6. Give an example of a 1×1 matrix.
7. Give an example of a 3×2 matrix.
8. If a symmetric matrix is also lower triangular, what type of matrix is it?
9. Write out the 2×3 matrix whose entries are $a_{ij} = i - j$.

Morgan worked for three weeks during the summer, earning money on Mondays, Tuesdays, Wednesdays, Thursdays, and Fridays. The following matrix represents his earnings:

$$\begin{bmatrix} 24 & 22 & 32 \\ 25 & 28 & 30 \\ 30 & 28 & 32 \\ 10 & 15 & 19 \\ 35 & 32 & 30 \end{bmatrix}$$

10. What do the rows and columns represent?
11. How much money did Morgan make in the 1st week?
12. How much money did Morgan make on Tuesdays?
13. What day of the week was most profitable?
14. What day of the week was least profitable?
15. Is the following a matrix? Explain.

$$\begin{bmatrix} \text{dogs} & 0 \\ \text{cats} & 3 \\ \text{sheep} & 0 \\ \text{ducks} & 4 \end{bmatrix}$$

Review (Answers)

Please see the Appendix.

10.5 Matrix Algebra

Learning Objectives

Learn to add, subtract and multiply matrices. As a result you will discover the algebraic properties of matrices.

Introduction

The local movie theater is running a special for the Martin Luther King Jr. holiday weekend. All matinee tickets are \$5, regardless of the age of the customer.

The following matrix shows the number of each type of matinee ticket the movie theater sold over the Martin Luther King Jr. holiday weekend:

Tickets Sold

$$\begin{matrix} \\ \text{Kids} \\ \text{Adults} \\ \text{Seniors} \end{matrix} \; 5 \cdot \begin{matrix} \text{Sat} & \text{Sun} & \text{Mon} \\ \end{matrix} \begin{bmatrix} 122 & 133 & 150 \\ 89 & 75 & 101 \\ 57 & 38 & 49 \end{bmatrix}$$

How much total money did the movie theater take in for the three-day weekend from kids' ticket sales?

Matrix Algebra

Algebra refers to the ability to manipulate variables and unknowns based on rules and properties. Matrix algebra is extremely similar to the algebra you already know for numbers, with a few important differences.

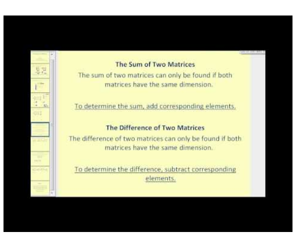
Two matrices of the same order can be added by summing the entries in the corresponding positions:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix}$$

Two matrices of the same order can be subtracted by subtracting the entries in the corresponding positions:

$$\begin{bmatrix} 10 & 9 & 8 \\ 7 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$

The following video further demonstrates how to add, subtract, and perform scalar multiplication with matrices:



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URL: http://www.ck12.org/flx/render/embeddedobject/61448

You can find the product of matrix A and matrix B if the number of columns in matrix A matches the number of rows in matrix B. Another way to remember this is to write the order (number of rows x number of columns) of matrix A and the order (number of rows x number of columns) of matrix B next to each other. You can find the product of the matrices if the inner numbers are the same. The matrix that results from multiplying matrix A and matrix B will have the same number of rows as matrix A, and the same number of columns as matrix B.

$(2\times 3)\cdot(3\times 5)=(2\times 5)$

To compute the 1st entry of the resulting 2×5 matrix, you should match the 1st row from the 1st matrix and the 1st column of the 2nd matrix. The arithmetic operation to combine these numbers is identical to taking the dot product between two vectors.

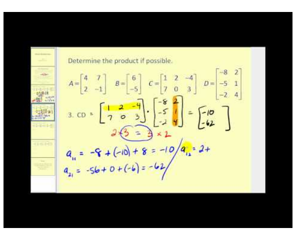
$$\begin{bmatrix} 1 & 4 & 3 \\ 5 & 6 & 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 2 & 0 & 0 & 2 & 1 \\ 1 & 1 & 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & ? & ? & ? & ? \\ 21 & ? & ? & ? & ? \end{bmatrix}$$

- The entry in the 1st row, 1st column of the new matrix is computed as $1\cdot 0+4\cdot 2+3\cdot 1=11$.
- The entry in the 2nd row, 1st column of the new matrix is computed as $5\cdot 0+6\cdot 2+9\cdot 1=21$.
- The rest of the entries of this product are left to Example 1, below.

Properties of Matrix Algebra

- The commutative property holds for matrix addition. This means that when matrices A and B can be added (when they have matching orders), then $A+B=B+A$.
- The commutative property **does not** hold in general for matrix multiplication. For this reason, we need to be careful to distribute without changing the order of the resulting multiplication.
- The associative property does hold for both multiplication and addition. $(AB)C=A(BC),(A+B)+C=A+(B+C)$.
- Distribution over addition and subtraction holds. $A(B\pm C)=AB\pm AC$.

The following video further explains how to multiply matrices:



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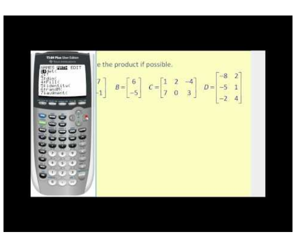
URL: http://www.ck12.org/flx/render/embeddedobject/61450

The following video demonstrates how to multiply matrices on the TI 83/84:



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/61452

Play, Learn, and Explore Matrix Algebra: www.ck12.org/a/2175884 .

Examples

Example 1

Complete the entries of the matrix multiplication introduced in the guidance section, above.

$$\begin{bmatrix} 1 & 4 & 3 \\ 5 & 6 & 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 2 & 0 & 0 & 2 & 1 \\ 1 & 1 & 3 & 0 & 1 \end{bmatrix}$$

Solution:

Two of the arithmetic operations are shown here:

$$c_{12} = 1 \cdot 1 + 4 \cdot 0 + 3 \cdot 1 = 4$$
$$c_{22} = 5 \cdot 1 + 6 \cdot 0 + 9 \cdot 1 = 14$$

When the rest are completed, the result is as follows:

$$C = \begin{bmatrix} 11 & 4 & 12 & 9 & 7 \\ 21 & 14 & 42 & 17 & 15 \end{bmatrix}$$

Example 2

Show the commutative property does not hold by demonstrating $AB \neq BA$ for:

$$A = \begin{bmatrix} 0 & -1 & 8 \\ 1 & 2 & 0 \\ 4 & 3 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 0 \end{bmatrix}$$

Solution:

$$AB = \begin{bmatrix} 30 & 22 & -1 \\ 5 & 9 & 3 \\ 58 & 62 & 7 \end{bmatrix}$$
$$BA = \begin{bmatrix} 9 & 12 & 20 \\ 6 & 5 & 28 \\ 3 & 2 & 32 \end{bmatrix}$$

Example 3

Compute the following matrix arithmetic, $10 \cdot (2A - 3C) \cdot B$, for the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}, C = \begin{bmatrix} 12 & 0 \\ 1 & 3 \end{bmatrix}$$

Solution:

When a matrix is multiplied by a scalar (such as with $2A$), multiply each entry in the matrix by the scalar.

$$2A = \begin{bmatrix} 2 & 4 \\ 8 & 10 \end{bmatrix}$$
$$-3C = \begin{bmatrix} -36 & 0 \\ -3 & -9 \end{bmatrix}$$
$$2A - 3C = \begin{bmatrix} -34 & 4 \\ 5 & 1 \end{bmatrix}$$

Since the associative property holds, you can either distribute the 10 or multiply by matrix B next.

$$(2A - 3C) \cdot B = \begin{bmatrix} 16 & -22 & -60 \\ 4 & 8 & 12 \end{bmatrix}$$
$$10 \cdot (2A - 3C) \cdot B = \begin{bmatrix} 160 & -220 & -600 \\ 40 & 80 & 120 \end{bmatrix}$$

Example 4

Recall the question from the Introduction: What is the total amount of money the movie theater took in from kids' ticket sales for the three-day weekend?

Solution:

This is a matrix multiplication problem where the matrix is multiplied by a scalar.

Tickets Sold

$$\begin{matrix} & \text{Sat} & \text{Sun} & \text{Mon} \\ \text{Kids} & & & \\ \text{Adults } 5\cdot & & & \\ \text{Seniors} & & & \end{matrix} \quad 5 \cdot \begin{bmatrix} 122 & 133 & 150 \\ 89 & 75 & 101 \\ 57 & 38 & 49 \end{bmatrix}$$

	Sat	Sun	Mon
Kids	122	133	150
Adults 5·	89	75	101
Seniors	57	38	49

Use your calculator to assist you. The resulting matrix is the following:

Tickets Sales ($)

	Sat	Sun	Mon
Kids	610	665	750
Adults	445	375	505
Seniors	285	190	245

From this matrix, we can see that the movie theater made \$610 from kids' ticket sales on Saturday, \$665 on Sunday, and \$750 on Monday. Therefore, the total amount made from kids' ticket sales was $\$610 + \$665 + \$750 = \$2,025$.

Example 5

Show that a 3×3 identity matrix works as the multiplicative identity.

Solution:

A 3×3 matrix multiplied by the identity should yield the original matrix.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$a_{11} = a \cdot 1 + b \cdot 0 + c \cdot 0 = a$$

$$a_{12} = a \cdot 0 + b \cdot 1 + c \cdot 0 = b$$

$$\vdots$$

Example 6

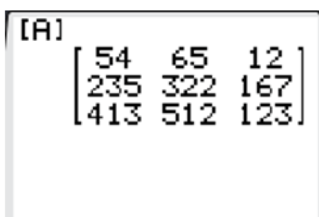
Use your calculator to input and compute the following matrix operations:

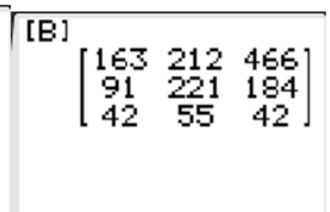
$$A = \begin{bmatrix} 54 & 65 & 12 \\ 235 & 322 & 167 \\ 413 & 512 & 123 \end{bmatrix}, \quad B = \begin{bmatrix} 163 & 212 & 466 \\ 91 & 221 & 184 \\ 42 & 55 & 42 \end{bmatrix}$$

$$A^T \cdot B \cdot A - 100A$$

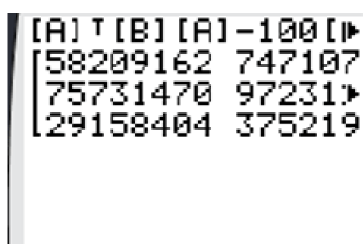
Solution:

Most graphing calculators, such as the TI-84, can do operations on matrices. Find where you can enter matrices and enter the following two matrices:





Then type in the appropriate operation and see the result. The TI-84 has a built-in Transpose button.

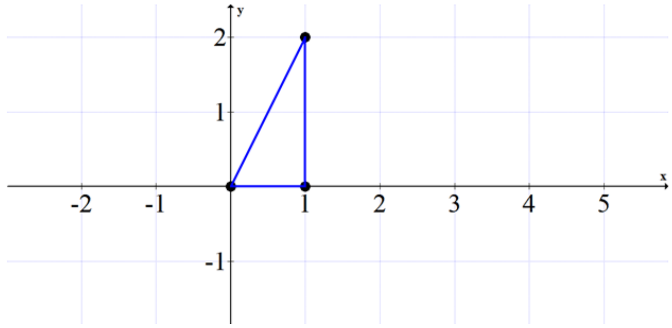


The actual numbers on this practice are less important than knowing that your calculator can perform all the matrix algebra demonstrated in this concept. It is useful to be aware of the capabilities of the tools at your disposal, but this recognition should not replace knowing why the calculator does what it does.

Example 7

Matrix multiplication can be used as a transformation in the coordinate system. Consider the triangle with coordinates (0, 0), (1, 2), and (1, 0), and the following matrix:

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix}$$



What does the new picture look like?

Solution:

The matrix simplifies to become:

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

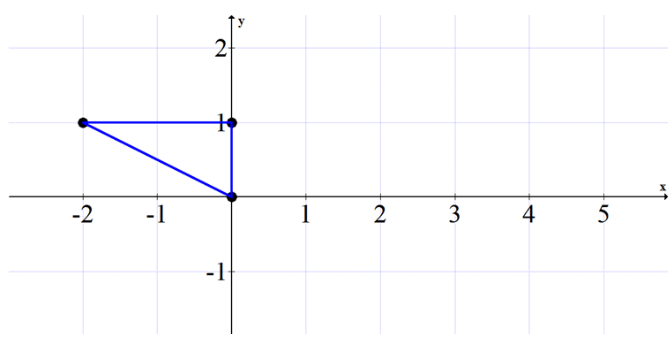
When applied to each point as a transformation, a new point is produced. Note that $\begin{bmatrix} x & y \end{bmatrix}$ is a matrix representing each original point, and $\begin{bmatrix} x' & y' \end{bmatrix}$ is the new point. The x' is read as "x prime," and is a common way to refer to a result after a transformation.

$$\begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} x' & y' \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$



Notice how the matrix transformation rotates graphs in a counterclockwise direction 90°.

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix} = \begin{bmatrix} -y & x \end{bmatrix}$$

The matrix transformation applied in the order below will rotate a graph clockwise 90°.

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$

Summary

- **Matrix operations** are addition, subtraction, and multiplication. Division involves a multiplicative inverse that we will learn about in a future section.
- The commutative property holds for matrix addition. $A+B=B+A$.
- The commutative property **does not** hold in general for matrix multiplication.
- The associative property holds for both multiplication and addition. $(AB)C=A(BC), (A+B)+C=A+(B+C)$.
- Distribution over addition and subtraction holds. $A(B\pm C)=AB\pm AC$.

Review

Attempt numbers 1-11 without your calculator.

$A=\begin{bmatrix} 2 & 7 \\ 3 & 8 \end{bmatrix}, B=\begin{bmatrix} 0 & 5 & 1 \\ 3 & 4 & 6 \end{bmatrix}, C=\begin{bmatrix} 14 & 6 \\ 1 & 2 \end{bmatrix}, D=\begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}$

1. Find AC. If not possible, explain.

2. Find BA. If not possible, explain.

3. Find CA. If not possible, explain.

4. Find $4B^T$. If not possible, explain.

5. Find $A+C$. If not possible, explain.

6. Find $D-A$. If not possible, explain.

7. Find $2(A+C-D)$. If not possible, explain.

8. Find $(A+C)B$. If not possible, explain.

9. Find $B(A+C)$. If not possible, explain.

10. Show that $A(C+D)=AC+AD$.

11. Show that $A(C-D)=AC-AD$.

Practice using your calculator for 12-15.

$E=\begin{bmatrix} 312 & 59 & 34 \\ 342 & 156 & 189 \\ 783 & 23 & 133 \end{bmatrix}, F=\begin{bmatrix} 33 & 72 & 21 \\ 93 & 41 & 94 \\ 62 & 75 & 72 \end{bmatrix}, G=\begin{bmatrix} 11 & 735 & 67 \\ 93 & 456 & 2 \\ 94 & 34 & 0 \end{bmatrix}$

12. Find $E+F+G$.

13. Find $2E$.

14. Find $4F$.

15. Find $(E+F)G$.

16. What are the differences between algebra and matrix algebra?

17. A movie theater tracks the sale of popcorn in three different sizes. The data collected over a weekend (Friday, Saturday, and Sunday nights) is shown in the matrix below. The price of each size is shown in a 2nd matrix. How much revenue did the theater take in each night for popcorn? How much did the theater take in for popcorn in total?

$$\begin{array}{c} \\ \text{Friday} \\ \text{Saturday} \\ \text{Sunday} \end{array} \begin{array}{c} \begin{array}{ccc} S & M & L \end{array} \\ \begin{bmatrix} 36 & 85 & 40 \\ 41 & 112 & 51 \\ 28 & 72 & 35 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ S \\ M \\ L \end{array} \begin{array}{c} \text{Price} \\ \begin{bmatrix} 5.50 \\ 6.25 \\ 7.25 \end{bmatrix} \end{array}$$

Review (Answers)

Please see the Appendix.

10.6 Row Operations and Row Echelon Forms

Learning Objectives

Learn to manipulate matrices using row operations into row echelon form and reduced row echelon form.

Introduction

Applying row operations to reduce a matrix is a procedural skill that takes lots of writing, rewriting, and careful arithmetic. The payoff for being able to transform a matrix into a simplified form will become clear later. For now, what does the simplified form mean for a matrix?

Row Operations

The process of solving a system of linear equations by elimination is similar to the process of solving matrices. Notice the connection between a system of linear expressions and its coefficient matrix.

$$\begin{array}{l} Ax+By+Cz \\ Dx+Ey+Fz \rightarrow \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & J \end{bmatrix} \\ Gx+Hy+Jz \end{array}$$

Thus, the operations that are permitted when solving a system of linear equations by elimination are the same as those permitted to act on matrices. Also, the goal for solving a matrix is the same as the goal of solving a system: solve for the unknown variables.

There are only three operations that are permitted to act on matrices:

1. Add a multiple of one row to another row.
2. Scale a row by multiplying through by a nonzero constant.
3. Swap two rows.

Using these three operations, your job is to simplify matrices into **row echelon form.** Row echelon form must meet three requirements:

1. The leading coefficient of each row must be a one.
2. All entries in a column below a leading one must be zero.
3. All rows that just contain zeros are at the bottom of the matrix.

Here are some examples of matrices in row echelon form:

$$\begin{bmatrix} 1 & 14 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 5 & 6 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced row echelon form also has one extra stipulation compared with row echelon form:

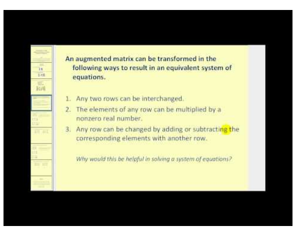
4. Every leading coefficient of one must be the only nonzero element in that column.

Here are some examples of matrices in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Putting a matrix into reduced row echelon form is a result of performing **Gauss-Jordan elimination.** The process illustrated in this concept is named after those mathematicians.

The following video discusses augmented matrices for the purpose of solving systems of equations, and row echelon and reduced row echelon forms of matrices:



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/61454

Examples

Example 1

Put the following matrix into reduced row echelon form:

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

Solution:

In each step of the solution, only one of the three row operations will be used. Specific shorthand will be introduced—namely, R_1 refers to the first row, and R_2 refers to the 2nd row.

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

$$3 \cdot R_2 \rightarrow \begin{bmatrix} 3 & 7 \\ 6 & 15 \end{bmatrix}$$

$$-2 \cdot R_1 + R_2 \rightarrow \begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix}$$

$$-7R_2 + R_1 \rightarrow \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R_1 \cdot \frac{1}{3} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Row reducing a 2×2 matrix to become the identity matrix is an exercise that illustrates the fact that the rows were linearly independent.

Example 2

Put the following matrix into reduced row echelon form:

$$\begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$R_1 \cdot -\frac{1}{2} + R_3 \rightarrow \begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$R_3 \div 4 \rightarrow \begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \div 2,\ R_2 \div 3 \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

Note that in the preceding step, two operations were used. This is acceptable when the operations do not interfere or interact with each other.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \cdot -\frac{1}{3} + R_2 \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \cdot -2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Again, row reducing a 3×3 matrix to become the identity matrix is just an exercise that illustrates the fact that the rows were linearly independent.

Example 3

In a single 3×3 matrix, describe the general approach of Gauss-Jordan elimination. In other words, which locations would you try to focus on first?

Solution:

One approach is to try to get a one in the A position. Then get a zero in position B and position C by multiplying by a multiple of row 1. Then try to get a zero in position D.

$$\begin{bmatrix} A & I & G \\ B & H & F \\ C & D & E \end{bmatrix}$$

Every matrix may have a different strategy, and as long as you use the three row operations, you will be on the right track. One thing to be very careful of is to try to avoid fractions within your matrix. Scale the row to eliminate the fraction.

Example 4

Recall the problem from the Introduction. There are two forms of a matrix that are most simplified. The most important is reduced row echelon form that follows the four stipulations from the guidance section. An example of a matrix in reduced row echelon form is:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 43 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 98 & 5 \end{bmatrix}$$

Example 5

Reduce the following matrix to reduced row echelon form:

$$\begin{bmatrix} 0 & 4 & 5 \\ 2 & 6 & 8 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0 & 4 & 5 \\ 2 & 6 & 8 \end{bmatrix}$$

$$\textbf{Switch Rows} \rightarrow \begin{bmatrix} 2 & 6 & 8 \\ 0 & 4 & 5 \end{bmatrix}$$

$$R_1 \div 2 \rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & 4 & 5 \end{bmatrix}$$

$$R_2 \cdot \frac{1}{4} \rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & \frac{5}{4} \end{bmatrix}$$

$$R_2 \cdot -3 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{5}{4} \end{bmatrix}$$

Example 6

Reduce the following matrix to row echelon form:

$$\begin{bmatrix} 3 & 6 \\ 2 & 4 \\ 5 & 17 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 3 & 6 \\ 2 & 4 \\ 5 & 17 \end{bmatrix}$$

$$R_1 \div 3, R_2 \div 2 \rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 5 & 17 \end{bmatrix}$$

$$R_1 \cdot -1 + R_2, R_1 \cdot -5 + R_3 \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\text{Switch } R_2 \text{ and } R_3 \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 7 \\ 0 & 0 \end{bmatrix}$$

$$R_2 \div 7 \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Example 7

Reduce the following matrix to reduced row echelon form:

$\begin{bmatrix} 3 & 4 & 1 & 0 \\ 5 & -1 & 0 & 1 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 3 & 4 & 1 & 0 \\ 5 & -1 & 0 & 1 \end{bmatrix}$$
$$R_1 \cdot 5, R_2 \cdot 3 \rightarrow \begin{bmatrix} 15 & 20 & 5 & 0 \\ 15 & -3 & 0 & 3 \end{bmatrix}$$
$$R_2 - R_1 \rightarrow \begin{bmatrix} 15 & 20 & 5 & 0 \\ 0 & -23 & -5 & 3 \end{bmatrix}$$
$$R_1 \div 15, R_2 \div -23 \rightarrow \begin{bmatrix} 1 & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{5}{23} & -\frac{3}{23} \end{bmatrix}$$
$$R_2 \cdot -\frac{4}{3} + R_1 \rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{23} & \frac{4}{23} \\ 0 & 1 & \frac{5}{23} & -\frac{3}{23} \end{bmatrix}$$

Summary

- **Row operations** are swapping rows, adding a multiple of one row to another, or scaling a row by multiplying through by a scalar.
- **Row echelon form** is a matrix that has a leading one at the start of every nonzero row, zeros below every leading one, and all rows containing only zeros at the bottom of the matrix.

- **Reduced row echelon form** is the same as row echelon form with one additional stipulation: that every other entry in a column with a leading one must be zero.
- There are only three operations that are permitted to act on matrices:
  - Add a multiple of one row to another row.
  - Scale a row by multiplying through by a nonzero constant.
  - Swap two rows.

Review

1. Give an example of a matrix in row echelon form.

2. Give an example of a matrix in reduced row echelon form.

3. What are the three row operations you are allowed to perform when reducing a matrix?

4. If a square matrix reduces to the identity matrix, what does that mean about the rows of the original matrix?

Use the following matrix for 5-6:

$$A = \begin{bmatrix} -3 & -4 & -12 \\ 4 & 4 & 12 \\ -11 & -12 & -35 \end{bmatrix}$$

5. Reduce matrix A to row echelon form.

6. Reduce matrix A to reduced row echelon form. Are the rows of matrix A linearly independent?

Use the following matrix for 7-8:

$$B = \begin{bmatrix} 3 & -4 & 8 \\ 9 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

7. Reduce matrix B to row echelon form.

8. Reduce matrix B to reduced row echelon form. Are the rows of matrix B linearly independent?

Use the following matrix for 9-10:

$$C = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 3 & 6 & -3 & 1 \\ 6 & 12 & -7 & 0 \end{bmatrix}$$

9. Reduce matrix C to row echelon form.

10. Reduce matrix C to reduced row echelon form. Are the rows of matrix C linearly independent?

Use the following matrix for 11-12:

$$D = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 2 & 3 \end{bmatrix}$$

11. Reduce matrix D to row echelon form.

12. Reduce matrix D to reduced row echelon form. Are the rows of matrix D linearly independent?

Use the following matrix for 13-14:

$$E = \begin{bmatrix} -5 & -6 & -12 \\ -1 & -1 & -2 \\ 2 & 2 & 4 \end{bmatrix}$$

13. Reduce matrix E to row echelon form.

14. Reduce matrix E to reduced row echelon form. Are the rows of matrix E linearly independent?

Use the following matrix for 15-16:

$$F = \begin{bmatrix} -23 & 6 & 3 \\ 2 & -\frac{1}{2} & 0 \\ -8 & 2 & 1 \end{bmatrix}$$

15. Reduce matrix F to row echelon form.

16. Reduce matrix F to reduced row echelon form. Are the rows of matrix F linearly independent?

Review (Answers)

Please see the Appendix.

10.7 Augmented Matrices

Learning Objectives

Learn to solve systems of equations using augmented matrices.

Introduction

The reason why the rules for row reducing matrices are the same as the rules for eliminating coefficients when solving a system of equations is because you are essentially doing the same thing in each case. When you write and rewrite the equation every time, you end up writing down lots of extra information. Matrices take care of this information by embedding it in the location of each entry. How would you use matrices to write the system of equations below?

$$\begin{aligned} 5x + y &= 6 \\ x + y &= 10 \end{aligned}$$

Augmented Matrix

To represent a system as a matrix equation, first write all the equations in standard form so that the coefficients of the variables line up in columns. Then copy down just the coefficients in a matrix array. Next, copy the variables in a variable matrix and the constants into a constant matrix.

$$\begin{aligned} x + y + z &= 9 \\ x + 2y + 3z &= 22 \\ 2x + 3y + 4z &= 31 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ 31 \end{bmatrix}$$

The reason why this works is because of the way matrix multiplication is defined.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 1y + 1z \\ 1x + 2y + 3z \\ 2z + 3y + 4z \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ 31 \end{bmatrix}$$

Notice how putting brackets around the two matrices on the right does very little to hide the fact that this is just a regular system of three equations and three variables.

Once you have your system represented as a matrix, you can solve it using an augmented matrix. An **augmented matrix** is two matrices that are joined together and operated on as if they were a single matrix. In the case of solving a system, you need to augment the coefficient matrix and the constant matrix. The vertical line indicates the separation between the coefficient matrix and the constant matrix.

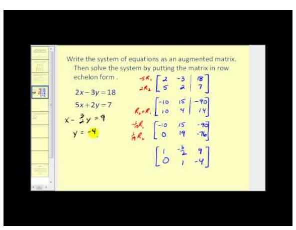
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 2 & 3 & 22 \\ 2 & 3 & 4 & 31 \end{array}\right]$$

$$R_1 \cdot -1 + R_2, R_1 \cdot -2 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 1 & 2 & 13 \end{array}\right]$$

$$R_2 \cdot -1 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduce the matrix to reduced row echelon form and you will find the solution to the system, if one exists.

The following video demonstrates how to transform an augmented matrix to row echelon form to solve a system of equations:



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/61456

Examples

Example 1

Solve the following system using an augmented matrix:

$$\begin{aligned} x+y+z &= 6 \\ x-y-z &= -4 \\ x+2y+3z &= 14 \end{aligned}$$

Solution:

$$\begin{array}{r}
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & -4 \\ 1 & 2 & 3 & 14 \end{array}\right] \\
R_1 \cdot -1 + R_2, R_1 \cdot -1 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & -2 & -10 \\ 0 & 1 & 2 & 8 \end{array}\right] \\
R_3 \cdot -1 + R_1, R_3 \cdot 3 + R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 4 & 14 \\ 0 & 1 & 2 & 8 \end{array}\right] \\
R_2 \cdot -1 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & -2 & -6 \end{array}\right] \\
R_3 \div -2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 1 & 3 \end{array}\right] \\
R_3 + R_1, R_3 \cdot -4 + R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]
\end{array}$$

Every matrix can be interpreted as its own linear system. The final augmented matrix can be interpreted as:

$$\begin{aligned}
1x + 0y + 0z &= 1 \\
0x + 1y + 0z &= 2 \\
0x + 0y + 1z &= 3
\end{aligned}$$

Thus, $x = 1, y = 2,$ and $z = 3$.

Example 2

Attempt to solve the system from the first section.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 2 & 3 & 22 \\ 2 & 3 & 4 & 31 \end{array}\right]$$

Solution:

$$\begin{array}{r}
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 2 & 3 & 22 \\ 2 & 3 & 4 & 31 \end{array}\right] \\
R_1 \cdot -1 + R_2, R_1 \cdot -2 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 1 & 2 & 13 \end{array}\right] \\
R_2 \cdot -1 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right]
\end{array}$$

This system is dependent, so there are infinitely many solutions.

Example 3

Solve the following system using augmented matrices:

$$\begin{aligned}w+x+z&=11\\w+x&=9\\x+y&=7\\y+z&=5\end{aligned}$$

Solution:

While substitution would work in this problem, the idea is to demonstrate how augmented matrices will work even with larger matrices.

$$\left[\begin{array}{cccc|c}1&1&0&1&11\\1&1&0&0&9\\0&1&1&0&7\\0&0&1&1&5\end{array}\right]$$

$$\text{Switch } R_2, R_3 \text{ and } R_4 \rightarrow \left[\begin{array}{cccc|c}1&1&0&1&11\\0&1&1&0&7\\0&0&1&1&5\\1&1&0&0&9\end{array}\right]$$

$$R_1\cdot -1+R_4 \rightarrow \left[\begin{array}{cccc|c}1&1&0&1&11\\0&1&1&0&7\\0&0&1&1&5\\0&0&0&-1&-2\end{array}\right]$$

$$R_4\cdot -1 \rightarrow \left[\begin{array}{cccc|c}1&1&0&1&11\\0&1&1&0&7\\0&0&1&1&5\\0&0&0&1&2\end{array}\right]$$

$$R_4\cdot -1+R_1, R_4\cdot -1+R_3 \rightarrow \left[\begin{array}{cccc|c}1&1&0&0&9\\0&1&1&0&7\\0&0&1&0&3\\0&0&0&1&2\end{array}\right]$$

$$R_3\cdot -1+R_2 \rightarrow \left[\begin{array}{cccc|c}1&1&0&0&9\\0&1&0&0&4\\0&0&1&0&3\\0&0&0&1&2\end{array}\right]$$

$$R_2\cdot -1+R_1 \rightarrow \left[\begin{array}{cccc|c}1&0&0&0&5\\0&1&0&0&4\\0&0&1&0&3\\0&0&0&1&2\end{array}\right]$$

Thus, $w=5, x=4, y=3,$ and $z=2$.

Example 4

Recall the problem from the Introduction. How would you use matrices to write the system of equations below?

$$\begin{aligned}5x+y&=6\\x+y&=10\end{aligned}$$

Solution:

$$\begin{bmatrix}5 & 1\\1 & 1\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}6\\10\end{bmatrix}$$

Example 5

Use an augmented matrix to solve the following system:

$$\begin{aligned}2x+y+z&=16\\2y+6z&=0\\x+y&=10\end{aligned}$$

Solution:

Only the initial and final augmented matrices are shown. The row reduction steps are not shown.

$$\left[\begin{array}{ccc|c}2 & 1 & 1 & 16\\0 & 2 & 6 & 0\\1 & 1 & 0 & 10\end{array}\right]\rightarrow\left[\begin{array}{ccc|c}1 & 0 & 0 & 7\\0 & 1 & 0 & 3\\0 & 0 & 1 & -1\end{array}\right]$$

Example 6

Use an augmented matrix to solve the following system:

$$\begin{aligned}3x+y&=-15\\x+2y&=15\end{aligned}$$

Solution:

$$\left[\begin{array}{cc|c}3 & 1 & -15\\1 & 2 & 15\end{array}\right]\rightarrow\left[\begin{array}{cc|c}1 & 0 & -9\\0 & 1 & 12\end{array}\right]$$

Example 7

Use an augmented matrix to solve the following system:

$$\begin{aligned}-a+b-c&=0\\2a-2b-3c&=25\\3a-4b+3c&=2\end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 2 & -2 & -3 & 25 \\ 3 & -4 & 3 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

Summary

- An **augmented matrix** is a matrix formed when two matrices are joined together and operated on as if they were a single matrix.
- In the case of solving a system, you need to augment the coefficient matrix and the constant matrix.
- The vertical line indicates the separation between the coefficient matrix and the constant matrix.
- Reduce the matrix to reduced row echelon form and you will find the solution to the system, if one exists.

Review

Solve the systems of equations below using augmented matrices. If one solution does not exist, explain why not.

1.

$$\begin{aligned} 4x-2y &= -20 \\ x-3y &= -15 \end{aligned}$$

2.

$$\begin{aligned} 3x+5y &= 33 \\ -x-2y &= -13 \end{aligned}$$

3.

$$\begin{aligned} x+4y &= 11 \\ 3x+12y &= 33 \end{aligned}$$

4.

$$\begin{aligned} -3x+y &= -7 \\ -x+4y &= 5 \end{aligned}$$

5.

$$\begin{aligned} 3x+y &= 6 \\ -6x-2y &= 10 \end{aligned}$$

6.

$$\begin{aligned} 2x - y + z &= 4 \\ 4x + 7y - z &= 38 \\ -x + 3y + 2z &= 23 \end{aligned}$$

7.

$$\begin{aligned} 4x + y - z &= -16 \\ -3x + 4y + z &= 18 \\ x + y - 3z &= -17 \end{aligned}$$

8.

$$\begin{aligned} 3x + 2y - 3z &= 7 \\ -x + 5y + 2z &= 29 \\ x + 2y + z &= 15 \end{aligned}$$

9.

$$\begin{aligned} 2x + y - 2z &= 4 \\ -4x - 2y + 4z &= -8 \\ 3x + y - z &= 5 \end{aligned}$$

10.

$$\begin{aligned} -x + 3y + z &= 11 \\ 3x + y + 2z &= 27 \\ 5x - y - z &= 5 \end{aligned}$$

11.

$$\begin{aligned} 3x + 2y + 4z &= 21 \\ -2x + 3y + z &= -11 \\ x + 2y - 3z &= -3 \end{aligned}$$

12.

$$\begin{aligned} -x + 2y - 6z &= 4 \\ 8x + 5y + 3z &= -8 \\ 2x - 4y + 12z &= 5 \end{aligned}$$

13.

$$\begin{aligned} 3x+5y+8z &= 37 \\ -6x+3y+z &= 42 \\ x+3y-2z &= 5 \end{aligned}$$

14.

$$\begin{aligned} 4x+y-6z &= -38 \\ 2x+7y+8z &= 108 \\ -3x+2y-3z &= -15 \end{aligned}$$

15.

$$\begin{aligned} 6x+3y-2z &= -22 \\ -4x-2y+4z &= 28 \\ 3x+3y+2z &= 7 \end{aligned}$$

Review (Answers)

Please see the Appendix.

10.8 Determinants

Learning Objectives

Learn to find the determinant of a matrix and use determinants to decide whether points are collinear.

Introduction

Suppose your math professor provided you with the coordinates of three points in the coordinate plane, $A = (2,1), B = (5,6)$, and $C = (9,-1)$. Your professor then asked you to determine if the points are collinear (lie on the same line), and, if not, to find the area of the triangle formed by the points. How would you answer your teacher's questions?

Determinant

A determinant is a number computed from the entries in a square matrix. The determinant of a matrix is defined only when a matrix is a square matrix. It has many properties and interpretations that you will explore in linear algebra. This concept is focused on the procedure of calculating determinants. Once you know how to calculate the determinant of a 2×2 matrix, you will be able to calculate the determinant of a 3×3 matrix. Once you know how to calculate the determinant of a 3×3 matrix, you can calculate the determinant of a 4×4, and so on.

The **determinant** of matrix A is written as $|A|$ or $\det A$. For a 2×2 matrix A, the value is calculated as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Notice how the elements on each diagonal are multiplied and then subtracted.

A logical question about determinants is, where does the procedure come from? Why are determinants defined the way they are? Determinants for 2×2 matrices are defined the way they are because of the general solution to a system of 2 variables and 2 equations:

$$ax + by = e$$
$$cx + dy = f.$$

To eliminate the x, scale the first equation by c and the second equation by a.

$$acx + bcy = ec$$
$$acx + ady = af$$

Subtract the second equation by the first, and solve for y.

$$\begin{aligned} ady - bcy &= af - ec \\ y(ad - bc) &= af - ec \\ y &= \frac{af - ec}{ad - bc} \end{aligned}$$

When you solve for x, you also get $ad - bc$ in the denominator of the general solution. This pattern led people to start using this strategy in solving systems of equations. The determinant is defined in this way so it will always be the denominator of the general solution of either variable.

The determinant of a 3×3 matrix is more involved.

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Usually you will start by looking at the top row, although any row or column will work. Then use the checkerboard pattern for signs (shown below) and create smaller 2×2 matrices.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

The smaller 2×2 matrices are the entries that remain when the row and column of the coefficient you are working with are ignored.

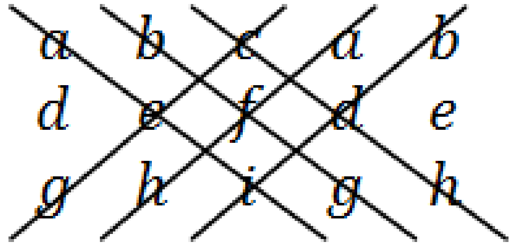
$$\det B = |B| = +a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Next, take the determinant of the smaller 2×2 matrices, and you get a long string of computations.

$$\begin{aligned} \det B &= +a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= aei - afh - bdi + bfg + cdh - ceg \\ &= aei + bfg + cdh - ceg - afh - bdi \end{aligned}$$

Most people do not remember this sequence. A French mathematician named Pierre Frédéric Sarrus demonstrated a great device to memorize the computation of the determinant for 3×3 matrices. The first step is simply to copy the first two columns to the right of the matrix. Then draw three diagonal lines going down and to the right.

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$



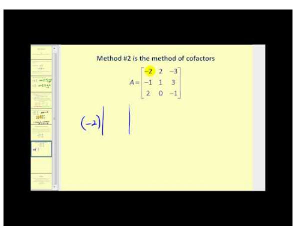
Recall the formula derived above:

$$\det B = aei + bfg + cdh - ceg - afh - bdi$$

These three diagonals run through the variables *aei, bfg,* and *cdh*. These are precisely the first three positive products in the formula.

Next, draw three diagonals going up and to the right. These three diagonals run through the variables *ceg, afh,* and *bdi.* These are precisely the last three negative products in the formula. It is important to note that Sarrus's Rule does not work for the determinants of matrices that are not of order 3×3.

The following video demonstrates how to evaluate 2×2 and 3×3 determinants:



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URL: http://www.ck12.org/flx/render/embeddedobject/61650

Examples

Example 1

Find $\det A$ for the matrix:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$$

Solution:

$$\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} = 3 \cdot 5 - 2 \cdot 1 = 15 - 2 = 13$$

Example 2

Find det B for the matrix:

$$B=\begin{bmatrix}3 & 2 & 1\\ 5 & 0 & 2\\ 2 & 1 & 5\end{bmatrix}$$

Solution:

$$\begin{aligned}\begin{vmatrix}3 & 2 & 1\\ 5 & 0 & 2\\ 2 & 1 & 5\end{vmatrix} &= 3\begin{vmatrix}0 & 2\\ 1 & 5\end{vmatrix}-2\begin{vmatrix}5 & 2\\ 2 & 5\end{vmatrix}+1\begin{vmatrix}5 & 0\\ 2 & 1\end{vmatrix}\\ &= 3(0\cdot 5-2\cdot 1)-2(5\cdot 5-2\cdot 2)+1(5\cdot 1-2\cdot 0)\\ &= -6-42+5\\ &= -43\end{aligned}$$

Example 3

Find the determinant of B from Example 2 using Sarrus's Rule.

Solution:

$$\begin{matrix}3 & 2 & 1 & 3 & 2\\ 5 & 0 & 2 & 5 & 0\\ 2 & 1 & 5 & 2 & 1\end{matrix}$$

$$\textbf{det } B = 0+8+5-0-6-50=-43$$

As you can see, Sarrus's Rule is efficient, and much of the calculations can be done mentally. Additionally, zero values make much of the multiplication easier.

Example 4

Recall the problem from the Introduction. Are the points $A=(2,1), B=(5,6),$ and $C=(9,-1)$ collinear (lie on the same line)? If not, what is the area of the triangle formed by the points?

Solution:

Finding the determinant of the matrix formed by the points could help you answer both questions.

Three points are collinear if and only if the determinant of the matrix with x-coordinates in the first column, y-coordinates in the second column, and ones in the third column is equal to zero.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 6 & 1 \\ 9 & -1 & 1 \end{bmatrix}$$

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & 1 & 1 \\ 5 & 6 & 1 \\ 9 & -1 & 1 \end{vmatrix} \\ &= 2\begin{vmatrix} 6 & 1 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 5 & 1 \\ 9 & 1 \end{vmatrix} + 1\begin{vmatrix} 5 & 6 \\ 9 & -1 \end{vmatrix} \\ &= 2(6 \cdot 1 - 1 \cdot -1) - 1(5 \cdot 1 - 1 \cdot 9) + 1(5 \cdot -1 - 6 \cdot 9) \\ &= 14 + 4 - 59 \\ &= -41 \end{aligned}$$

Since the det $A \neq 0$, the given points are not collinear.

The area of the triangle formed by three points is half the absolute value of the determinant of this matrix.

$$\begin{aligned} \text{Area} &= \frac{1}{2} \cdot |\det A| \\ &= \frac{1}{2} \cdot |-41| \\ &= \frac{41}{2} \\ &= 20.5 \end{aligned}$$

Thus, the area of the triangle formed by the three given points is 20.5.

Example 5

1) Find the determinant of the following matrix:

$$C = \begin{bmatrix} -4 & 12 \\ 1 & -3 \end{bmatrix}$$

Solution:

$$\det C = \begin{vmatrix} -4 & 12 \\ 1 & -3 \end{vmatrix} = 12 - 12 = 0$$

2) Find the determinant of the following matrix:

$$D = \begin{bmatrix} 4 & 8 & 3 \\ 0 & 1 & 7 \\ 12 & 5 & 13 \end{bmatrix}$$

Solution:

$$\det D = \begin{vmatrix} 4 & 8 & 3 \\ 0 & 1 & 7 \\ 12 & 5 & 13 \end{vmatrix} = 4\cdot 13 + 8\cdot 7\cdot 12 + 0 - 3\cdot 12 - 5\cdot 7\cdot 4 - 0 = 548$$

3) Find the determinant of the following 4×4 matrix by carefully choosing the row or column to work with:

$$E = \begin{bmatrix} 4 & 5 & 0 & 2 \\ -1 & -3 & 0 & 3 \\ 4 & 8 & 1 & 5 \\ -3 & 2 & 0 & 9 \end{bmatrix}$$

Solution:

Notice that the third column is made up with zeros and a one. Choose this column to make up the coefficients, because then instead of having to evaluate the determinant of four individual 3×3 matrices, you need to do only one.

$$\begin{aligned} |E| &= \begin{vmatrix} 4 & 5 & 0 & 2 \\ -1 & -3 & 0 & 3 \\ 4 & 8 & 1 & 5 \\ -3 & 2 & 0 & 9 \end{vmatrix} \\ &= 0\cdot \begin{vmatrix} -1 & -3 & 3 \\ 4 & 8 & 5 \\ -3 & 2 & 9 \end{vmatrix} - 0\cdot \begin{vmatrix} 4 & 5 & 2 \\ 4 & 8 & 5 \\ -3 & 2 & 9 \end{vmatrix} + 1\cdot \begin{vmatrix} 4 & 5 & 2 \\ -1 & -3 & 3 \\ -3 & 2 & 9 \end{vmatrix} - 0\cdot \begin{vmatrix} 4 & 5 & 2 \\ -1 & -3 & 3 \\ 4 & 8 & 5 \end{vmatrix} \\ &= \begin{vmatrix} 4 & 5 & 2 \\ -1 & -3 & 3 \\ -3 & 2 & 9 \end{vmatrix} \\ &= 4\cdot(-3)\cdot 9 + 5\cdot 3\cdot(-3) + 2\cdot(-1)\cdot 2 - 2\cdot(-3)\cdot(-3) - 4\cdot 3\cdot 2 - 5\cdot(-1)\cdot 9 \\ &= -154 \end{aligned}$$

Summary

- The **determinant** of a matrix is a number calculated from the entries in a matrix. The procedure is derived from solving linear systems.
- The determinant of matrix A can be expressed as $\det A$ or $|A|$.
- **Sarrus's Rule** is a memorization technique that enables you to compute the determinant of 3×3 matrices efficiently.

Review

Determine the determinants of each of the following matrices:

1. $\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$
2. $\begin{bmatrix} -3 & 6 \\ 2 & 5 \end{bmatrix}$
3. $\begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 6 & 5 \\ 2 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 \\ 6 & 3 \end{bmatrix}$

7. $\begin{bmatrix} -1 & 3 & -4 \\ 4 & 2 & 1 \\ 1 & 2 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 4 & 5 & 8 \\ 9 & 0 & 1 \\ 0 & 3 & -2 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 7 & -1 \\ 2 & -3 & 1 \\ 6 & 8 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 2 & -3 \\ 2 & 4 & 5 \\ 1 & 8 & 0 \end{bmatrix}$

11. $\begin{bmatrix} -2 & -6 & -12 \\ -1 & -5 & -2 \\ 2 & 3 & 4 \end{bmatrix}$

12. $\begin{bmatrix} -2 & 6 & 3 \\ 2 & 4 & 0 \\ -8 & 2 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 6 & 4 & 6 \\ 0 & 1 & 0 & 1 \\ 2 & 4 & 2 & 0 \\ -6 & 2 & 3 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 5 & 0 & 0 & 1 \\ 2 & 1 & 8 & 3 \\ 9 & 3 & 2 & 6 \\ -4 & 2 & 5 & 1 \end{bmatrix}$

15. Can you find the determinant for any matrix? Explain.

16. The following matrix has a determinant of zero: $\begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$. If the determinant of a matrix is zero, what does that say about the rows of the matrix?

Determine if the given points below are collinear. If not, then determine the area of each triangle with vertices given below.

17. (2, -1), (-5, 2), and (0, 6)

18. (-8, 12), (10, 5), and (1, -4)

19. (-7, 2), (8, 0), and (3, -4)

Review (Answers)

Please see the Appendix.

10.9 Cramer's Rule

Learning Objectives

Learn to solve systems of equations using Cramer's Rule.

Introduction

Suppose Jane and John both have the same cell phone provider. They've both just received their monthly bill. Assume Jane and John both know the total amount of their bill, how many text messages they sent, and how many minutes they talked. They could create a system of equations to solve for the amount that the cell phone provider charges per text and per minute. Previously, we've used elimination to solve systems of equations. In this section we'll learn a new method called Cramer's Rule for solving systems of equations. Is Cramer's Rule the most efficient means of solving a system of equations?

Cramer's Rule

Examining the general solution for a 2×2 matrix helps us understand the definition of the determinant for a 2×2 matrix.

$$ax + by = e$$
$$cx + dy = f$$

When you solve the system above for y and x, you get the following:

$$y = \frac{af - ce}{ad - bc}$$
$$x = \frac{ed - fb}{ad - bc}$$

A system of equations can be represented a matrix, and the solutions can be written as ratios of two determinants as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$
$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$
$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

The denominator of both the solution for x and the solution for y is of the determinant of the coefficient matrix. Note that if the determinant of the coefficient matrix is 0, then unique solutions for the variables cannot be found. The numerator of the x solution is the determinant of a matrix formed with the y coefficients in column 2, and the solution coefficients in column 1. The numerator of the y solution is the determinant of a matrix formed with the x coefficients in column 1, and the solution coefficients in column 2.

Cramer's Rule also works with larger order matrices. For a system of three variables and three equations, the reasoning is identical.

$$\begin{aligned} ax+by+cz &= j \\ dx+ey+fz &= k \\ gx+hy+iz &= l \end{aligned}$$

The system can be represented as a matrix.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} j \\ k \\ l \end{bmatrix}$$

The three solutions can be represented as a ratio of determinants.

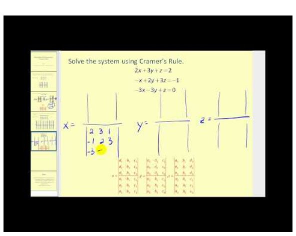
$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

Remember that evaluating the determinants of 3×3 matrices using Sarrus's Rule is very efficient.

The following video demonstrates how to solve a system of equations using Cramer's Rule:



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/61652

Play, Learn, and Explore Cramer's Rule: www.ck12.org/a/2054556 .

Examples

Example 1

Represent the system of equations below as a matrix equation, and use Cramer's Rule to solve for x and y.

$$\begin{aligned} y-13 &= -3x \\ x &= 19-4y \end{aligned}$$

Solution:

First write each equation in standard form.

$$\begin{aligned} 3x+y &= 13 \\ x+4y &= 19 \end{aligned}$$

Then write as a coefficient matrix times a variable matrix equal to a solution matrix.

$$\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 19 \end{bmatrix}$$

Now use Cramer's Rule.

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{\begin{vmatrix} 13 & 1 \\ 19 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}} = \frac{13\cdot 4 - 19\cdot 1}{3\cdot 4 - 1\cdot 1} = \frac{33}{11} = 3$$

$$y = \frac{\begin{vmatrix} 3 & 13 \\ 1 & 19 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}} = \frac{3\cdot 19 - 13}{11} = \frac{44}{11} = 4$$

Example 2

What is y equal to in the following system?

$$\begin{aligned} x+2y-z &= 0 \\ 7x-0y+z &= 14 \\ 0x+y+z &= 10 \end{aligned}$$

Solution:

If you attempted to solve this using elimination, it would take over a page of writing and rewriting to solve. Cramer's Rule speeds up the solving process.

$$\begin{bmatrix} 1 & 2 & -1 \\ 7 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ 10 \end{bmatrix}$$

$$y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 0 & -1 \\ 7 & 14 & 1 \\ 0 & 10 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 7 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}} = \frac{14+0+(-70)-0-10-0}{0+0+(-7)-0-1-14} = \frac{-66}{-22} = 3$$

Example 3

Recall the question from the Introduction: Is Cramer's Rule the most efficient means of solving a system of equations? Example 2 reminds you that a problem done with traditional coefficient elimination can take over a page of writing and rewriting. Efficiency partly means requiring less time and space. If this was all that efficiency meant, then it would not make sense to solve systems of two equations with two unknowns using matrices, because the solution could be found more quickly using substitution. However, the other part of efficiency is minimizing the number of decisions that have to be made. A computer is very good at adding, subtracting, and multiplying numbers, but not very good at deciding whether eliminating x or eliminating y would be better. This is why a definite algorithm using matrices and Cramer's Rule is more efficient.

Example 4

1) Solve the following system using Cramer's Rule:

$$\begin{aligned} 5x + 12y &= 72 \\ 18x - 12y &= 108 \end{aligned}$$

Solution:

$$x = \frac{\begin{vmatrix} 72 & 12 \\ 108 & -12 \end{vmatrix}}{\begin{vmatrix} 5 & 12 \\ 18 & -12 \end{vmatrix}} = \frac{72 \cdot (-12) - 12 \cdot 108}{5 \cdot (-12) - 12 \cdot 18} = \frac{-2160}{-276} = \frac{180}{23}$$

$$y = \frac{\begin{vmatrix} 5 & 72 \\ 18 & 108 \end{vmatrix}}{\begin{vmatrix} 5 & 12 \\ 18 & -12 \end{vmatrix}} = \frac{5 \cdot 108 - 72 \cdot 18}{-276} = \frac{-756}{-276} = \frac{63}{23}$$

2) Solve the following system using Cramer's Rule and your calculator:

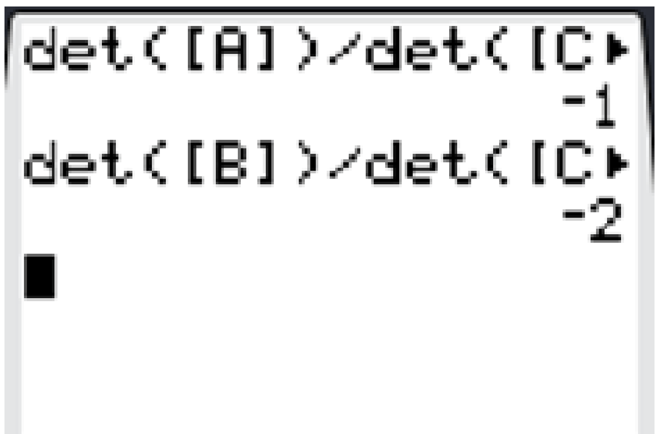
$$\begin{aligned} 70x + 21y &= -112 \\ 27x - 21y &= 15 \end{aligned}$$

Solution:

Input the three matrices below into your calculator. Matrix A has columns that are the constants and the y coefficients. Matrix B has columns that are x coefficients and the constants. Matrix C is just the coefficient matrix.

$$A = \begin{bmatrix} -112 & 21 \\ 15 & -21 \end{bmatrix}$$
$$B = \begin{bmatrix} 70 & -112 \\ 27 & 15 \end{bmatrix}$$
$$C = \begin{bmatrix} 70 & 21 \\ 27 & -21 \end{bmatrix}$$

Then compute $x = \frac{\det A}{\det C}$ and $y = \frac{\det B}{\det C}$.



Thus, the solution is $x = -1$ and $y = -2$.

3) What is the value of z in the system below?

$$3x + 2y + z = 7$$
$$4x + 0y + z = 6$$
$$6x - y + 0z = 5$$

Solution:

$$z = \frac{\begin{vmatrix} 3 & 2 & 7 \\ 4 & 0 & 6 \\ 6 & -1 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & 1 \\ 4 & 0 & 1 \\ 6 & -1 & 0 \end{vmatrix}}$$
$$= \frac{0 + 2 \cdot 6 \cdot 6 + 7 \cdot 4 \cdot (-1) - 0 - (-1) \cdot 6 \cdot 3 - 5 \cdot 4 \cdot 2}{0 + 2 \cdot 1 \cdot 6 + 1 \cdot 4 \cdot (-1) - 0 - (-1) \cdot 1 \cdot 3 - 0}$$
$$= \frac{22}{11}$$
$$= 2$$

Summary

- A **matrix equation** represents a system of equations by multiplying a coefficient matrix and a variable matrix to get a solution matrix.
- **Cramer's Rule** is a method that uses determinants to solve a system of equations.
- Cramer's Rule can be more efficient than the elimination method for solving a system of equations.
- Cramer's Rule can be used only when the determinant of the coefficient matrix is nonzero.

Review

Solve the systems of equations below using Cramer's Rule. If one solution does not exist, explain.

1.

$$\begin{aligned} 4x - 2y &= -20 \\ x - 3y &= -15 \end{aligned}$$

2.

$$\begin{aligned} 3x + 5y &= 33 \\ -x - 2y &= -13 \end{aligned}$$

3.

$$\begin{aligned} x + 4y &= 11 \\ 3x + 12y &= 33 \end{aligned}$$

4.

$$\begin{aligned} -3x + y &= -7 \\ -x + 4y &= 5 \end{aligned}$$

5.

$$\begin{aligned} 3x + y &= 6 \\ -6x - 2y &= 10 \end{aligned}$$

6. Use Cramer's Rule to solve for x in the following system:

$$\begin{aligned} 2x - y + z &= 4 \\ 4x + 7y - z &= 38 \\ -x + 3y + 2z &= 23 \end{aligned}$$

7. Use Cramer's Rule to solve for y in the following system:

$$\begin{aligned} 4x+y-z &= -16 \\ -3x+4y+z &= 18 \\ x+y-3z &= -17 \end{aligned}$$

8. Use Cramer's Rule to solve for z in the following system:

$$\begin{aligned} 3x+2y-3z &= 7 \\ -x+5y+2z &= 29 \\ x+2y+z &= 15 \end{aligned}$$

9. Use Cramer's Rule to solve for x in the following system:

$$\begin{aligned} 2x+y-2z &= -5 \\ -4x-2y+3z &= 2 \\ 3x+y-z &= 3 \end{aligned}$$

10. Use Cramer's Rule to solve for y in the following system:

$$\begin{aligned} -x+3y+z &= 11 \\ 3x+y+2z &= 27 \\ 5x-y-z &= 5 \end{aligned}$$

11. Use Cramer's Rule to solve for z in the following system:

$$\begin{aligned} 3x+2y+4z &= 21 \\ -2x+3y+z &= -11 \\ x+2y-3z &= -3 \end{aligned}$$

Solve the systems of equations below using Cramer's Rule. Practice using your calculator to help with at least one problem. If one solution does not exist, explain.

12.

$$\begin{aligned} -x+2y-6z &= 4 \\ 8x+5y+3z &= -8 \\ 2x-4y+12z &= 5 \end{aligned}$$

13.

$$\begin{aligned} 3x+5y+8z&=37\\ -6x+3y+z&=42\\ x+3y-2z&=5 \end{aligned}$$

14.

$$\begin{aligned} 4x+y-6z&=-38\\ 2x+7y+8z&=108\\ -3x+2y-3z&=-15 \end{aligned}$$

15.

$$\begin{aligned} 6x+3y-2z&=-22\\ -4x-2y+4z&=28\\ 3x+3y+2z&=7 \end{aligned}$$

16. When using Cramer's Rule to solve a system of equations, you will occasionally find that the determinant of the coefficient matrix is zero. When this happens, how can you tell whether your system has no solution or infinite solutions?

17. At the university book store, paperback books cost one price and hardcover book cost another price. You buy 3 paperback and 2 hardcover books for a total cost of $54. Your friend buys 2 paperback and 4 hardcover books for a total cost of $76. Use Cramer's Rule to determine the price of each type of book.

18. The Smith and Jamison families go to the county fair. The Smiths purchase 6 corndogs and 3 cotton candies for $21.75. The Jamisons purchase 3 corndogs and 4 cotton candies for $15.25. Use Cramer's Rule to determine the price of each type of food.

Review (Answers)

Please see the Appendix.

10.10 Inverse Matrices

Learning Objectives

Learn how to find the inverse of a matrix and how to solve a system of equations using an inverse matrix.

Introduction

Cryptography, the study of encoding and decoding important information, is used today to protect data like credit card numbers, computer passwords, and electronic transactions. Before modern times, cryptography was primarily used as a means of encrypting messages that needed to remain secret.

The invention of computers has made breaking codes incredibly easy, so encryption and cryptography have become increasingly complex. One method of more advanced cryptography is the use of a matrix for encoding a message. By assigning each letter of the alphabet a value between 1 and 26, the message to be passed is translated to a series of numbers, with the zero being used as a space between words. The series of numbers is converted into a matrix with any extra spaces filled in with zeros. The matrix is then multiplied by an agreed-upon encoding matrix, and the coded matrix is sent. To decrypt the original message, the recipient needs to find the inverse of the encoded matrix and work backwards to decipher the message.

Two numbers are multiplicative inverses if their product is 1, which is the multiplicative identity for the set of real numbers. Every number besides the number 0 has a multiplicative inverse. For matrices, two matrices are inverses of each other if they multiply to be the identity matrix.

What kinds of matrices do not have inverses?

Inverting Matrices

Consider a matrix A that has **inverse** A^{-1}. How do you find matrix A^{-1} if you just have matrix A?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, A^{-1} = ?$$

We can find A^{-1} by creating an augmented matrix of matrix A and the identity matrix seen below.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

The next step is to perform row reduction. The row reduction of this matrix can be found in Example 1 below. The right part of the augmented matrix is the inverse matrix A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & 0 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Fractions are often unavoidable when computing inverses.

One reason why inverses are so powerful is that they allow you to solve systems of equations with the same logic as you would solve a single linear equation. Consider the following system based on the coefficients of matrix A from above:

$$x + 2y + 3z = 96$$
$$x + 0y + z = 36$$
$$0x + 2y - z = -12$$

By writing this system as a matrix equation, you get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

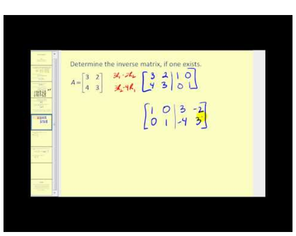
If this were a normal linear equation where you had a constant times the variable equals a constant, you would multiply both sides by the multiplicative inverse of the coefficient. Do the same in this case.

$$A^{-1} \cdot A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

All that is left is for you to perform the matrix multiplication to get the solution. See Example 2.

The following video demonstrates how to determine the inverse of a matrix using augmented matrices.



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Examples

Example 1

Show the steps for finding the inverse matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix},\ A^{-1} = ?$$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \cdot -1 + R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 + R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 0 & -3 & -1 & 1 & 1 \end{array}\right]$$

$$R_2 \div -2, R_3 \div -3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

$$R_3 \cdot -3 + R_1, R_3 \cdot -1 + R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

$$R_2 \cdot -2 + R_1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

The matrix on the right is the inverse matrix A^{-1}.

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Example 2

Solve the following system of equations using inverse matrices:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

Solution:

Note: Use the inverse matrix you found in Example 1 since A is the same.

$$A\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}96\\36\\-12\end{bmatrix}$$
$$A^{-1}\cdot A\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=A^{-1}\cdot\begin{bmatrix}96\\36\\-12\end{bmatrix}$$
$$\begin{bmatrix}x\\y\\z\end{bmatrix}=A^{-1}\cdot\begin{bmatrix}96\\36\\-12\end{bmatrix}$$
$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-\frac{1}{3}&\frac{4}{3}&\frac{1}{3}\\\frac{1}{6}&-\frac{1}{6}&\frac{1}{3}\\\frac{1}{3}&-\frac{1}{3}&-\frac{1}{3}\end{bmatrix}\cdot\begin{bmatrix}96\\36\\-12\end{bmatrix}$$
$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-\frac{1}{3}\cdot 96+\frac{4}{3}\cdot 36+\frac{1}{3}\cdot(-12)\\\frac{1}{6}\cdot 96-\frac{1}{6}\cdot 36+\frac{1}{3}\cdot(-12)\\\frac{1}{3}\cdot 96-\frac{1}{3}\cdot 36-\frac{1}{3}\cdot(-12)\end{bmatrix}$$
$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}12\\6\\24\end{bmatrix}$$

Example 3

Find the inverse of the following matrix:

$$\begin{bmatrix}1&6\\4&24\end{bmatrix}$$

Solution:

$$\left[\begin{array}{cc|cc}1&6&1&0\\4&24&0&1\end{array}\right]$$
$$R_1\cdot -4+R_2\rightarrow\left[\begin{array}{cc|cc}1&6&1&0\\0&0&-4&1\end{array}\right]$$

This matrix is not invertible because its rows are not linearly independent. To test to see if a square matrix is invertible, check whether or not the determinant is zero. If the determinant is zero, then the matrix is not invertible because the rows are not linearly independent.

Example 4

Recall the question from the Introduction: What kinds of matrices do not have inverses?

Solution:

Non-square matrices do not have inverses. Square matrices that have determinants equal to zero do not have inverses.

Example 5

1) Confirm matrix A and A^{-1} are inverses by computing $A^{-1}\cdot A$ and $A\cdot A^{-1}$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Solution:

$$A^{-1} \cdot A = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} =$$

$$a_{11} = -\frac{1}{3} \cdot 1 + \frac{4}{3} \cdot 1 + \frac{1}{3} \cdot 0 = 1$$

$$a_{22} = \frac{1}{6} \cdot 2 - \frac{1}{6} \cdot 0 + \frac{1}{3} \cdot 2 = 1$$

$$a_{33} = \frac{1}{3} \cdot 3 - \frac{1}{3} \cdot 1 - \frac{1}{3}(-1) = 1$$

Note that the rest of the entries turn out to be zero. This is left for you to confirm.

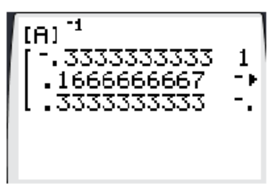
2) Use a calculator to compute A^{-1}, compute $A^{-1} \cdot A$, compute $A \cdot A^{-1}$ and compute

$$A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

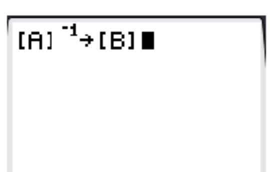
Solution:

Start by entering just matrix A into the calculator.

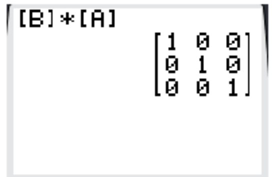
To compute matrix A^{-1}, use the Inverse button programmed into the calculator. Do not try to raise the matrix to the negative one exponent. This will not work.



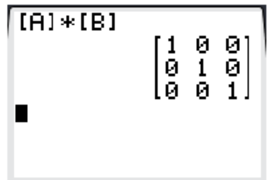
Note that the calculator may return decimal versions of the fractions, and will not show the entire matrix on its limited display. You will have to scroll to the right to confirm that A^{-1} matches what you have already found. Once you have found A^{-1}, go ahead and store it as matrix B so you do not need to type in the entries.



$A^{-1} \cdot A = B \cdot A$

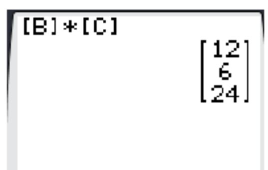


$A \cdot A^{-1} = A \cdot B$



$$A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} = B \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} = B \cdot C$$

You need to create matrix $C = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$



Using a calculator effectively should improve your understanding of matrices and allow you to check all the work you do by hand.

3) The identity matrix happens to be its own inverse. Find another matrix that is its own inverse.

Solution:

Mathematician Friedrich Helmert came up with a very clever matrix that happens to be its own inverse. Here are the 2×2 and the 3×3 versions:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

Summary

- **Multiplicative inverses** are two numbers or matrices whose product is one or the identity matrix.
- Non-square matrices do not have inverses.
- Square matrices that have determinants equal to zero do not have inverses.

Review

Find the inverse of each of the matrices below, if possible. Make sure to do some by hand and some with your calculator.

1. $\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$

2. $\begin{bmatrix} -3 & 6 \\ 2 & 5 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 6 & 5 \\ 2 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$

7. $\begin{bmatrix} -1 & 3 & -4 \\ 4 & 2 & 1 \\ 1 & 2 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 4 & 5 & 8 \\ 9 & 0 & 1 \\ 0 & 3 & -2 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 7 & -1 \\ 2 & -3 & 1 \\ 6 & 8 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 2 & -3 \\ 2 & 4 & 5 \\ 1 & 8 & 0 \end{bmatrix}$

11. $\begin{bmatrix} -2 & -6 & -12 \\ -1 & -5 & -2 \\ 2 & 3 & 4 \end{bmatrix}$

12. $\begin{bmatrix} -2 & 6 & 3 \\ 2 & 4 & 0 \\ -8 & 2 & 1 \end{bmatrix}$

13. Show that Helmert's 2×2 matrix is its own inverse: $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

14. Show that Helmert's 3×3 matrix is its own inverse: $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$.

15. Non-square matrices sometimes have left inverses, where $A^{-1} \cdot A = I$, or right inverses, where $A \cdot A^{-1} = I$. Why can't non-square matrices have regular inverses?

Review (Answers)

Please see the Appendix.

10.11 Partial Fraction Decomposition

Learning Objectives

Learn to apply what you know about systems and matrices to decompose rational expressions into the sum of several partial fractions.

Introduction

Suppose you are painting a very large mural on the side of a building. You need to calculate the area under a curve in the mural, so you know how much paint to buy. In calculus, you'll learn how to use a method called "integration" to determine the area under a curve. If the curve is very complex, breaking it into parts will simplify the integration process. Partial fractions are used to break rational equations into parts.

When given a rational expression such as $\frac{4x-9}{x^2-3x}$, we can break it into the sum of two simpler fractions. The challenging part is trying to get from the initial rational expression to the simpler fractions. You may know how to add fractions and go from two or more separate fractions to a single fraction, but how do you reverse this process?

Partial Fractions

Partial fraction decomposition, also referred to as **partial fraction expansion,** is a procedure that reverses adding fractions with unlike denominators. The most challenging part is coming up with the denominators of each individual partial fraction. See if you can spot the pattern.

$$\frac{6x-1}{x^2(x-1)(x^2+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+2}$$

Notice how, in this example, each individual factor was represented. When performing partial fraction decomposition, linear factors that are raised to a power greater than one must have each successive power included as a separate denominator. These factors are called **repeated factors.** When the denominator of the rational expression has a repeated factor, a factor for each power needs to be included in the partial fraction expansion. For example,

$$\frac{1}{(x-2)^3} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}.$$

Also from the initial example, quadratic terms that do not factor to be linear terms are included with a numerator that is a linear function of x. For example,

$$\frac{1}{x(x^2+6)} = \frac{A}{x} + \frac{Bx+C}{x^2+6}.$$

In general, a rational function $\frac{P(x)}{Q(x)}$ can be rewritten using what is known as partial fraction decomposition. This procedure often allows integration to be performed on each term separately by inspection. For each factor of $Q(x)$, the form $(ax+b)^m$ introduces terms

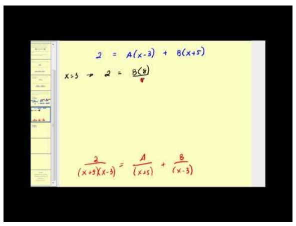
$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_m}{(ax+b)^m}.$$

Then write

$$\frac{P(x)}{Q(x)}=\frac{A_1}{ax+b}+\ldots+\frac{A_2x+B_2}{ax^2+bx+c}+\ldots$$

and solve for all A_i and B_i.

The following video demonstrates how to perform partial fraction decomposition:



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/189953

Examples

Example 1

Add the partial fractions to show both sides are equal.

$$\frac{2x+4}{(x-1)(x+3)}=\frac{1}{x+1}+\frac{1}{x+3}$$

Solution:

$$\frac{1}{x+1}+\frac{1}{x+3}=\frac{x+3}{(x+1)(x+3)}+\frac{x+1}{(x+1)(x+3)}=\frac{2x+4}{(x+1)(x+3)}$$

Example 2

Recall the problem from the Introduction: What is the partial fraction decomposition of the given rational expression?

$$\frac{4x-9}{x^2-3x}$$

Solution:

To decompose the rational expression into the sum of two simpler fractions, you need to use partial fraction decomposition.

$$\frac{4x-9}{x^2-3x}=\frac{A}{x}+\frac{B}{x-3}$$
$$4x-9=A(x-3)+Bx$$

When $x = 3$, the factor $(x-3)$ equals 0, and we can solve for B.

$$\begin{aligned} 4 \cdot 3 - 9 &= A(3-3) + B \cdot 3 \\ 3 &= 3B \\ B &= 1 \end{aligned}$$

When $x = 0$, we can solve for A.

$$\begin{aligned} 4 \cdot 0 - 9 &= A(0-3) + B \cdot 0 \\ -9 &= -3A \\ A &= 3 \end{aligned}$$

Solving this system yields $A = 3$ and $B = 1$. Therefore,

$$\frac{4x-9}{x^2-3x} = \frac{3}{x} + \frac{1}{x-3}.$$

Example 3

Use partial fractions to decompose the rational expression.

$$\frac{7x^2+x+6}{x^3+3x}$$

Solution:

First, factor the denominator and identify the denominators of the partial fractions.

$$\frac{7x^2+x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

When the fractions are eliminated by multiplying through by the LCD, the equation becomes

$$\begin{aligned} 7x^2+x+6 &= A(x^2+3) + x(Bx+C) \\ 7x^2+x+6 &= Ax^2+3A+Bx^2+Cx. \end{aligned}$$

Notice that the squared term, linear term, and constant term form a system of three equations with three variables.

$$\begin{aligned} A+B &= 7 \\ C &= 1 \\ 3A &= 6 \end{aligned}$$

In this case, it is easy to see that $A = 2, B = 5, C = 1$. Often, the resulting system of equations is more complex and would benefit from your knowledge of solving systems using matrices.

$$\frac{7x^2+x+6}{x(x^2+3)} = \frac{2}{x} + \frac{5x+1}{x^2+3}$$

Example 4

Decompose the rational expression.

$$\frac{5x^4-3x^3-x^2+4x-1}{(x-1)^3x^2}$$

Solution:

First, identify the denominators of the partial fractions. Note that there are repeated factors in the denominator of the given rational expression.

$$\frac{5x^4-3x^3-x^2+4x-1}{(x-1)^3x^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x} + \frac{E}{x^2}$$

When the entire fraction is multiplied through by $(x-1)^3x^2$, the equation results to

$$\begin{aligned}
&5x^4-3x^3-x^2+4x-1 \\
&= A(x-1)^2x^2 + B(x-1)x^2 + Cx^2 + D(x-1)^3x + E(x-1)^3.
\end{aligned}$$

Multiplication of each term can be done separately to be extra careful.

$$\begin{aligned}
&Ax^4 - 2Ax^3 + Ax^2 \\
&Bx^3 - Bx^2 \\
&Cx^2 \\
&Dx^4 - 3Dx^3 + 3Dx^2 - Dx \\
&Ex^3 - 3Ex^2 + 3Ex - E
\end{aligned}$$

Group terms with the same power of x, and set equal to the corresponding term.

$$\begin{aligned}
5x^4 &= Ax^4 + Dx^4 \\
-3x^3 &= -2Ax^3 + Bx^3 - 3Dx^3 + Ex^3 \\
-x^2 &= Ax^2 - Bx^2 + Cx^2 + 3Dx^2 - 3Ex^2 \\
4x &= -Dx + 3Ex \\
-1 &= -E
\end{aligned}$$

From these 5 equations, every x can be divided out. Assume that $x \neq 0$ because if it were, then the original expression would be undefined.

$$\begin{aligned}5 &= A+D\\ -3 &= -2A+B-3D+E\\ -1 &= A-B+C+3D-3E\\ 4 &= -D+3E\\ -1 &= -E\end{aligned}$$

This is a system of equations of 5 variables and 5 equations. Some of the equations can be solved using logic and substitution, like $E=1,\ D=-1,\ A=6$. You can use any method involving determinants or matrices. In this case, it is easiest to substitute known values into equations with one unknown value to get more known values and repeat.

$$\begin{aligned}B &= 5\\ C &= 4\\ \frac{5x^4-3x^3-x^2+4x-1}{(x-1)^3x^2} &= \frac{6}{x-1}+\frac{5}{(x-1)^2}+\frac{4}{(x-1)^3}+\frac{-1}{x}+\frac{1}{x^2}\end{aligned}$$

Example 5

Use matrices to complete the partial fraction decomposition of the rational expression.

$$\frac{2x+4}{(x-1)(x+3)}$$

Solution:

$$\begin{aligned}\frac{2x+4}{(x-1)(x+3)} &= \frac{A}{x+1}+\frac{B}{x+3}\\ 2x+4 &= Ax+3A+Bx+B\end{aligned}$$

$$\begin{aligned}2 &= A+B\\ 4 &= 3A+B\end{aligned}$$

$$\left[\begin{array}{cc|c}1 & 1 & 2\\ 3 & 1 & 4\end{array}\right]\begin{array}{c}\rightarrow\\ -3\cdot R_1\end{array}\rightarrow\left[\begin{array}{cc|c}1 & 1 & 2\\ 0 & -2 & -2\end{array}\right]\begin{array}{c}\rightarrow\\ \div -2\end{array}\rightarrow\left[\begin{array}{cc|c}1 & 1 & 2\\ 0 & 1 & 1\end{array}\right]\begin{array}{c}-R_2\\ \rightarrow\end{array}\rightarrow\left[\begin{array}{cc|c}1 & 0 & 1\\ 0 & 1 & 1\end{array}\right]$$

$$A=1, B=1$$

$$\frac{2x+4}{(x-1)(x+3)} = \frac{1}{x+1} + \frac{1}{x+3}$$

Example 6

Use matrices to help you decompose the rational expression.

$$\frac{5x-2}{(2x-1)(3x+4)}$$

Solution:

$$\begin{aligned}\frac{5x-2}{(2x-1)(3x+4)} &= \frac{A}{2x-1} + \frac{B}{3x+4}\\ 5x-2 &= A(3x+4)+B(2x-1)\\ 5x-2 &= 3Ax+4A+2Bx-B\\ 5 &= 3A+2B\\ -2 &= 4A-B\end{aligned}$$

$$\left[\begin{array}{cc|c} 3 & 2 & 5\\ 4 & -1 & -2\end{array}\right]\begin{array}{c}\cdot 4\\ \cdot 3\end{array}\begin{array}{c}\rightarrow\\ \rightarrow\end{array}\left[\begin{array}{cc|c} 12 & 8 & 20\\ 12 & -3 & -6\end{array}\right]\begin{array}{c}\rightarrow\\ -R_1\end{array}\begin{array}{c}\\ \rightarrow\end{array}\left[\begin{array}{cc|c} 12 & 8 & 20\\ 0 & -11 & -26\end{array}\right]$$

$$\begin{array}{c}\rightarrow\\ \rightarrow\end{array}\begin{array}{c}\cdot 11\\ \cdot 8\end{array}\begin{array}{c}\rightarrow\\ \rightarrow\end{array}\left[\begin{array}{cc|c} 132 & 88 & 220\\ 0 & -88 & -208\end{array}\right]\begin{array}{c}+R_2\\ \rightarrow\end{array}\begin{array}{c}\rightarrow\\ \end{array}\left[\begin{array}{cc|c} 132 & 0 & 12\\ 0 & -88 & -208\end{array}\right]$$

$$\begin{array}{c}\rightarrow\\ \rightarrow\end{array}\begin{array}{c}\div 132\\ \div -88\end{array}\begin{array}{c}\rightarrow\\ \rightarrow\end{array}\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{11}\\ 0 & 1 & \frac{26}{11}\end{array}\right]$$

$$\begin{aligned}A = \frac{1}{11}&, B = \frac{26}{11}\\ \frac{5x-2}{(2x-1)(3x+4)} &= \frac{\frac{1}{11}}{2x-1} + \frac{\frac{26}{11}}{3x+4}\end{aligned}$$

Example 7

Add the partial fractions to check Example 6.

$$\frac{5x-2}{(2x-1)(3x+4)} = \frac{\frac{1}{11}}{2x-1} + \frac{\frac{26}{11}}{3x+4}$$

Solution:

$$\frac{5x-2}{(2x-1)(3x+4)} = \frac{\frac{1}{11}}{2x-1} + \frac{\frac{26}{11}}{3x+4}$$
$$5x-2 = \frac{1}{11}(3x+4) + \frac{26}{11}(2x-1)$$
$$55x-22 = 3x+4+26(2x-1)$$
$$55x-22 = 3x+4+52x-26$$
$$55x-22 = 55x-22$$

Summary

- **Partial fraction decomposition** is a procedure that undoes the operation of adding fractions with unlike denominators.
- Partial fraction decomposition separates a rational expression into the sum of rational expressions with unlike denominators.
- When performing partial fraction decomposition, **repeated factors**, linear factors that are raised to a power greater than one, must have each successive power included as a separate denominator.
- When performing partial fraction decomposition, quadratic terms that do not factor to be linear terms are included with a numerator that is a linear function of x.

Review

Decompose the rational expressions below. Practice using matrices with at least one of the problems.

1. $\frac{3x-4}{(x-1)(x+4)}$
2. $\frac{2x+1}{x^2(x-3)}$
3. $\frac{x+1}{x(x-5)}$
4. $\frac{x^2+3x+1}{x(x-3)(x+6)}$
5. $\frac{3x^2+2x-1}{x^2(x+2)}$
6. $\frac{x^2+1}{x(x-1)(x+1)}$
7. $\frac{4x^2-9}{x^2(x-4)}$
8. $\frac{2x-4}{(x+7)(x-3)}$
9. $\frac{3x-4}{x^2(x^2+1)}$
10. $\frac{2x+5}{(x-3)(x^2+4)}$
11. $\frac{3x^2+2x-5}{x^2(x-3)(x^2+1)}$
12. Confirm your answer to Number 1 by adding the partial fractions.
13. Confirm your answer to Number 3 by adding the partial fractions.
14. Confirm your answer to Number 6 by adding the partial fractions.
15. Confirm your answer to Number 9 by adding the partial fractions.

Review (Answers)

Please see the Appendix.

10.12 Project: Systems and Matrices

Chapter Project: Matrices

In Gotham City, avenues run north-south and streets run east-west. Many of the roads alternate direction and are one-way streets. In this example, Avenue A is west of Avenue B and they are parallel. Both are perpendicular to 2nd Street and 3rd Street (which are parallel to each other with 2nd Street north of 3rd Street). Avenue A traffic runs north only, and Avenue B traffic runs south only. Cars on 2nd Street can only drive west. Cars on 3rd Street can only drive east. City planners have requested data to improve the lane use and light timing. Here are the data provided for these four intersections:

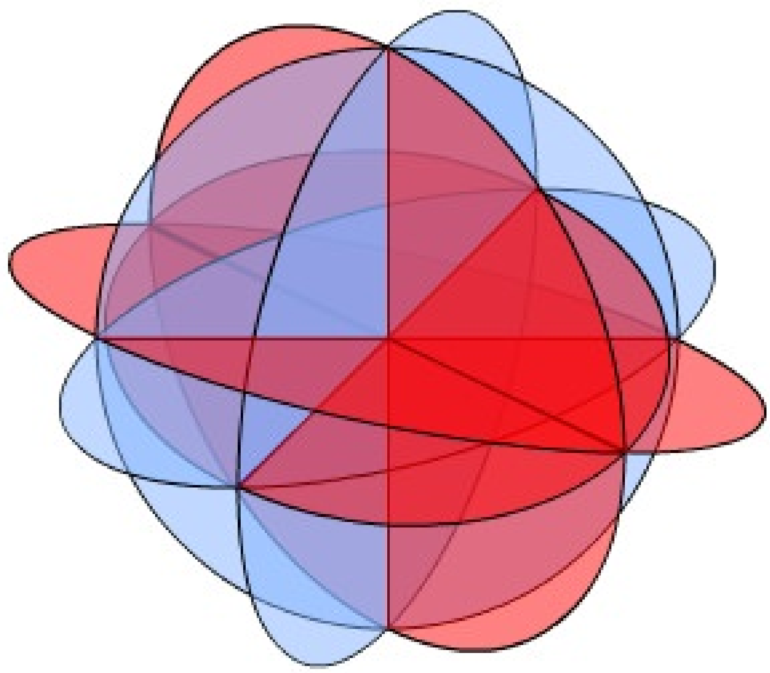
- Four hundred cars enter the intersection of Avenue A and 3rd Street from the south on Avenue A.
- Two hundred cars exit the intersection of Avenue A and 2nd Street headed north on Avenue A.
- Nine hundred cars enter the intersection of Avenue B and 2nd Street headed south on Avenue B.
- Two hundred cars exit the intersection of Avenue B and 3rd Street on Avenue B headed south.
- Five hundred cars enter the intersection of Avenue B and 2nd Street headed west on 2nd Street.
- Seven hundred cars exit the intersection of Avenue A and 2nd Street headed west on 2nd Street.
- Five hundred cars enter the intersection of Avenue A and 3rd Street headed east on 3rd Street.
- Three hundred cars exit the intersection of Avenue B and 3rd Street headed east on 3rd Street.

Project Steps:

- Create a diagram that models the situation described.
- Label the number of cars that exit at each intersection with the variables c_1, c_2, c_3, c_4.
- Create four equations to represent the traffic in and out of the intersections.
- Create an augmented matrix to represent the traffic patterns.
- Determine the restrictions on the variables.
- Does the system have a unique solution? Can you determine at least one solution? Explain.

10.13 Summary: Systems and Matrices

A system of equations is a collection of two or more equations with multiple unknown variables. We solve a system of equations by finding the values for the variables that satisfy all the equations in the system. Graphically, the solution to a system of equations is the point(s) of intersection between the lines, curves, or planes represented by the equations. In this chapter, we learned that a matrix is a rectangular array of numbers that corresponds to the coefficients in a system. Using matrices to represent systems is incredibly powerful because it makes it easier to solve for the variables. We also reviewed techniques for solving systems in two and three dimensions, developed an understanding for the concept of matrices, practiced applying a variety of properties of matrices, and used matrices to solve systems.



Chapter Summary

In this chapter we learned:

- The standard form of linear equations is $Ax+By=C$.
- Equations in standard form are most easily translated into matrices.
- Systems can be solved a number of ways, such as substitution, elimination, and graphically.
- Solving a system of equations by the process of elimination is most helpful to the understanding of solving matrices.
- A minimum of three equations is necessary to solve a system with three variables.
- The linear combination method can be used to solve a system of three variables in three equations.
- There can be three possible results when solving systems of three linear equations: one solution, infinitely many solutions, and no solution.

- A matrix is a rectangular array of numbers.
- Square matrices have the same number of rows as columns.
- The order of a matrix describes the number of rows and the number of columns in the matrix.
- A symmetric matrix is a special type of square matrix that has reflection symmetry across the main diagonal. The identity matrix is an example of a symmetric matrix.
- The identity matrix of order $n \times n$ has zeros everywhere, except along the main diagonal, where it has ones. Just as the number 1 has an important property with numbers, the identity matrix of any order has special properties as well.
- Matrix operations are addition, subtraction, and multiplication.
- The commutative property holds for matrix addition: $A+B=B+A$.
- The commutative property does not hold in general for matrix multiplication.
- The associative property holds for both multiplication and addition: $(AB)C=A(BC),(A+B)+C=A+(B+C)$.
- Distribution over addition and subtraction holds: $A(B \pm C)=AB \pm AC$. The order of the matrix multiplication matters, so be careful to preserve the order of operations.
- Row operations include swapping rows, adding a multiple of one row to another, or scaling a row by multiplying through by a scalar.
- Row echelon form is a matrix that has a leading one at the start of every nonzero row, zeros below every leading one, and all rows containing only zeros at the bottom of the matrix.
- Reduced row echelon form is the same as row echelon form with one additional stipulation: every other entry in a column with a leading one must be zero.
- Only three operations are permitted to act on matrices: Add a multiple of one row to another row; scale a row by multiplying through by a nonzero constant; and swap two rows.
- The determinant of a matrix is a number calculated from the entries in a matrix. The procedure is derived from solving linear systems.
- The determinant of matrix A can be expressed in either of the following ways: det A or $|A|$.
- Sarrus's rule is a memorization technique that enables you to compute the determinant of 3 x 3 matrices efficiently.
- A matrix equation represents a system of equations by multiplying a coefficient matrix and a variable matrix to get a solution matrix.

- Cramer's Rule is a method that uses determinants to solve a system of equations.
- Cramer's Rule can be more efficient that the elimination method for solving a system of equations.
- Cramer's Rule can be used only when the determinant of the coefficient matrix is nonzero.

- Multiplicative inverses are two numbers or matrices whose product is one or the identity matrix.
- Non-square matrices do not generally have inverses.
- Square matrices that have determinants equal to zero do not have inverses.

- Partial fraction decomposition is a procedure that undoes the operation of adding fractions with unlike denominators.
- Partial fraction decomposition separates a rational expression into the sum of rational expressions with unlike denominators.
- When performing partial fraction decomposition, linear factors that are raised to a power greater than one must have each successive power included as a separate denominator.
- When performing partial fraction decomposition, quadratic terms that do not factor to be linear terms are included with a numerator that is a linear function of x.

Review

Try the following cumulative review problems to practice the concepts we studied in this chapter:

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URL: http://www.ck12.org/flx/render/embeddedobject/195287

10.14 References

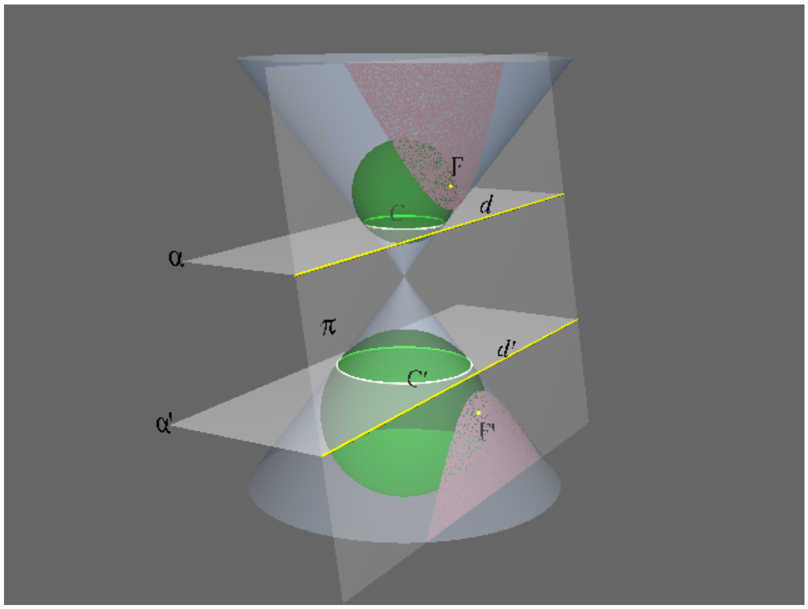
1. SophieSchieli. https://www.flickr.com/photos/sophieschieli/6037327058/in/photolist-acuSZQ-qxmEWC-8R32zo-fs8mh2-4SDdKe-iezrk-mtPyLF-2N46c-hm68U2-HnhZC-GFKHP6-GWiek-7GiQEy-pu6UT2-5qjD46-iezqN-guNwXN-kTKAb-u5DJj-8yLdYZ-pu9xhj-9K7CZi-5vM3LX-dkLz3M-FLdHMG-cc7nis-a44Ks-ikTuY-6U2sAv-eSHzG-5rtsCi-fsnFpC-7EiVv5-sw75k-5rh7jF-6XPJXK-4pXchz-6uQM-3Vjt1y-6U6tx7-6cbgDx-5gi8CJ-d76Bv-5PbUuP-oNgyVe-5gi99d-xkars-5n25xS-5gi93A-miPjg .
2. flickr: rick. https://www.flickr.com/photos/spine/558333048 .
3. CK-12 Foundation. CK-12 Foundation .
4. Tim Patterson. https://www.flickr.com/photos/timpatterson/313655887/in/photolist-tHyY8-a1XDe4-53ZdGY-5AZeky-7Pz9vZ-9aKwTL-d2t9N-8xbkK-4BEbqj-cthCm-6mkbW6-6q38o-aNXewX-9tgSWu-db5Pnx-acTrMM-89VHuJ-4k5gTu-iKGsc-62KctJ-cQqjp-opT3uC-4k9oQo-33kVn-34KHNL-6FmtJx-3UW5y-xN7hS-kPiu-bEk64-D3Nk9-8bPZNB-6GkLw3-7vqNu2-4kSgse-9qUKLb-bqPw-bCAJSG-8eZeDw-do2MPo-7AviWp-ofWus-T68Px-6mppmd-KEWSE-zEax3n-5ijeyT-9sqACQ-AdNAF-6FSEfZ .
5. Tony Webster. https://commons.wikimedia.org/wiki/File:Bakery_(15725767689).jpg .
6. By TaranVH (talk) (Uploads) (Own work) [Public domain], via Wikimedia Commons. https://commons.wikimedia.org/wiki/File%3A3sphereprojection.jpg .

CHAPTER 11 Conics

Chapter Outline

11.1 Introduction: Conics

Earth was once thought to be the center of our universe. The study of astronomy and the positions of cosmic bodies, satellites, and rays have been important in understanding our own world. The orbits and trajectories in the universe can be modeled by curves called conic sections. Conics are an application of analytic geometry. In this chapter, we will explore the attributes and equations of conics.

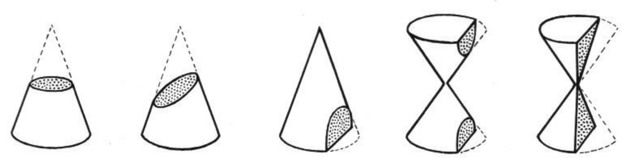


11.2 General Form of a Conic

Learning Objectives

Learn to use the standard form of a conic to determine the type of conic.

Introduction

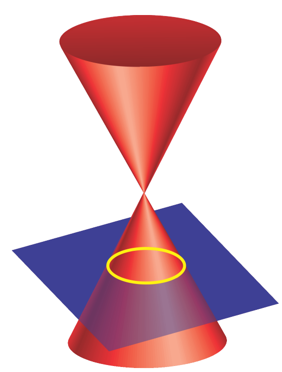


Conics are a family of graphs that include parabolas, circles, ellipses, and hyperbolas. All of these graphs are derived from the same general equation. By manipulating this specific equation, you can determine the type of conic and graph it, using key information.

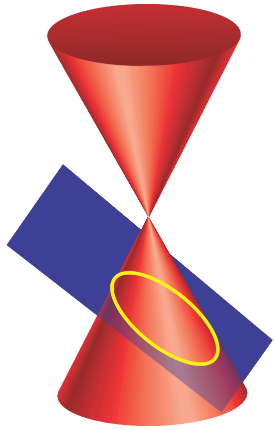
Conics

The word "conic" comes from the word "cone," which is where the shapes of parabolas, circles, ellipses, and hyperbolas originate. Consider a double cone, defined as the form generated when one of two intersecting lines is rotated about the other. The lines that pass through the vertex to form this cone are called the **generators**. The fixed line corresponding to the height of the cone is called the **axis** of the cone. **Conic sections** are the nondegenerate curves generated by intersecting one or both pieces of the cone with a plane. One piece of a double cone is called a **nappe.**

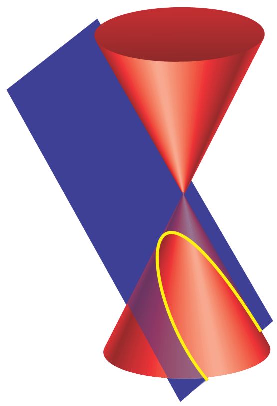
When a plane is perpendicular to the axis of the cone, a circle is generated.



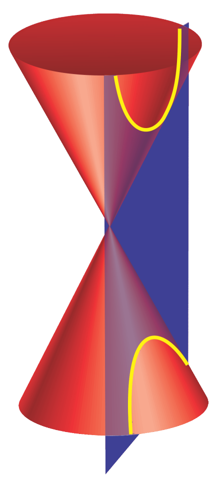
When a plane is not perpendicular to the axis of the cone but does intersect one nappe of the cone, either an ellipse or a parabola is generated. An ellipse is generated when the plane is tilted so that it intersects both generator lines.



A parabola is generated when the plane is tilted so that it is parallel to one generator and intersects the other generator.



When a plane intersects both nappes of the cone, a hyperbola is generated.



In the coordinate plane, each of the above conics can be graphed using the same general formula and by applying important information about the size and place of the points.

Standard Form of a Conic

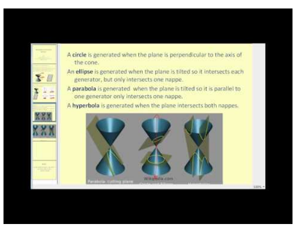
$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

In this form, examining the two coefficients A and C will expose the type of conic.

- For **circles,** the coefficients of x^2 and y^2 are the same sign and the same value: $A = C$.
- For **ellipses,** the coefficients of x^2 and y^2 are the same sign and different values: $A, C > 0$, $A \neq C$.
- For **hyperbolas,** the coefficients of x^2 and y^2 are opposite signs: $C < 0 < A$ or $A < 0 < C$.
- For **parabolas,** either the coefficient of x^2 or y^2 must be zero: $A = 0$ or $C = 0$.

Each specific type of conic has its own graphing form, but in all cases the technique of completing the square is essential. The examples below review completing the square and recognizing conics.

The following video demonstrates how you can generate a circle, ellipse, parabola, and hyperbola by intersecting a cone with a plane:



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Examples

Example 1

Complete the square in the expression $x^2 + 6x$. Demonstrate graphically what completing the square represents.

Solution:

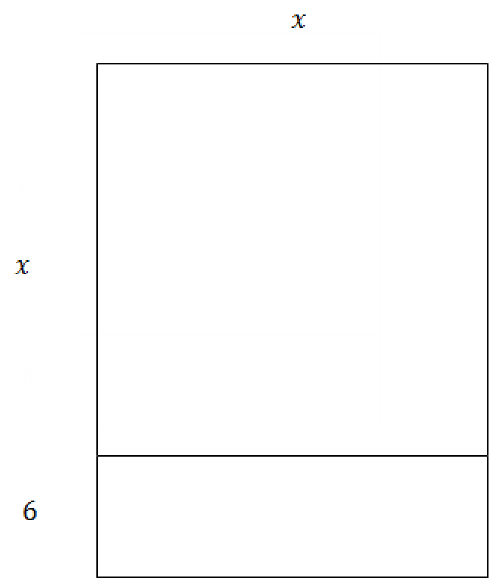
Algebraically, completing the square just requires you to divide the coefficient of x by 2 and square the result. In this case, $\left(\frac{6}{2}\right)^2 = 3^2 = 9$. Since you cannot add 9 to an expression without changing its value, you must simultaneously add 9 and subtract 9 so the net change will be zero.

$$x^2 + 6x + 9 - 9$$

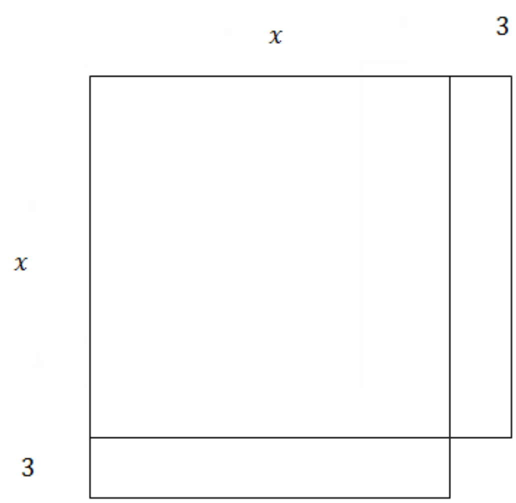
Now you can factor by recognizing a perfect square.

$$(x+3)^2 - 9$$

Graphically, the original expression $x^2 + 6x$ can be represented by the area of a rectangle with sides x and $(x+6)$.



The term "complete the square" has visual meaning as well algebraic meaning. The rectangle can be rearranged to be more square-like, so that instead of small rectangle of area $6x$ at the bottom, there is a rectangle of area $3x$ on two sides of the x^2 square.



What is missing to complete this shape as a perfect square? A little corner square of 9 is missing, which is why the 9 should be added to make the perfect square of $(x+3)(x+3)$.

Example 2

What type of conic is each of the following relations?

1. $5y^2 - 2x^2 = -25$
2. $x = -\frac{1}{2}y^2 - 3$
3. $4x^2 + 6y^2 = 36$
4. $x^2 - \frac{1}{4}y = 1$
5. $-\frac{x^2}{8} + \frac{y^2}{4} = 1$
6. $-x^2 + 99y^2 = 12$

Solutions:

1. Hyperbola because the x^2 and y^2 coefficients are different signs.
2. Parabola (sideways) because the x^2 term is missing.
3. Ellipse because the x^2 and y^2 coefficients are different values but the same sign.
4. Parabola (upright) because the y^2 term is missing.
5. Hyperbola because the x^2 and y^2 coefficients are different signs.
6. Hyperbola because the x^2 and y^2 coefficients are different signs.

Example 3

Complete the square for both the x and y terms in the equation below.

$$x^2+6x+2y^2+16y=0$$

Solution:

First, write out the equation with space, so there is room for the terms to be added to both sides. Since this is an equation, it is appropriate to add the values to both sides instead of adding and subtracting the same value simultaneously. As you rewrite with spaces, factor out any coefficient of the x^2 or y^2 terms, since your algorithm for completing the square works only when this coefficient is 1.

$$x^2+6x+\_\_+2(y^2+8y+\_\_)=0$$

Next, complete the square by adding a 9 and what looks like a 16 on the left. (It is actually a 32, since it is inside the parentheses.)

$$x^2+6x+9+2(y^2+8y+16)=9+32$$

Factor.

$$(x+3)^2+2(y+4)^2=41$$

Example 4

Identify the type of conic in each of the following relations:

1. $3x^2=3y^2+18$
2. $y=4(x-3)^2+2$
3. $x^2+y^2=4$
4. $y^2+2y+x^2-6x=12$
5. $\frac{x^2}{6}+\frac{y^2}{12}=1$
6. $x^2-y^2+4=0$

Solutions:

1. The relation is a hyperbola because when you move the $3y^2$ to the lefthand side of the equation, it becomes negative, and then the coefficients of x^2 and y^2 have opposite signs.

2. Parabola

3. Circle

4. Circle

5. Ellipse

6. Hyperbola

Example 5

Complete the square in the following expression:

$$6y^2 - 36y + 4.$$

Solution:

$$\begin{aligned}
&6y^2 - 36y + 4\\
&6(y^2 - 6y + __) + 4\\
&6(y^2 - 6y + 9) + 4 - 54\\
&6(y-3)^2 - 50
\end{aligned}$$

Example 6

Complete the square for both x and y in the equation below.

$$-3x^2 - 24x + 4y^2 - 32y = 8$$

Solution:

$$\begin{aligned}
-3x^2 - 24x + 4y^2 - 32y &= 8\\
-3(x^2 + 8x + __) + 4(y^2 - 8y + __) &= 8\\
-3(x^2 + 8x + 16) + 4(y^2 - 8y + 16) &= 8 - 48 + 64\\
-3(x+4)^2 + 4(y-4)^2 &= 24
\end{aligned}$$

Summary

- **Conic sections** are the nondegenerate curves generated by intersecting one or both pieces of the cone with a plane.
- There are four conic sections: circles, ellipses, parabolas, and hyperbolas.
- **Standard form of a Conic**: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

Review

Identify the type of conic in each of the following relations:

1. $3x^2 + 4y^2 = 12$

2. $x^2 + y^2 = 9$

3. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

4. $y^2 + x = 11$

5. $x^2 + 2x - y^2 + 6y = 15$

6. $x^2 = y - 1$

Complete the square for x or y in each of the following expressions:

7. $x^2 + 4x$

8. $y^2 - 8y$

9. $3x^2 + 6x + 4$

10. $3y^2 + 9y + 15$

11. $2x^2 - 12x + 1$

Complete the square for x and/or y in each of the following equations:

12. $4x^2 - 16x + y^2 + 2y = -1$

13. $9x^2 - 54x + y^2 - 2y = -81$

14. $3x^2 - 6x - 4y^2 = 9$

15. $y = x^2 + 4x + 1$

Review (Answers)

Please see the Appendix.

11.3 Parabolas

Learning Objectives

Learn to define a parabola in terms of its directrix and focus, graph parabolas vertically and horizontally, and use a new graphing form of the parabola equation.

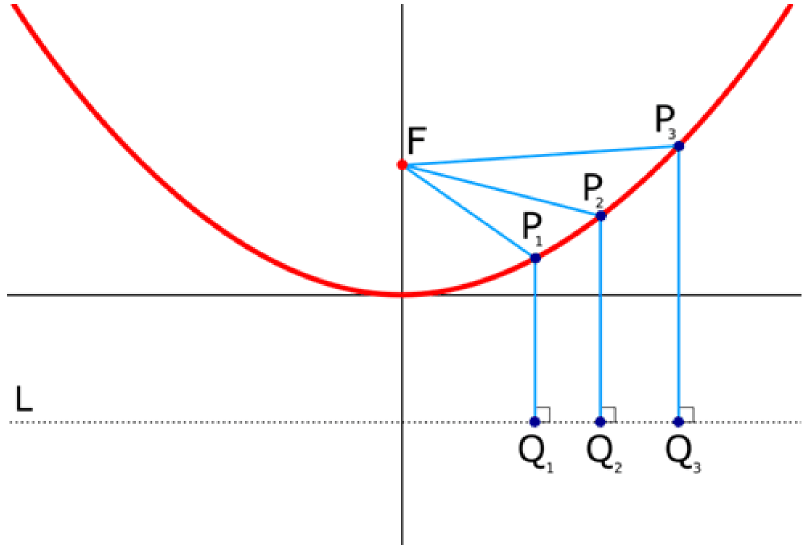
Introduction

One of the many uses of parabolic shapes in the real world is satellite dishes. With these shapes, it is vital to know where the receptor point should be placed so it can absorb all the signals being reflected from the dish.

Where should the receptor be located on a satellite dish that is 4 feet wide and 9 inches deep?

The Parabola

Recall that a parabola is generated when a plane is tilted so that it is parallel to one generator and intersects the other generator and one nappe of the cone. The definition of a **parabola** is the collection of points equidistant from a point called the focus and a line called the **directrix**.



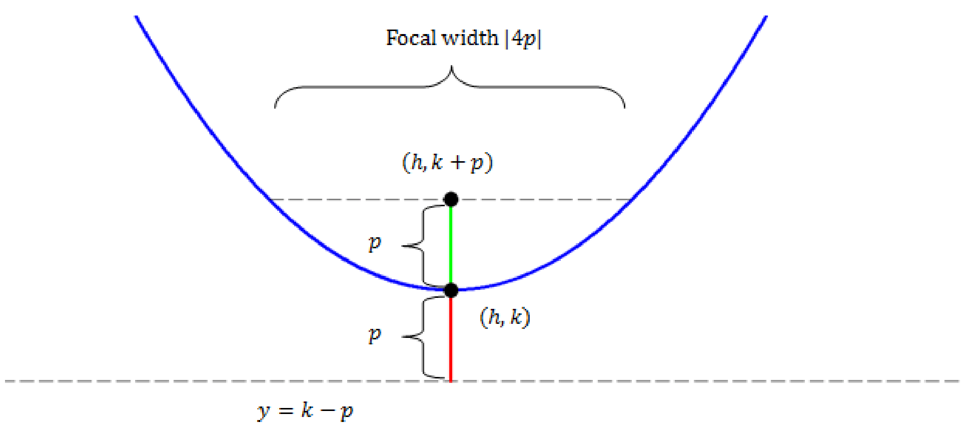
Notice how the three points, P_1, P_2, P_3, are each connected by a blue line to the focus point $F$and the directrix line L.

$$\overline{FP_1} = \overline{P_1Q_1}$$
$$\overline{FP_2} = \overline{P_2Q_2}$$
$$\overline{FP_3} = \overline{P_3Q_3}$$

There are two graphing equations for parabolas that will be used in this concept. The only difference is that one equation graphs parabolas opening vertically, and one equation graphs parabolas opening horizontally. You can recognize the parabolas opening vertically because they have an x^2 term. Likewise, parabolas opening horizontally have a y^2 term.

Standard Equation of Parabola

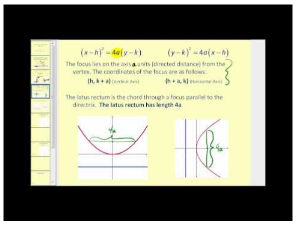
The general equation for a parabola opening vertically is $(x-h)^2 = \pm 4p(y-k)$.
The general equation for a parabola opening horizontally is $(y-k)^2 = \pm 4p(x-h)$.



Note that the vertex is still (h, k). The parabola opens upwards or to the right if the $4p$ is positive. The parabola opens down or to the left if the $4p$ is negative. The focus is just a point that is a distance of p units away from the vertex. The directrix is a line that is a distance of p units away from the vertex in the opposite direction. You can sketch how wide the parabola is by noting the focal width is $|4p|$.

Once you put the parabola into this graphing form, you can sketch the parabola by plotting the vertex, identifying p and plotting the focus and directrix, and lastly determining the focal width and sketching the curve.

The following video defines a parabola and explains how to graph a parabola in standard form:



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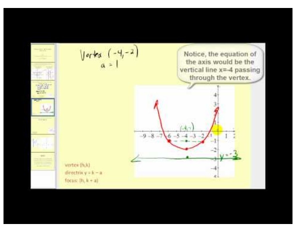
The following video explains how to graph a parabola given in general form by rewriting it in standard form:



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URL: http://www.ck12.org/flx/render/embeddedobject/61847

Play, Learn, and Explore Parabolas: www.ck12.org/a/2056819

Examples

Example 1

Identify the conic below. Then put it into graphing form and identify its vertex, focal length (p), focus, directrix, and focal width.

$2x^2+16x+y=0$

Solution:

This is a parabola because the y^2 coefficient is zero.

$$\begin{aligned} x^2+8x &= -\frac{1}{2}y \\ x^2+8x+16 &= -\frac{1}{2}y+16 \\ (x+4)^2 &= -\frac{1}{2}(y-32) \\ (x+4)^2 &= -4\cdot\frac{1}{8}(y-32) \end{aligned}$$

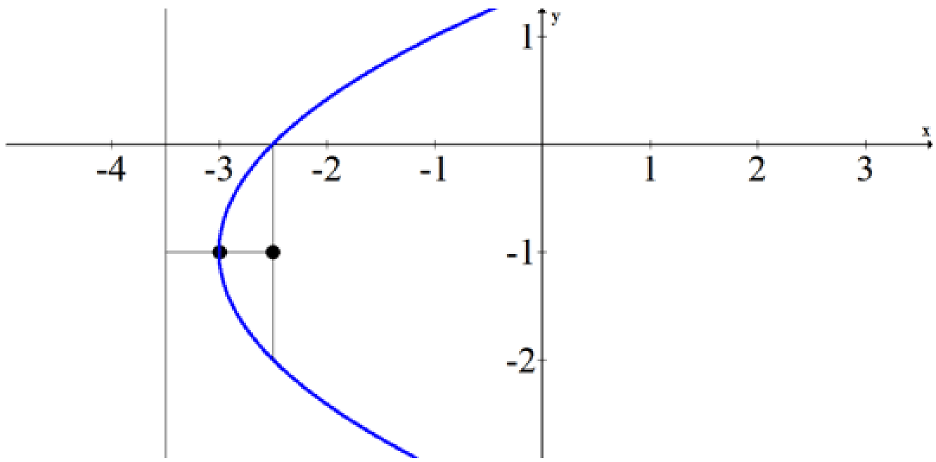
The vertex is (-4, 32). The length from the focus to the vertex is $p = \frac{1}{8}$. This parabola opens down, which means that the focus is at $\left(-4, 32 - \frac{1}{8}\right)$, and the directrix is horizontal at $y = 32 + \frac{1}{8}$. The focal width is $\frac{1}{2}$.

Example 2

Sketch the following parabola and identify the important pieces of information:

$$(y+1)^2 = 4 \cdot \frac{1}{2} \cdot (x+3).$$

Solution:



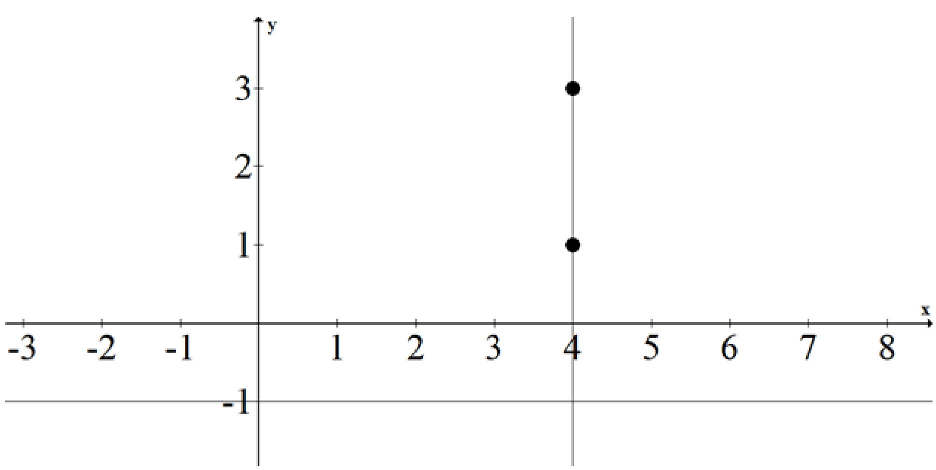
The vertex is at (-3, -1). The parabola is sideways because there is a y^2 term. The parabola opens to the right because the $4p$ is positive. The length from the focus to the vertex is $p = \frac{1}{2}$,which means the focus is $\frac{1}{2}$ to the right of the vertex at (-2.5, -1), and the directrix is $\frac{1}{2}$ to the left of the vertex at $x = -3.5$. The focal width is 2, which is why the width of the parabola stretches from (-2.5, 0) to (-2.5, -2).

Example 3

What is the equation of a parabola that has a focus at (4, 3) and a directrix of $y = -1$?

Solution:

It would probably be useful to graph the information you have in order to reason about where the vertex is.



The vertex must be halfway between the focus and the directrix. This places it at (4, 1). The length from the focus to the vertex is 2. The parabola opens upwards. This is all the information you need to create the equation.

$$(x-4)^2 = 4 \cdot 2 \cdot (y-1)$$
$$(x-4)^2 = 8(y-1)$$

Example 4

Recall the question from the Introduction: Where should the receptor be located on a satellite dish that is 4 feet wide and 9 inches deep?

Solution:

Convert feet to inches to get 48 inches, and then taking half for either side, use 24 in your calculations.

$$\begin{aligned}(x-0)^2 &= 4p(y-0)\\ (24-0)^2 &= 4p(9-0)\\ \frac{24^2}{4\cdot 9} &= p\\ 16 &= p\end{aligned}$$

The receptor should be 16 inches away from the vertex of the parabolic dish.

Example 5

What is the equation of a parabola with focus at (2, 3) and directrix at $y = 5$?

Solution:

The vertex must lie directly between the focus and the directrix, so it must be at (2, 4). The focal length is therefore equal to 1. The parabola opens downwards.

$$(x-2)^2 = -4 \cdot 1 \cdot (y-4)$$
$$(x-2)^2 = -4(y-4)$$

Example 6

What is the equation of a parabola that opens to the right with focal width from (6, -7) to (6, 12)?

Solution:

The focus is in the middle of the focal width. The focus is $\left(6, \frac{5}{2}\right)$. The focal width is 19, which is 4 times the length from the focus to the vertex. The length from the focus to the vertex must be $\frac{19}{4}$. The vertex must be a focal length to the left of the focus, so the vertex is at $\left(6-\frac{19}{4}, \frac{5}{2}\right)$. This is enough information to write the equation of the parabola.

$$\left(y-\frac{5}{2}\right)^2 = 4 \cdot \frac{19}{4} \cdot \left(x-6+\frac{19}{4}\right)$$
$$\left(y-\frac{5}{2}\right)^2 = 19\left(x-\frac{5}{4}\right)$$

Example 7

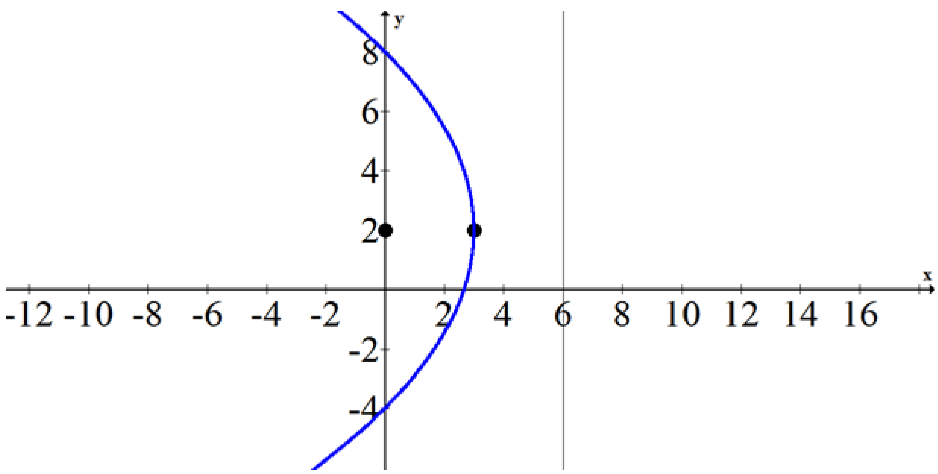
Sketch the following conic by putting it into graphing form and identifying important information:

$$y^2 - 4y + 12x - 32 = 0.$$

Solution:

$$y^2 - 4y + 12x - 32 = 0$$

$$\begin{aligned} y^2 - 4y &= -12x + 32 \\ y^2 - 4y + 4 &= -12x + 32 + 4 \\ (y-2)^2 &= -12(x-3) \\ (y-2)^2 &= -4 \cdot 3 \cdot (x-3) \end{aligned}$$



The vertex is at (3, 2). The focus is at (0, 2). The directrix is at $x = 6$.

Summary

- A ***parabola*** is the collection of points that are equidistant from a fixed focus and directrix.
- The ***focus*** of a parabola is the point that the parabola seems to curve around.
- The ***directrix*** of a parabola is the line that the parabola seems to curve away from.
- The general equation for a parabola opening vertically is $(x-h)^2 = \pm 4p(y-k)$.
- The general equation for a parabola opening horizontally is $(y-k)^2 = \pm 4p(x-h)$.

Review

1. What is the equation of a parabola with focus at (1, 4) and directrix at $y = -2$?

2. What is the equation of a parabola that opens to the left with focal width from (-2, 5) to (-2, -7)?

3. What is the equation of a parabola that opens to the right with vertex at (5, 4) and focal width of 12?

4. What is the equation of a parabola with vertex at (1, 8) and directrix at $y = 12$?

5. What is the equation of a parabola with focus at (-2, 4) and directrix at $x = 4$?

6. What is the equation of a parabola that opens downward with a focal width from (-4, 9) to (16, 9)?

7. What is the equation of a parabola that opens upward with vertex at (1, 11) and focal width of 4?

Sketch the following parabolas by putting them into graphing form and identifying important information:

8. $y^2+2y-8x+33=0$

9. $x^2-8x+20y+36=0$

10. $x^2+6x-12y-15=0$

11. $y^2-12y+8x+4=0$

12. $x^2+6x-4y+21=0$

13. $y^2+14y-2x+59=0$

14. $x^2+12x-\frac{8}{3}y+\frac{92}{3}=0$

15. $x^2+2x-\frac{4}{5}y+1=0$

Review (Answers)

Please see the Appendix.

11.4 Circles

Learning Objectives

Learn the formal definition of a circle, translate a conic from standard form into graphing form, and graph circles.

Introduction

A circle is the collection of points that are the same distance from a single point. What is the connection between the Pythagorean Theorem and a circle?

The Circle

Recall that a circle is generated when a plane is perpendicular to the axis of the cone. As a result, a circle is the collection of points that are equidistant from a single point. This single point is called the center of the circle. A circle does not have a focus or a directrix, instead it simply has a center.

Circles can be recognized immediately from the general equation of a conic when the coefficients of x^2 and y^2 are the same sign and the same value. Circles are not functions because they do not pass the vertical line test. The distance from the center of a circle to the edge of the circle is called the radius of the circle. The distance from one end of the circle through the center to the other end of the circle is called the diameter. The diameter is equal to twice the radius.

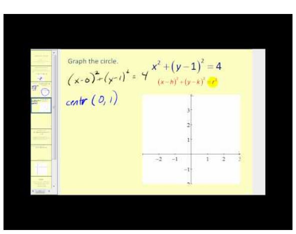
The graphing form of a circle is $(x-h)^2+(y-k)^2=r^2$. The center of the circle is at (h,k), and the radius of the circle is r. Note that this looks remarkably like the Pythagorean Theorem.

Standard Form of the Equation of a Circle

$$(x-h)^2+(y-k)^2=r^2,$$

where (h,k) is the center of the circle and r is the radius.

The following video explains how to graph a circle in standard form and general form:



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/61851

Play, Learn, and Explore Circles: www.ck12.org/a/1824219

Examples

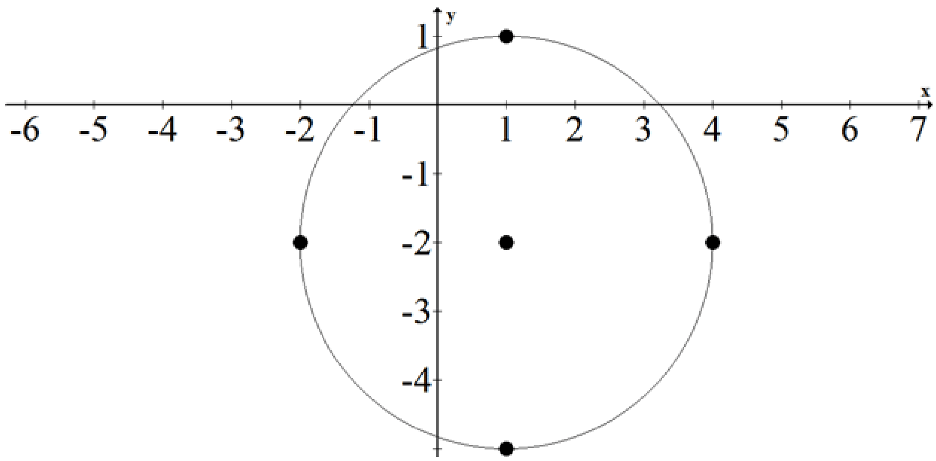
Example 1

Graph the following circle:

$$(x-1)^2+(y+2)^2=9.$$

Solution:

Plot the center and the four points that are exactly 3 units from the center.



Example 2

Translate the following conic from standard form to graphing form. Identify the center and the radius.

$$36x^2+36y^2-24x+36y-275=0$$

Solution:

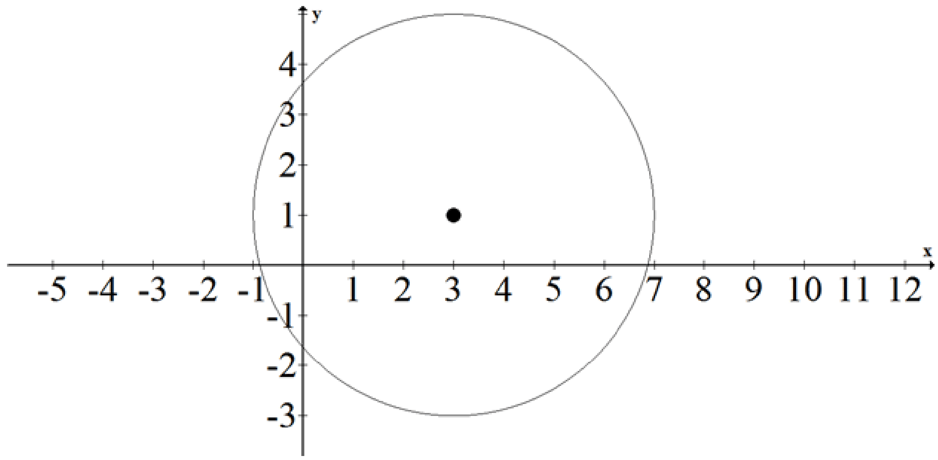
Complete the square and then divide by the coefficient of x^2 and y^2.

$$\begin{aligned}
&36x^2-24x+36y^2+36y=275\\
&36\left(x^2-\frac{2}{3}x+__\right)+36(y^2+y+__)=275\\
&36\left(x^2-\frac{2}{3}x+\frac{1}{9}\right)+36\left(y^2+y+\frac{1}{4}\right)=275+4+9\\
&36\left(x-\frac{1}{3}\right)^2+36\left(y+\frac{1}{2}\right)^2=288\\
&\left(x-\frac{1}{3}\right)^2+\left(y+\frac{1}{2}\right)^2=8
\end{aligned}$$

The center is $\left(\frac{1}{3},-\frac{1}{2}\right)$. The radius is $\sqrt{8}=2\sqrt{2}$.

Example 3

Write the equation of the following circle:



Solution:

The center of the circle is at (3, 1), and the radius of the circle is $r = 4$. The equation is $(x-3)^2 + (y-1)^2 = 16$.

Example 4

Recall the question from the Introduction: What is the connection between the Pythagorean Theorem and a circle?

Solution:

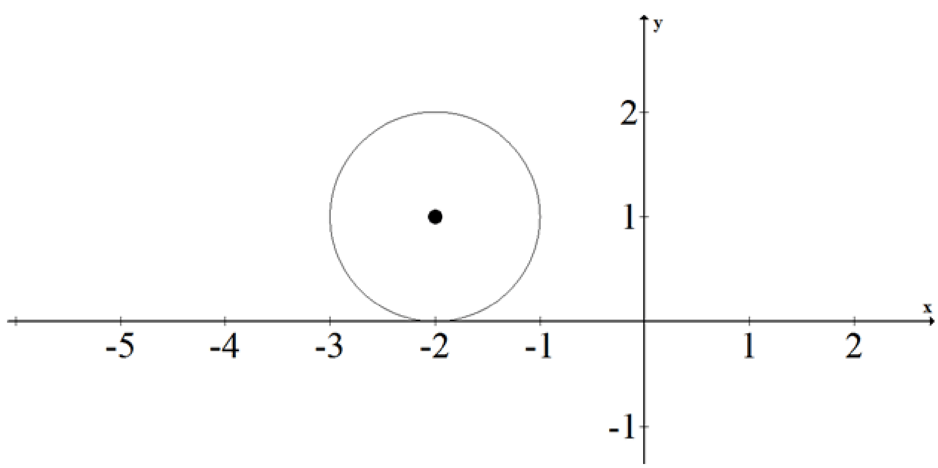
The reason why the graphing form of a circle looks like the Pythagorean Theorem is because each x and y coordinate along the outside of the circle forms a perfect right triangle with the radius as the hypotenuse.

Example 5

Graph the following conic section:

$$(x+2)^2 + (y-1)^2 = 1.$$

Solution:



Example 6

Translate the following conic from standard form to graphing form:

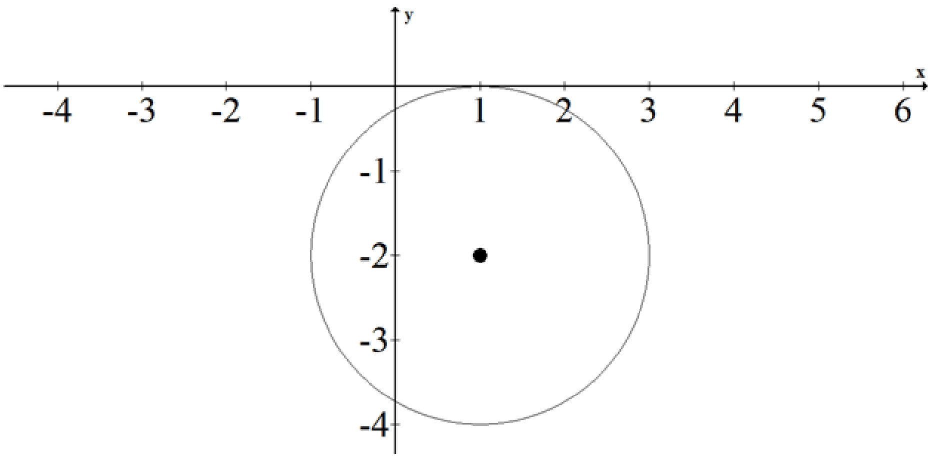
$$x^2 - 34x + y^2 + 24y + \frac{749}{2} = 0.$$

Solution:

$$\begin{aligned}
x^2-34x+y^2+24y+\frac{749}{2}&=0\\
x^2-34x+y^2+24y&=-\frac{749}{2}\\
x^2-34x+289+y^2+24y+144&=-\frac{749}{2}+289+144\\
(x-17)^2+(y+12)^2&=\frac{117}{2}
\end{aligned}$$

Example 7

Write the equation for the following circle:



Solution:

Note key information from the graph. The center of the circle is located at (1, -2). The radius of the circle is 2. Substitute these key points into the general equation $(x-h)^2+(y-k)^2=r^2$.

The equation of this graph will be $(x-1)^2+(y+2)^2=4$.

Summary

- A **circle** is the collection of points that are equidistant from a given point.
- The **radius** of a circle is the distance from the center of the circle to the outside edge.
- The **center** of a circle is the point that defines the location of the circle.
- The standard equation of a circle is $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center of the circle and r is the radius.

Review

Graph the following conics:

1. $(x+4)^2+(y-3)^2=1$
2. $(x-7)^2+(y+1)^2=4$
3. $(y+2)^2+(x-1)^2=9$
4. $x^2+(y-5)^2=8$

5. $(x-2)^2+y^2=16$

Translate the following conics from standard form to graphing form:

6. $x^2-4x+y^2+10y+18=0$

7. $x^2+2x+y^2-8y+1=0$

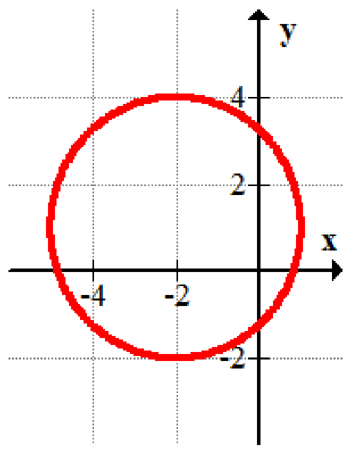
8. $x^2-6x+y^2-4y+12=0$

9. $x^2+2x+y^2+14y+25=0$

10. $x^2-2x+y^2-2y=0$

Write the equations for the following circles:

11.



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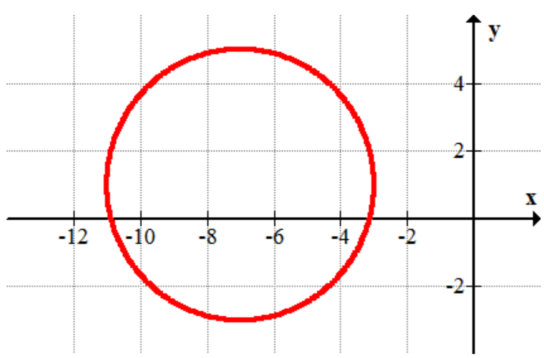

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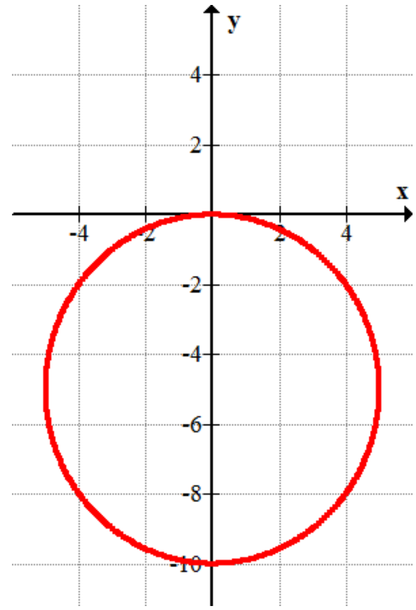

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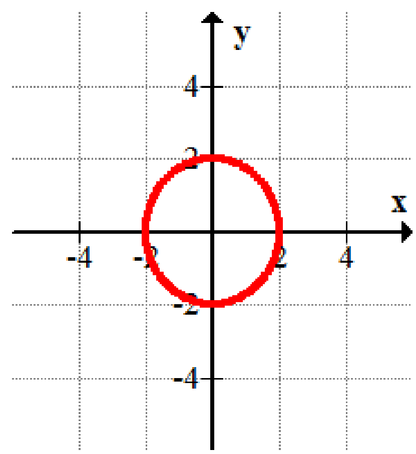

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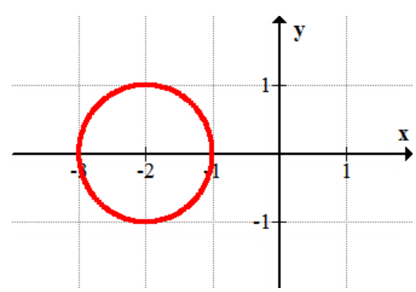


Review (Answers)

Please see the Appendix.

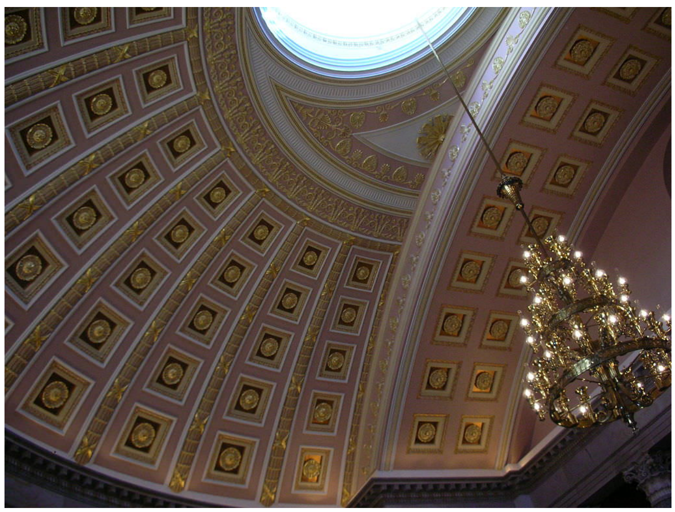
11.5 Ellipses

Learning Objectives

Learn to translate ellipse equations from standard conic form to graphing form, graph ellipses, and identify the different axes.

Introduction

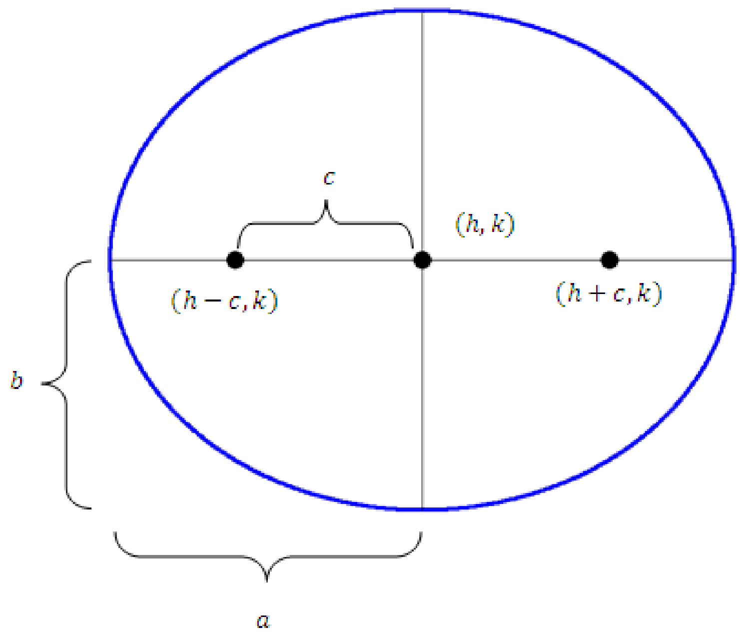
An **ellipse** is commonly known as an oval. In the real world, ellipses are just as common as parabolas, with their own uses. Rooms that have elliptical-shaped ceilings are called "whispering rooms," because if you stand at one focus point and whisper, someone standing at the other focus point will be able to hear you.



Ellipses look similar to circles, but there are a few key differences between these shapes. Ellipses have both an x-radius and a y-radius, while circles have only one radius. Another difference between circles and ellipses is that an ellipse is defined as the collection of points a set distance from two focal points, while circles are defined as the collection of points a set distance from one center point. A 3rd difference between ellipses and circles is that not all ellipses are similar to each other, while all circles are similar to each other. Some ellipses are narrow and some are almost circular. How do you measure and graph an ellipse?

The Ellipse

Recall that an ellipse is generated when a plane is tilted so that it intersects both generator lines of one nappe of the cone. Thus, an ellipse has two foci. For every point on the ellipse, the sum of the distances to each foci is constant. This is what defines an ellipse. Another way of thinking about the definition of an ellipse is to allocate a set amount of string and fix the two ends of the string so there is some slack between them. Then use a pencil to pull the string taut and trace the curve all the way around both fixed points. You will trace an ellipse, and the fixed end points of the string will be the foci. (Foci is the plural form of focus.) In the picture below, (h,k) is the center of the ellipse, and the other two marked points are the foci.



Standard Equation for an Ellipse

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \qquad \frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1,$$

where (h,k) is the center of the circle, and $a > b$ are the lengths of the semi-major and semi-minor axes.

In the example shown above, the major axis is the horizontal axis and the minor axis is the vertical axis because $a > b$. In this case, the first standard equation for an ellipse would be used. If the y-radius were larger than the x-radius, the major axis would be the vertical axis and the minor axis would be the horizontal axis. In this case, the second standard equation would be used. In general, the coefficient a always comes from the length of the semi-major axis (half of the longer axis), and the coefficient b always comes from the length of the semi-minor axis (half of the shorter axis).

To find the locations of the two foci, you will need to find the focal length represented as c using the following relationship: $a^2 - b^2 = c^2$.

Add the focal length to the corresponding coordinate of the center along the major axis to locate the foci. The general shape of an ellipse is measured using eccentricity. Eccentricity is a measure of how oval or how circular the shape is. Ellipses can have an eccentricity between 0 and 1, where a number close to 0 is extremely circular, and a number close to 1 is more elongated or flatter. Eccentricity is calculated by $e = \frac{c}{a}$.

Eccentricity

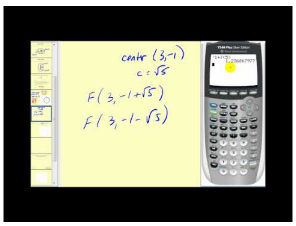
$e = \frac{c}{a}$, where a is the length of the semi-major axis and $c = \sqrt{a^2 - b^2}$.

Ellipses also have two directrix lines that correspond to each focus, but on the outside of the ellipse. The distance from the center of the ellipse to each directrix line is $x = \frac{a^2}{c}$.

Directrix line

$x = \frac{a^2}{c}$, where a is the length of the semi-major axis and $c = \sqrt{a^2 - b^2}$.

The following video defines an ellipse and explains how to graph an ellipse in standard form:



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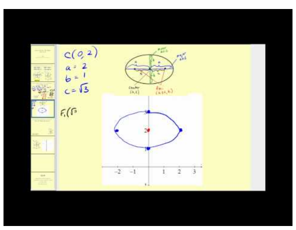
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/61865

The following video explains how to graph an ellipse in general form:



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/61867

Play, Learn, and Explore Ellipses: www.ck12.org/a/2102374 .

Examples

Example 1

Find the endpoints of the major axis, foci, and eccentricity of the following ellipse:

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

Solution:

The center of this ellipse is at (0, 0). The semi-major axis is horizontal with $a = 5$. This means that the endpoints are at (5, 0) and (-5, 0). The semi-minor axis is vertical with $b = 4$.

$$\begin{aligned} 25 - 16 &= c^2 \\ 3 &= c \end{aligned}$$

The focal radius is 3. This means that the foci are at (3, 0) and (-3, 0).

The eccentricity is $e = \frac{3}{5}$.

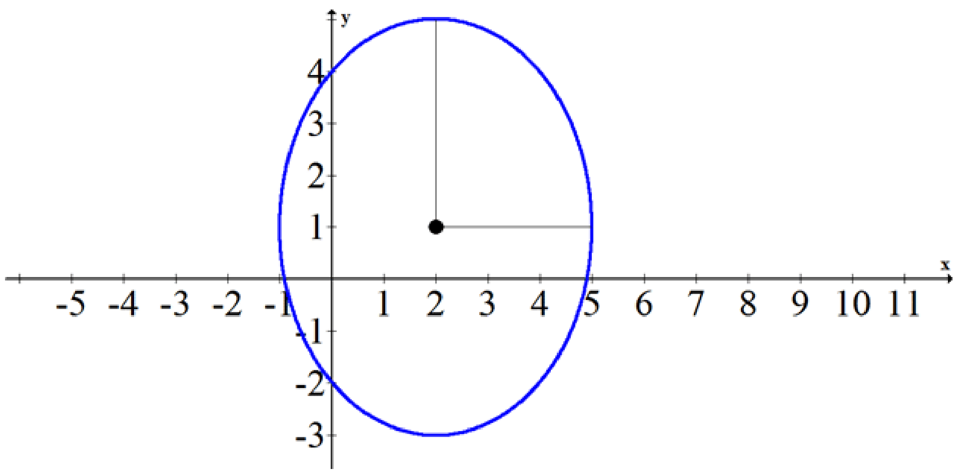
Example 2

Sketch the following ellipse:

$$\frac{(y-1)^2}{16} + \frac{(x-2)^2}{9} = 1.$$

Solution:

Plotting the foci is usually important, but in this case the question simply asks you to sketch the ellipse. All you need are the center and the lengths of the semi-major and semi-minor axes. The center is $(2, 1)$ with $a = 4$ and $b = 3$.



Example 3

Put the following conic into graphing form:

$$25x^2 - 150x + 36y^2 + 72y - 639 = 0.$$

Solution:

Complete the square to put in graphing form.

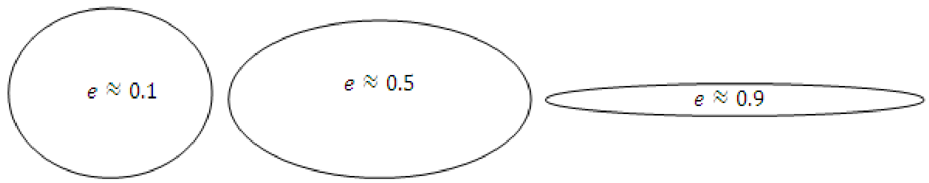
$$\begin{aligned} 25x^2 - 150x + 36y^2 + 72y - 639 &= 0 \\ 25(x^2 - 6x) + 36(y^2 + 2y) &= 639 \\ 25(x^2 - 6x + 9) + 36(y^2 + 2y + 1) &= 639 + 225 + 36 \\ 25(x-3)^2 + 36(y+1)^2 &= 900 \\ \frac{25(x-3)^2}{900} + \frac{36(y+1)^2}{900} &= \frac{900}{900} \\ \frac{(x-3)^2}{36} + \frac{(y+1)^2}{25} &= 1 \end{aligned}$$

Example 4

Recall the problem from the Introduction: How do you measure and graph an ellipse?

Solution:

Ellipses are measured using their eccentricity. Here are three ellipses with estimated eccentricity for you to compare.



Eccentricity is the ratio of the focal radius to the semi-major axis: $e = \frac{c}{a}$.

You can graph ellipses using the foci, axes, and center.

Example 5

Find the endpoints of the major axis, foci, and eccentricity of the following ellipse:

$$\frac{(x-2)^2}{4} + \frac{(y+1)^2}{16} = 1.$$

Solution:

The center of the ellipse is at (2, -1). The major axis is vertical, which means the semi-major axis is $a = 4$. The endpoints are (2, 3) and (2, -5).

$$\begin{aligned} 16^2 - 4^2 &= c^2 \\ 4\sqrt{15} = \sqrt{240} &= c \end{aligned}$$

Thus, the foci are $(2, -1 + 4\sqrt{15})$ and $(2, -1 - 4\sqrt{15})$.

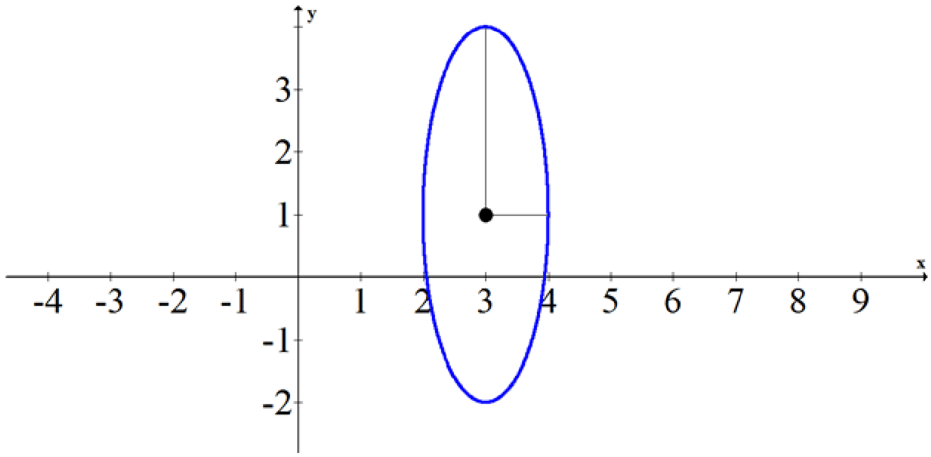
The eccentricity is $e = \frac{c}{a} = \frac{4\sqrt{15}}{4} = \sqrt{15}$.

Example 6

Sketch the following ellipse:

$$(x-3)^2 + \frac{(y-1)^2}{9} = 1.$$

Solution:



Example 7

Convert the following conic into graphing form:

$$9x^2-9x+4y^2+12y+\frac{9}{4}=-8.$$

Solution:

Complete the square to convert.

$$\begin{aligned}
9x^2-9x+4y^2+12y+\frac{9}{4}&=-8\\
9x^2-9x+\frac{9}{4}+4y^2+12y&=-8\\
9\left(x^2-x+\frac{1}{4}\right)+4(y^2+3y)&=-8\\
9\left(x-\frac{1}{2}\right)^2+4\left(y^2+3y+\frac{9}{4}\right)&=-8+4\cdot\frac{9}{4}\\
9\left(x-\frac{1}{2}\right)^2+4\left(y+\frac{3}{2}\right)^2&=1\\
\frac{(x-\frac{1}{2})^2}{\frac{1}{9}}+\frac{(y+\frac{3}{2})^2}{\frac{1}{4}}&=1
\end{aligned}$$

Summary

- An **ellipse** is the collection of points whose sum of distances from two foci is constant.
- The **foci** in an ellipse are the two points that the ellipse curves around.
- **Eccentricity** is a measure of how oval or how circular the shape is. It is the ratio of the focal radius to the semi-major axis: $e=\frac{c}{a}$.
- The **major axis** of an ellipse is its longest diameter: a line segment that runs through the center and both foci, with ends at the widest points of the perimeter.
- The **semi-major axis** is one half of the major axis. It runs from the center, through a focus, and to the endpoints. It is the longest radius of an ellipse.
- The **semi-minor axis** is the shortest radius of an ellipse.

Review

Find the vertices, foci, and eccentricity for each of the following ellipses:

1. $\frac{(x-1)^2}{4}+\frac{(y+5)^2}{16}=1$
2. $\frac{(x+1)^2}{9}+\frac{(y+2)^2}{4}=1$
3. $(x-2)^2+\frac{(y-1)^2}{4}=1$

Sketch each of the ellipses below. (Note they are the same as the ellipses in 1-3 above.)

4. $\frac{(x-1)^2}{4}+\frac{(y+5)^2}{16}=1$
5. $\frac{(x+1)^2}{9}+\frac{(y+2)^2}{4}=1$
6. $(x-2)^2+\frac{(y-1)^2}{4}=1$

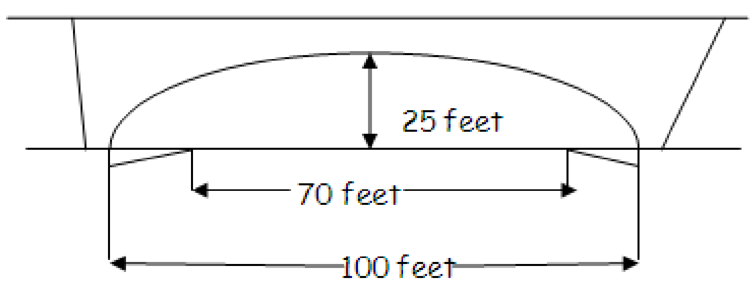
Put each of the following equations into graphing form:

7. $x^2+2x+4y^2+56y+197=16$
8. $x^2-8x+9y^2+18y+25=9$
9. $9x^2-36x+4y^2+16y+52=36$

Find the equation for each ellipse based on the description:

10. An ellipse with vertices (4, -2) and (4, 8), and minor axis of length 6.
11. An ellipse with minor axis from (4, -1) to (4, 3), and major axis of length 12.
12. An ellipse with minor axis from (-2, 1) to (-2, 7), and one focus at (2, 4).
13. An ellipse with one vertex at (6, -15), and foci at (6, 10) and (6, -14).

A bridge over a roadway is to be built with its bottom the shape of a semi-ellipse 100 feet wide and 25 feet high at the center. The roadway is to be 70 feet wide.



14. Find one possible equation of the ellipse that models the bottom of the bridge.
15. What is the clearance between the roadway and the overpass at the edge of the roadway?

Review (Answers)

Please see the Appendix.

11.6 Hyperbolas

Learning Objectives

Learn to translate conic equations into graphing form and graph hyperbolas.

Introduction

Hyperbolas can be oriented so they open side to side or up and down. How can we determine the direction of the opening of a hyperbola by the equation? Hyperbolas are relations that have asymptotes. When graphing rational functions, you often produce a hyperbola. In this concept, hyperbolas will not be oriented in the same way as with rational functions, but the basic shape of a hyperbola will still be there.

The Hyperbola

Recall that a hyperbola is generated when a plane intersects both nappes of the cone. As a result, a hyperbola has two foci. For every point on the hyperbola, the difference of the distances to each foci is constant.

Standard Equation of a Hyperbola

The graphing form of a hyperbola that opens side to side is

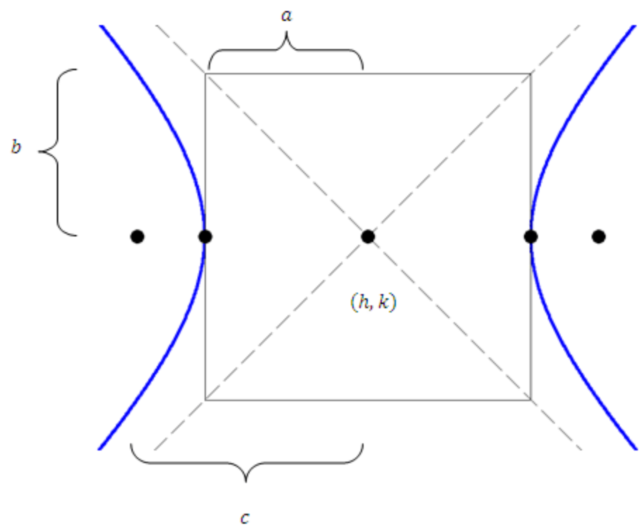
$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1,$$

where (h,k) is the center of the hyperbola, a is the semi-major axis, and b is the semi-minor axis.
A hyperbola that opens up and down is

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1,$$

where (h,k) is the center of the hyperbola, a is the semi-major axis, and b is the semi-minor axis.

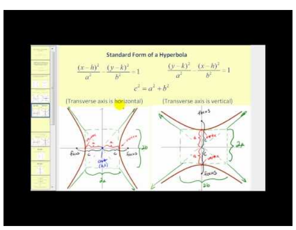
Notice that for hyperbolas, a goes with the positive term and b goes with the negative term. It does not matter which constant is larger.



When graphing, the constants a and b enable you to draw a rectangle around the center. The **transverse axis** travels from vertex to vertex and has length $2a$. The conjugate axis travels perpendicular to the transverse axis through the center and has length $2b$. The foci lie beyond the vertices so the eccentricity, which is measured as $e = \frac{c}{a}$, is larger than 1 for all hyperbolas. Hyperbolas also have two directrix lines that are $\frac{a^2}{c}$ away from the center (not shown on the image).

The focal radius is given by c where $a^2 + b^2 = c^2$.

The following video defines a hyperbola and explains how to graph a hyperbola given in standard form:



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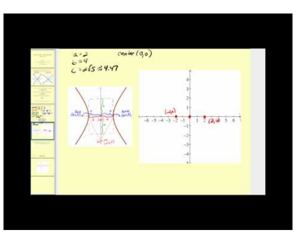
URL: http://www.ck12.org/flx/render/embeddedobject/61869

The following video explains how to graph a hyperbola in general form:



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URL: http://www.ck12.org/flx/render/embeddedobject/61871

Play, Learn, and Explore Hyperbolas: www.ck12.org/a/2104561

Examples

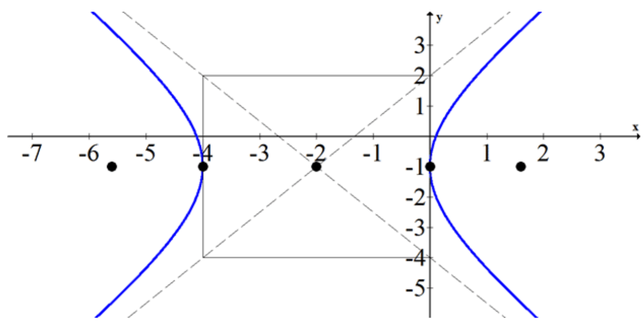
Example 1

Put the following hyperbola into graphing form and sketch it:

$$9x^2 - 4y^2 + 36x - 8y - 4 = 0.$$

Solution:

$$\begin{aligned}
9(x^2 + 4x) - 4(y^2 + 2y) &= 4 \\
9(x^2 + 4x + 4) - 4(y^2 + 2y + 1) &= 4 + 36 - 4 \\
9(x+2)^2 - 4(y+1)^2 &= 36 \\
\frac{(x+2)^2}{4} - \frac{(y+1)^2}{9} &= 1
\end{aligned}$$



Example 2

Find the equation of the hyperbola with foci at (-3, 5) and (9, 5) and asymptotes with slopes of $\pm\frac{4}{3}$.

Solution:

The center is between the foci at (3, 5). The focal radius is $c = 6$. The slope of the asymptotes is always the rise over run inside the box. In this case, since the hyperbola is horizontal and a is in the x direction, the slope is $\frac{b}{a}$. This makes a system of equations.

$$\begin{aligned}
\frac{b}{a} &= \pm\frac{4}{3} \\
a^2 + b^2 &= 6^2
\end{aligned}$$

When you solve, you get $a = 3.6$ and $b = 4.8$.

$$\frac{(x-3)^2}{\frac{324}{25}} - \frac{(y-5)^2}{\frac{576}{25}} = 1$$

Example 3

Find the equation of the conic that has a focus point at (1, 2), a directrix at $x = 5$, and an eccentricity equal to $\frac{3}{2}$. Use the property that the distance from a point on the hyperbola to the focus is equal to the eccentricity times the distance from that same point to the directrix:

$$\overline{PF} = e\overline{PD}.$$

Solution:

This relationship bridges the gap between ellipses, which have eccentricity less than 1, and hyperbolas, which have eccentricity greater than 1. When eccentricity is equal to 1, the shape is a parabola.

$$\sqrt{(x-1)^2+(y-2)^2} = \frac{3}{2}\sqrt{(x-5)^2}$$

Square both sides and rearrange terms so that it becomes a hyperbola in graphing form.

$$\begin{aligned}
x^2-2x+1+(y-2)^2 &= \frac{9}{4}(x^2-10x+25) \\
x^2-2x+1-\frac{9}{4}x^2+\frac{90}{4}x-\frac{225}{4}+(y-2)^2 &= 0 \\
4x^2-8x+4-9x^2+90x-225+4(y-2)^2 &= 0 \\
-5x^2+82x-221+4(y-2)^2 &= 0 \\
-5\left(x^2+\frac{82}{5}x+\frac{1681}{25}\right)+4(y-2)^2 &= 221-\frac{1681}{5} \\
-5\left(x+\frac{41}{5}\right)^2+4(y-2)^2 &= -\frac{576}{5} \\
\frac{25\left(x+\frac{41}{5}\right)^2}{576}-\frac{5(y-2)^2}{144} &= 1
\end{aligned}$$

Example 4

Recall the problem from the Introduction: How can we determine the direction of the opening of a hyperbola by the equation?

Solution:

Consider the hyperbola $x^2 - y^2 = 1$. This hyperbola opens side to side because x can never be equal to zero. This example demonstrates that when the coefficient of the y value is negative, the hyperbola opens up side to side.

Example 5

Identify the shape, center, foci, vertices, equations of asymptotes, and equations of directrices of the following conic:

$$9x^2-16y^2-18x+96y+9=0.$$

Solution:

$$9x^2-16y^2-18x+96y+9=0$$

$$9(x^2-2x)-16(y^2-6y)=-9$$
$$9(x^2-2x+1)-16(y^2-6y+9)=-9+9-144$$
$$9(x-1)^2-16(y-3)^2=-144$$
$$-\frac{(x-1)^2}{16}+\frac{(y-3)^2}{9}=1$$

Shape: Hyperbola that opens vertically

Center: (1, 3)

$a=3$

$b=4$

$c=5$

$e=\frac{c}{a}=\frac{5}{3}$

$d=\frac{a^2}{c}=\frac{9}{5}$

Foci: (1, 8), (1, -2)

Vertices: (1, 6), (1, 0)

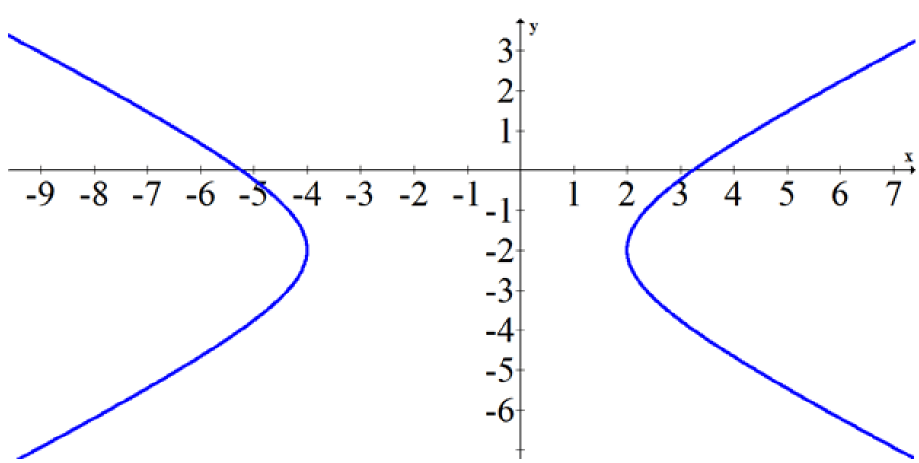
Equations of asymptotes: $(x-1)=\pm\frac{3}{4}(y-3)$

Note that it is easiest to write the equations of the asymptotes in point-slope form using the center and the slope.

Equations of directrices: $y=3\pm\frac{9}{5}$

Example 6

Given the graph below, estimate the equation of the conic.



Solution:

Since exact points are not marked, you will need to estimate the slope of asymptotes to get an approximation for a and b. The slope seems to be about $\pm\frac{2}{3}$. The center seems to be at (-1, -2). The transverse axis is 6, which means $a=3$.

$\frac{(x+1)^2}{9}-\frac{(y+2)^2}{4}=1$

Example 7

Find the equation of the hyperbola that has foci at (13, 5) and (-17, 5), with asymptote slopes of $\pm\frac{3}{4}$.

Solution:

The center of the conic must be at (-2, 5). The focal radius is $c = 15$. The slopes of the asymptotes are $\pm\frac{3}{4} = \frac{b}{a}$.

$a^2 + b^2 = c^2$

Since 3, 4, 5 is a well-known Pythagorean triple, it should be clear to you that $a = 12, b = 9$.

$\frac{(x+2)^2}{12^2} - \frac{(y-5)^2}{9^2} = 1$.

Summary

- A hyperbola is the collection of points that share a constant difference between the distances between two focus points.
- **Eccentricity** is the ratio between the length of the focal radius and the length of the semi-transverse axis. For hyperbolas, the eccentricity is greater than 1.
- The graphing form of a hyperbola that opens side to side is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.
- The graphing form of a hyperbola that opens up and down is $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.

Review

Use the following equation for 1-5: $x^2 + 2x - 4y^2 - 24y - 51 = 0$.

1. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.

2. Identify whether the hyperbola opens side to side or up and down.

3. Find the location of the vertices.

4. Find the equations of the asymptotes.

5. Sketch the hyperbola.

Use the following equation for 6-10: $-9x^2 - 36x + 16y^2 - 32y - 164 = 0$.

6. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.

7. Identify whether the hyperbola opens side to side or up and down.

8. Find the location of the vertices.

9. Find the equations of the asymptotes.

10. Sketch the hyperbola.

Use the following equation for 11-15: $x^2 - 6x - 9y^2 - 54y - 81 = 0$.

11. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.

12. Identify whether the hyperbola opens side to side or up and down.

13. Find the location of the vertices.

14. Find the equations of the asymptotes.

15. Sketch the hyperbola.

Review (Answers)

Please see the Appendix.

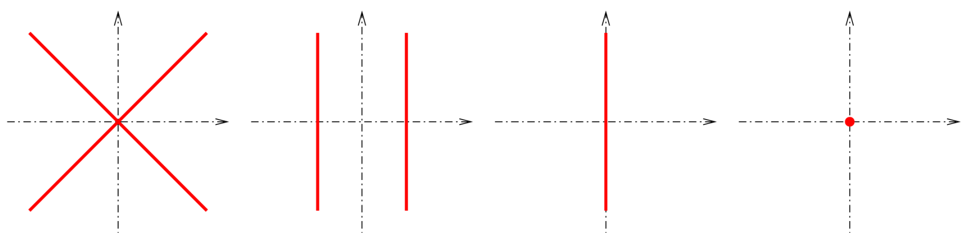
11.7 Degenerate Conics

Learning Objectives

Learn what happens when a conic equation can't be put into graphing form.

Introduction

The general equation of a conic is $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$. This form is so general that it encompasses all regular lines and curves, singular points, and degenerate hyperbolas that look like an X. There are a few special cases of how a plane can intersect a cone. How are these degenerate shapes formed?

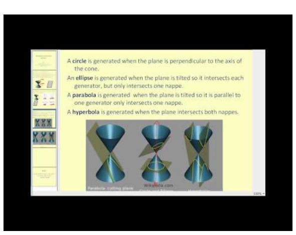


Degenerate Conics

A **degenerate conic** is generated when a plane intersects the vertex of the cone. There are three types of degenerate conics:

1. The degenerate form of a circle or an ellipse is a **singular point**. At the vertex of the cone, the radius is 0, $r=0$. Thus, the standard equation is $\frac{(x-h)^2}{a}+\frac{(y-k)^2}{b}=0$.
2. The degenerate form of a parabola is a **line** or **two parallel lines**. For this conic section, the coefficients $A=B=C=0$ in the general equation. Thus, the resulting general equation is $Dx+Ey+F=0$.
3. The degenerate form of a hyperbola is **two intersecting lines**. At the vertex of the cone, $r=0$, so the standard equation is $\frac{(x-h)^2}{a}-\frac{(y-k)^2}{b}=0$. Thus, the slopes of the intersecting lines are $m=\pm\frac{b}{a}$, because b corresponds to the y portion of the equation, and a corresponds to the x portion of the equation.

The following video reviews the four conic sections: circle, ellipse, parabola, and hyperbola:



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Examples

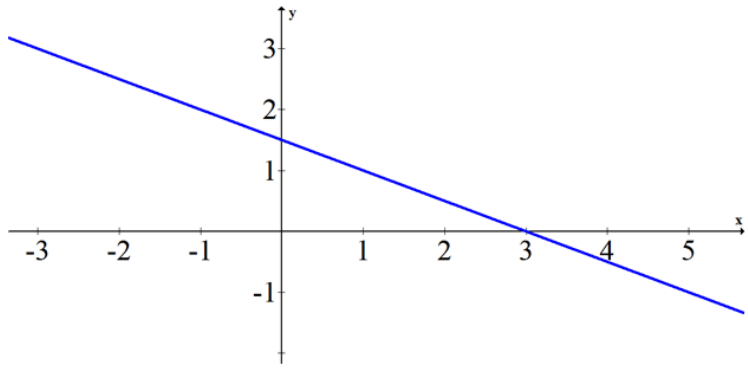
Example 1

Transform the conic equation into standard form and sketch.

$$0x^2 + 0xy + 0y^2 + 2x + 4y - 6 = 0$$

Solution:

This is the line $y = -\frac{1}{2}x + \frac{3}{2}$.



Example 2

Transform the conic equation into standard form and sketch.

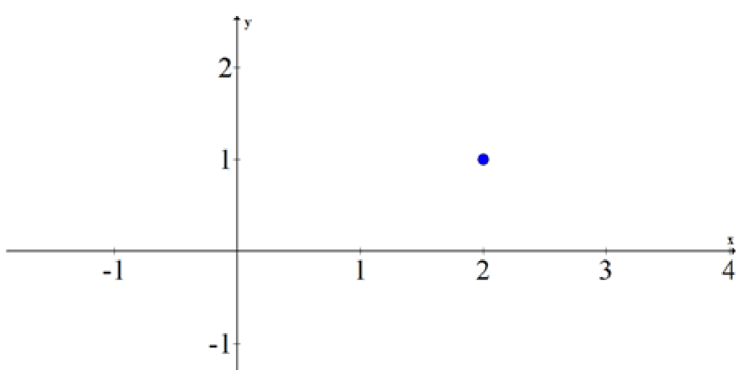
$$3x^2 - 12x + 4y^2 - 8y + 16 = 0$$

Solution:

$$3x^2 - 12x + 4y^2 - 8y + 16 = 0$$

$$\begin{aligned} 3(x^2 - 4x) + 4(y^2 - 2y) &= -16 \\ 3(x^2 - 4x + 4) + 4(y^2 - 2y + 1) &= -16 + 12 + 4 \\ 3(x-2)^2 + 4(y-1)^2 &= 0 \\ \frac{(x-2)^2}{4} + \frac{(y-1)^2}{3} &= 0 \end{aligned}$$

The point (2, 1) is the result of this degenerate conic.



Example 3

Transform the conic equation into standard form and sketch.

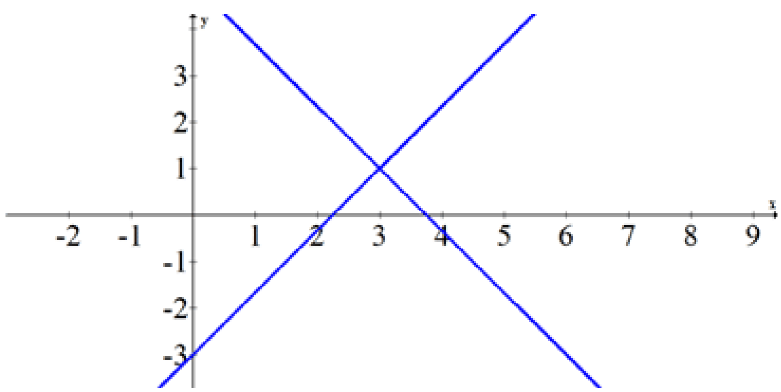
$$16x^2-96x-9y^2+18y+135=0$$

Solution:

$$16x^2-96x-9y^2+18y+135=0$$

$$\begin{aligned}16(x^2-6x)-9(y^2-2y)&=-135\\16(x^2-6x+9)-9(y^2-2y+1)&=-135+144-9\\16(x-3)^2-9(y-1)^2&=0\\\frac{(x-3)^2}{9}-\frac{(y-1)^2}{16}&=0\end{aligned}$$

This is a degenerate hyperbola.



Example 4

Recall the question from the Introduction: How are these degenerate shapes formed?

Solution:

When you intersect a plane with a cone at the vertex where the two cones touch, the intersection is a single point. When you intersect a plane with the edge of one cone, passing through the vertex point, and continuing to touch the edge of the other conic, this produces a line or two parallel lines. When you intersect a plane with a cone so that the plane passes vertically through the vertex, it produces two intersecting lines.

Example 5

Create a conic that describes just the point (4, 7).

Solution:

$$(x-4)^2+(y-7)^2=0$$

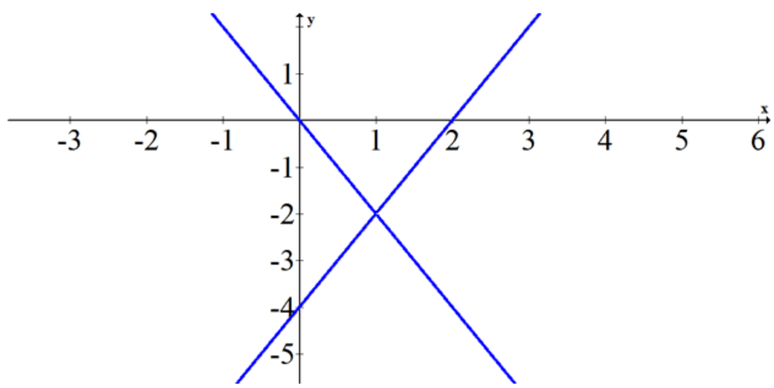
Example 6

Transform the conic equation into standard form and sketch.

$$-4x^2+8x+y^2+4y=0$$

Solution:

$$\begin{aligned}
-4x^2+8x+y^2+4y&=0\\
-4(x^2-2x)+(y^2+4y)&=0\\
-4(x^2-2x+1)+(y^2+4y+4)&=-4+4\\
-4(x-1)^2+(y+2)^2&=0\\
\frac{(x-1)^2}{1}-\frac{(y+2)^2}{4}&=0
\end{aligned}$$



Example 7

How can you tell just by looking at a conic in general form if it is a degenerate conic?

Solution:

In general, you cannot tell if a conic is degenerate from the general form of the equation. You can tell that the degenerate conic is a line if there are no x^2 or y^2 terms. However, you should always try to put the conic equation into graphing form to see whether it equals zero, because that is the best way to identify degenerate conics.

Summary

- A **degenerate conic** is generated when a plane intersects the vertex of the cone.
- There are three types of degenerate conics: a single point, a line or two parallel lines, or two intersecting lines.

Review

1. What are the three degenerate conics?

Change each equation into graphing form and state what type of conic or degenerate conic it is:

2. $x^2 - 6x - 9y^2 - 54y - 72 = 0$
3. $4x^2 + 16x - 9y^2 + 18y - 29 = 0$
4. $9x^2 + 36x + 4y^2 - 24y + 72 = 0$
5. $9x^2 + 36x + 4y^2 - 24y + 36 = 0$
6. $0x^2 + 5x + 0y^2 - 2y + 1 = 0$
7. $x^2 + 4x - y + 8 = 0$
8. $x^2 - 2x + y^2 - 6y + 6 = 0$
9. $x^2 - 2x - 4y^2 + 24y - 35 = 0$
10. $x^2 - 2x + 4y^2 - 24y + 33 = 0$

Sketch each conic or degenerate conic:

11. $\frac{(x+2)^2}{4} + \frac{(y-3)^2}{9} = 0$
12. $\frac{(x-3)^2}{9} + \frac{(y+3)^2}{16} = 1$
13. $\frac{(x+2)^2}{9} - \frac{(y-1)^2}{4} = 1$
14. $\frac{(x-3)^2}{9} - \frac{(y+3)^2}{4} = 0$
15. $3x + 4y = 12$

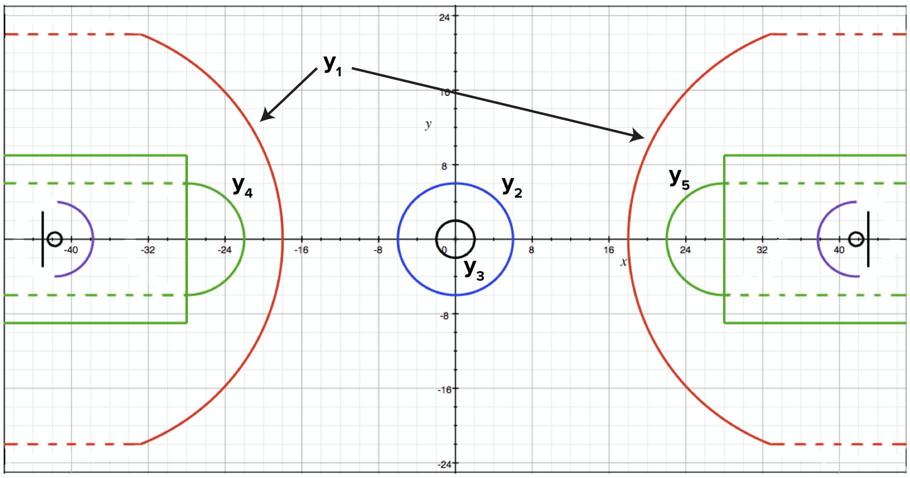
Review (Answers)

Please see the Appendix.

11.8 Project: Conics

Conics Project

Use your knowledge of conics to recreate the following rendering of this basketball court:



Step 1: Identify the type of conic for each relation. For y_1, use the conic that most closely matches both parts of the solid curve. For y_4 and y_5, note what part of a conic you are using.

Step 2: Determine the domain and range for each relation.

Step 3: Match each relation with the general equation.

Step 4: Determine key points for each conic (center, foci, vertex, etc.).

Step 5: Develop the equations for each part of the graph.

Step 6: Enter your equations into graphing software to check your equations.

11.9 Summary: Conics

In this chapter, you learned that a conic section is the family of shapes that are formed by the different ways a flat plane intersects a two-sided cone in three-dimensional space. Parabolas, circles, ellipses, and hyperbolas each have precise definitions that are important to their shapes, as well as equations that can be written in a form that provides key information for sketching and graphing. Each conic has important points and measurements. Relationships between these key numbers are used to understand how the conic is formed, create equations, and sketch the graphs of the equations.

Chapter Summary

- Completing the square is a procedure that enables you to combine squared and linear terms of the same variable into a perfect square of a binomial.
- Conics are a family of graphs (not usually functions) that come from the same general equation. This family is the intersection of a double cone and a plane in three-dimensional space.
- The standard form of a conic is $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.

- A parabola is the collection of points that are equidistant from a fixed focus and directrix.
- The focus of a parabola is the point that the parabola seems to curve around.
- The directrix of a parabola is the line that the parabola seems to curve away from.
- The general equation for a parabola opening vertically is $(x-h)^2 = \pm 4p(y-k)$.
- The general equation for a parabola opening horizontally is $(y-k)^2 = \pm 4p(x-h)$.

- A circle is the collection of points that are equidistant from a given point.
- The radius of a circle is the distance from the center of the circle to the outside edge.
- The center of a circle is the point that defines the location of the circle.
- The general equation of a circle is $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center of the circle and r is the radius.

- An ellipse is the collection of points whose sum of distances from two foci is constant.
- The foci in an ellipse are the two points that the ellipse curves around.
- Eccentricity is a measure of how oval or how circular the shape is. It is the ratio of the focal radius to the semi major axis: $e=\frac{c}{a}$.
- The major axis of an ellipse is its longer line segment that runs through the center and both foci, with ends at the widest endpoints.
- The semi-major axis is one half of the major axis, and thus runs from the center, through a focus, and to the perimeter. It is the longest radius of an ellipse.
- The minor axis of an ellipse is its shorter line segment, which runs through the center, with ends at the more narrow endpoints.
- The semi-minor axis is the shortest radius of an ellipse.

- A hyperbola is the collection of points that share a constant difference between the distances between two focus points.
- Eccentricity is the ratio between the length of the focal radius and the length of the semi-transverse axis. For hyperbolas, the eccentricity is greater than 1.
- The graphing form of a hyperbola that opens side to side is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.
- The graphing form of a hyperbola that opens up and down is $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.

- A degenerate conic is generated when a plane intersects the vertex of the cone.
- There are three types of degenerate conics: a single point, a line or two parallel lines, or two intersecting lines.

Review

Try the following cumulative review problems to practice the concepts we studied in this chapter:

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Click image to the left or use the URL below.

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11.10 References

1. By ja.wiki user Tsukapee (Rendered using "POV-Ray") (Own work) [Public domain], via Wikimedia Commons. https://commons.wikimedia.org/wiki/File%3AConicSection_hyperbola.PNG .
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4. CK-12 Foundation. .

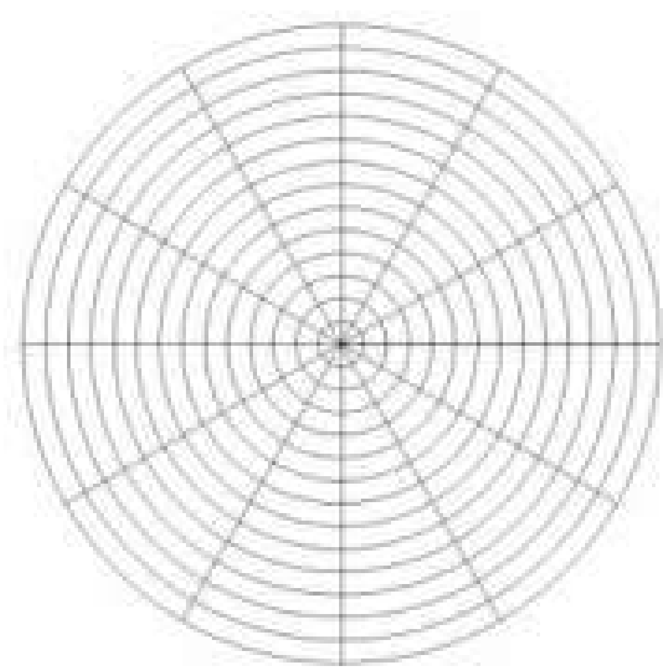
CHAPTER 12 Polar Coordinates and Parametric Equations

Chapter Outline

12.1 Introduction: Polar Coordinates and Parametric Equations

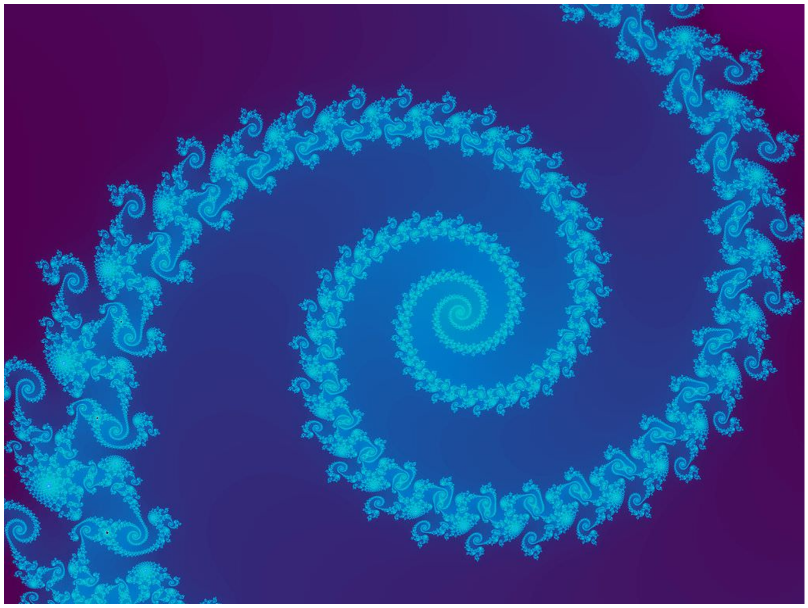
In earlier mathematics, graphing took place in the rectangular plane, also known as the Cartesian plane or the *xy*-plane. The study of this plane has an impact on the understanding of points as they move from arbitrary space into specific locations. The rectangular plane is extremely useful in measurement and placement, and in understanding distance, length, area, and the attributes of functions.

We'll now begin to explore a new idea in which the graph is no longer rectangular but circular. This type of graphing is called "polar," where all points are graphed in relation to a "pole," and concentric circles represent placements along the graph. The pole is the center of the graph, representing the starting point for the graph.



The coordinates of the points in polar form no longer represent a horizontal and vertical place, but now denote the distance from the pole and the measure of the angle formed. We can convert rectangular coordinates to polar coordinates and plot equations on the polar grid as well. In this chapter, we'll explore the types of equations and their resultant graphs in the polar coordinate system.

In addition, we will determine how complex numbers are converted to polar form. There are also special formulas for finding the product, quotient, and powers of complex numbers in polar form. These formulas have applications within fractal geometry and in physics.



Finally, this chapter will explore the modeling of movement through the use of parametric equations. Parameters are the independent variables in a set of equations that are being applied to situations like motion. While you may have already explored distance equals rate multiplied by time or $d = rt$, we need to add a third dimension to our description of motion: direction.

12.2 Polar Coordinate System

Learning Objectives

Learn to graph polar equations.

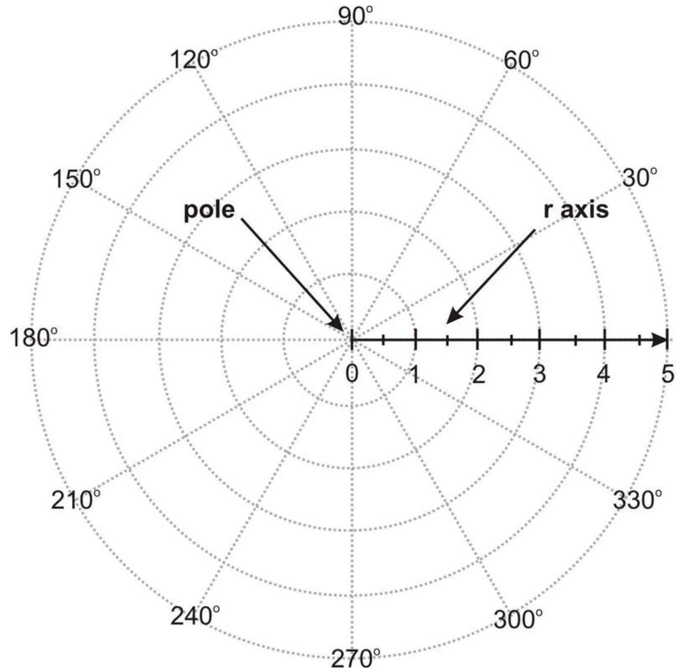
Introduction

When traveling a long distance from one place to another, walking, biking, or driving takes considerably more time than flying. One obvious reason is the faster speed of air travel, as well as the extra traffic on the roadways. Another reason is that the coordinate system that most roads are based upon limits the movement to north, south, east, and west. For instance, when walking or driving, you may go east two blocks, turn left, go north six blocks, and then have to wait for a train to pass. Then you may turn right, go east three more blocks, turn left, go north four more blocks, and then finally park. However, when flying you may fly 30^o east of north for a little less than $11\frac{1}{4}$ blocks, and then you land. Your flight is based upon the polar coordinate system.

Polar Coordinate System

The **polar coordinate system** is an alternative to the Cartesian (rectangular) coordinate system. Where the Cartesian system identifies position east and north of a fixed reference point (the origin), the polar coordinate system measures location using direction and distance from a fixed reference point (the pole).

Points in the polar coordinate system are given in the form (r, θ). The polar axis, or r-axis, is referring to the polar radius r. To plot a specific point, first find the point that is r units from the origin on the polar axis. Then, rotate counterclockwise by the given angle, commonly represented as θ. Be certain to use the correct units for the angle measure (either radians or degrees). Angles are identified by traveling counterclockwise around the circular graph from the 0° line, or polar axis (where the positive-axis would be), to a specified angle. Note that you can choose to travel θ° counterclockwise, or you can reach the same location by traveling $(360-\theta)^\circ$in the clockwise direction. For example, traveling 230° counterclockwise is the same as traveling 130° clockwise.

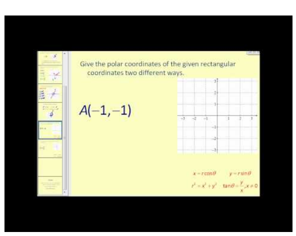


When using polar coordinates, the angle is usually stated using either degree measure or radian measure. On the Cartesian coordinate system, you can define a radian as the angle created by the *x*-axis and another radius of a circle centered at the origin where the arclength on the edge of the circle equals the radius. Similarly, a radian is the angle formed between the polar axis and the terminal side lying on the line passing through the point and the pole.

Graphing Using Technology

Polar equations can be graphed using online graphing software (e.g. Desmos or GeoGebra) or a graphing calculator. With the graphing calculator, go to **MODE**. There, select **RADIAN** for the angle measure, and **POL** (for polar) on the **FUNC** (function) line. When Y = is pressed, note that the equation has changed from y = to r = . There, input the polar equation. After pressing graph, if you can't see the full graph, adjust *x*- and *y*- max/min, etc. in **WINDOW.** In Desmos, choose **SETTINGS** (the icon that looks like a wrench). In the graph paper section, you can change the grid and axes. Click the green circular icons to choose between Cartesian and polar grids, and show or hide axes labels. You can also select whether you would like the graph to show radians or degrees at the bottom of this menu.

The video below distinguishes between the rectangular coordinate system and the polar coordinate system. It also demonstrates how to plot points using the polar coordinate system.



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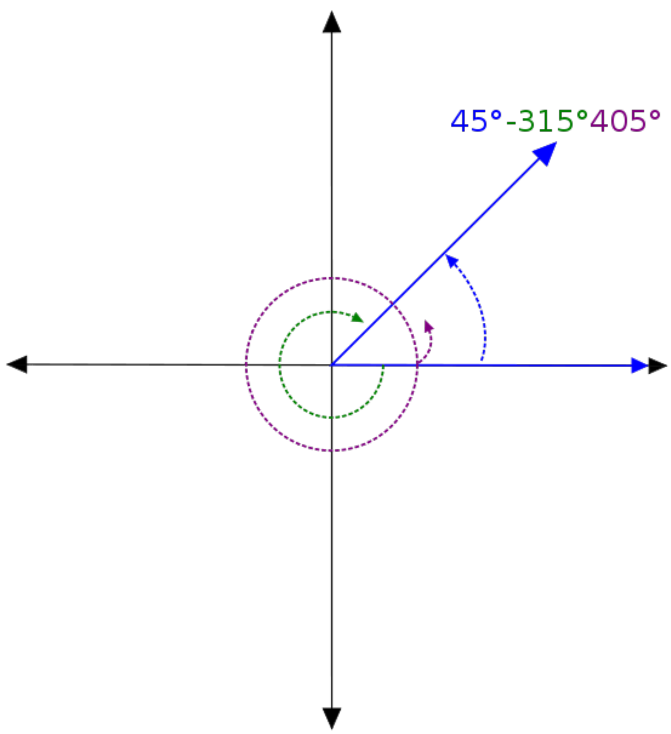
Multiple equivalent representations of (r, θ)

Angles can be represented in more than one way. For example, angles 35° and 395° are coterminal. In the polar plane, a point can be represented by more than one pair of polar coordinates.

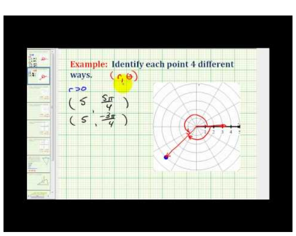
When (r, θ) and (r', θ') are polar coordinates where $r \neq 0$ and $r' \neq 0$, then (r, θ) and (r', θ') determine the same point P, if and only if one of the following is true:

- $r' = r$ and $\theta' = \theta + 2\pi k$ for some integer k.
- $r' = -r$ and $\theta' = \theta + (2k+1)\pi$ for some integer k.

The following illustrates that a 45° angle is equivalent to a -315° angle, as well as equivalent to a 405° angle on the polar coordinate plane:



The following video provides an example of the different ways to identify a point with polar coordinates using radians:



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URL: http://www.ck12.org/flx/render/embeddedobject/183686

Explore, Play, and Learn with Polar Coordinates: www.ck12.org/a/2173455 .

Examples

Example 1

Plot the points on a polar coordinate graph:

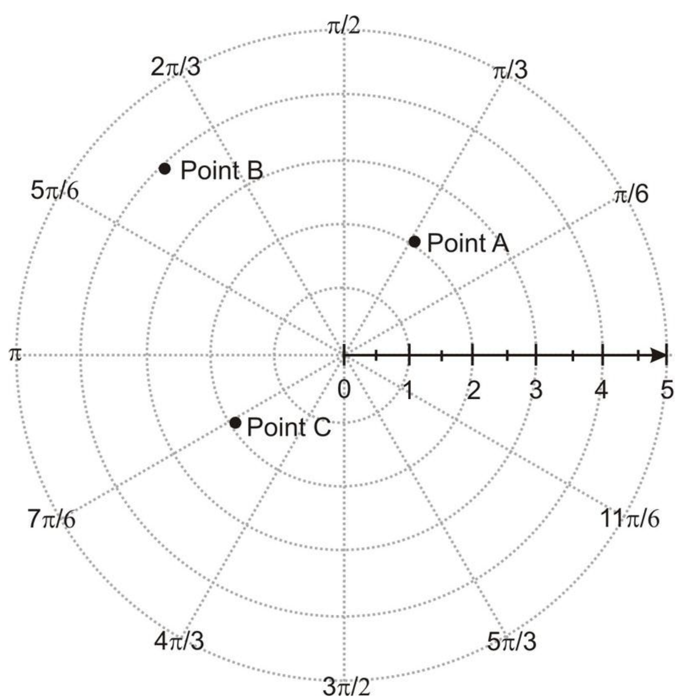
Point A $\left(2, \frac{\pi}{3}\right)$

Point B $(4, 135^o)$

Point C $\left(-2, \frac{\pi}{6}\right)$

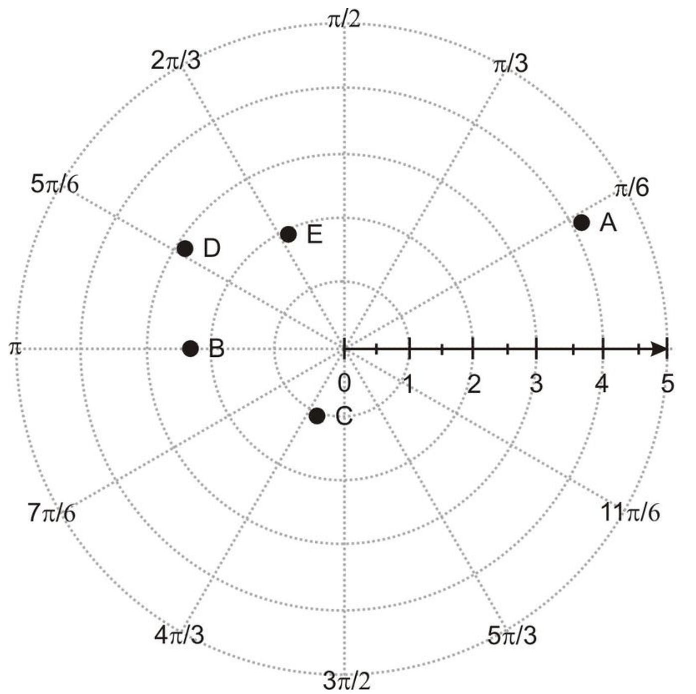
Solution:

Below is the pole, polar axis, and the points A, B, and C.



Example 2

Determine the coordinates of the following points using radians:



Solution:

A. $\left(4, \frac{\pi}{6}\right)$
B. $(2.5, \pi)$
C. $\left(1, \frac{4\pi}{3}\right)$
D. $\left(3, \frac{5\pi}{6}\right)$
E. $\left(2, \frac{2\pi}{3}\right)$

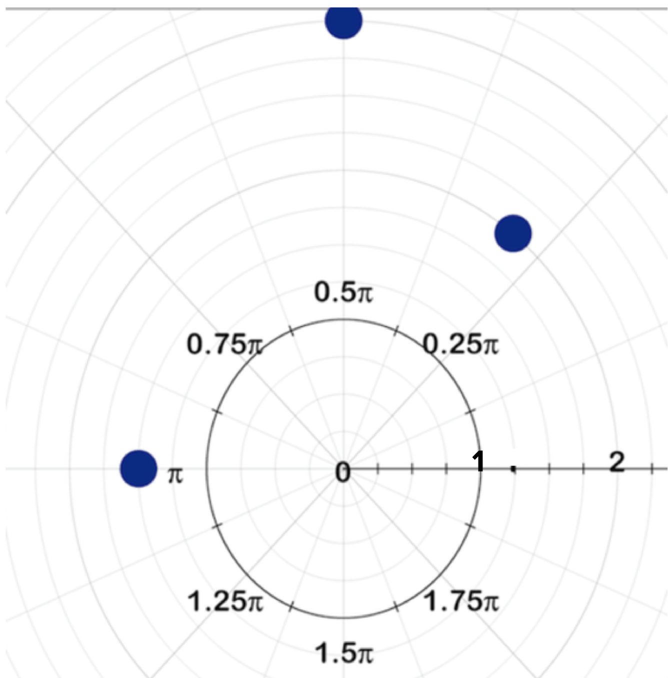
Example 3

Plot the points on a polar graph:

a) $\left(2, \frac{\pi}{3}\right)$
b) $(3, 90^o)$
c) $(1.5, \pi)$

Solution:

The points are plotted on the graph below:



Example 4

Recall the situation in the Introduction: Your flight is traveling 30° east of north for a little less than $11\frac{1}{4}$ blocks prior to landing. What is the polar coordinate point that best represents this situation? Graph this point.

Solution:

The polar radius for the flight is 11.25 blocks.

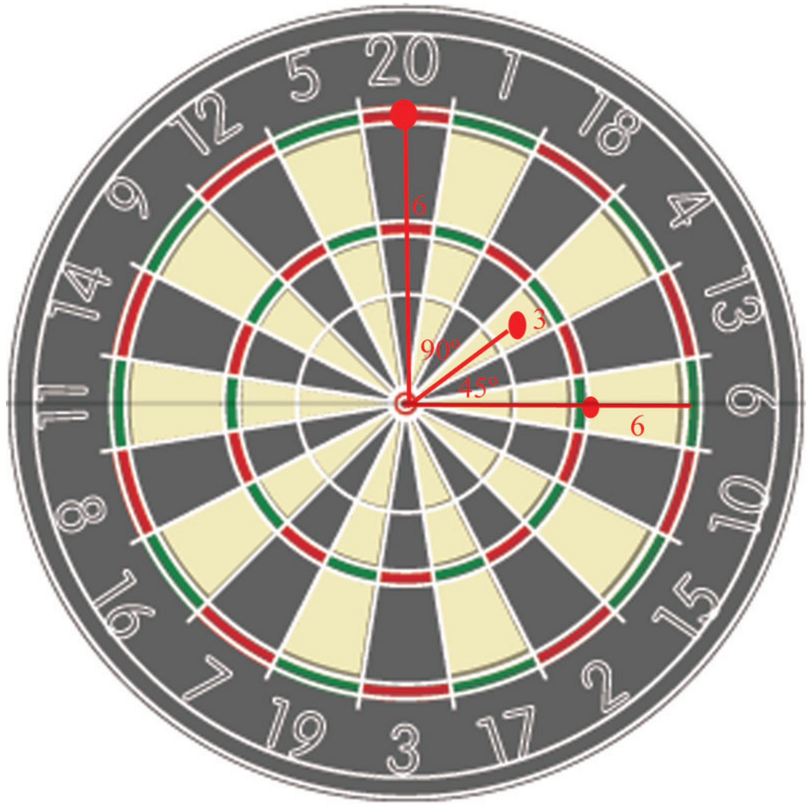
The polar axis corresponds to the east direction, and angles are identified by traveling counter-clockwise around the circular graph from the polar axis. The flight is traveling 30° east of north, which means the flight is 30° clockwise from north. To get the corresponding angle traveling counterclockwise from east (or the polar axis), subtract the 30° from 90°, because north is 90° from the polar axis.

$$\theta = 90^\circ - 30^\circ = 60^\circ$$

Thus, the polar coordinate is $(11.25, 60^\circ)$.

Example 5

While playing a game of darts with your friend, you decide to see if you can plot the coordinates of where your darts land. The dartboard looks like this:



Solution:

Since you have the positions of the darts on the board with both the distance from the origin and the angle they make with the horizontal, you can describe them using polar coordinates: $(4,0°), (3,45°)$, and $(6,90°)$.

Summary

- The **polar coordinate system** is a circular system used to visualize and plot angles.
- Coordinates in the polar system come in the form (r,θ), where r is the polar radius and θ represents the number of degrees or radians from the polar axis.

Review

1. Why can a point on the plane not be labeled using a unique ordered pair (r,θ)?
2. Explain how to graph (r,θ) if $r < 0$ and/or $\theta > 360°$.

Graph each point on the polar plane:

3. A $(6,145°)$
4. B $\left(-2,\frac{13\pi}{6}\right)$
5. C $\left(\frac{7}{4},-210°\right)$
6. D $\left(5,\frac{\pi}{2}\right)$
7. E $\left(3.5,\frac{-\pi}{8}\right)$

Name two other pairs of polar coordinates for each point:

8. $(1.5,170°)$

9. $\left(-5,-\frac{\pi}{3}\right)$
10. $(3,305^\circ)$
11. $\left(4,-\frac{5\pi}{6}\right)$

12. Suppose you are traveling north in a boat. A heavy wind moves the boat 15° west of north while traveling 500 m. What is the polar coordinate point that best represents this situation?
13. Graph the polar coordinate point from 12 above.
14. The planets move in a circular motion around the sun. The planet Mercury travels around the sun in an elliptical orbit given by

$$r=\frac{34{,}420{,}000}{1-0.206\cos\theta},$$

where r is measured in miles, and the sun is at the pole. Determine the polar radius for the following θ values:

   a. 30°
   b. −60°
   c. 135°

15. Determine the polar coordinate points for parts a-c from number 14.
16. Graph the polar coordinate points from 14 above.

Review (Answers)

Please see the Appendix.

12.3 Polar Equations

Learning Objectives

Learn to graph polar equations.

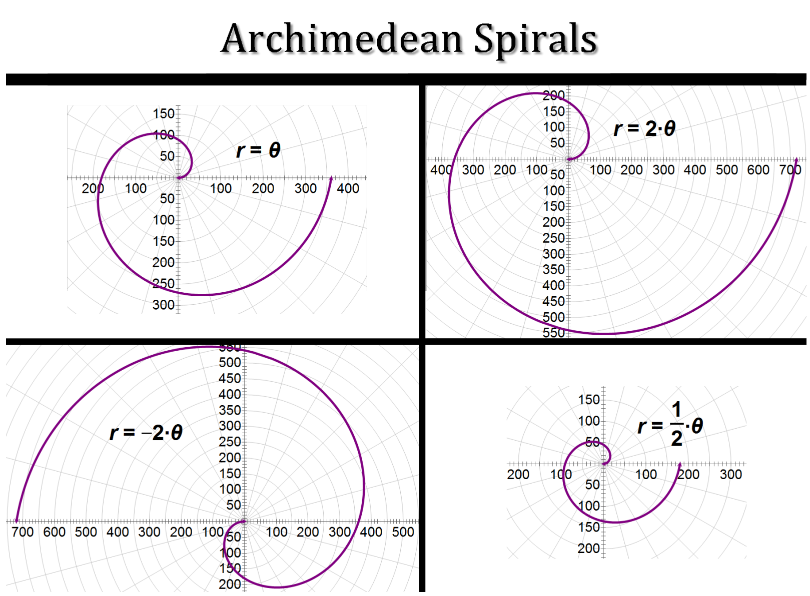
Introduction

Max is in charge of the sound system for his big sister's wedding reception. He can't get into the hall until the morning of the wedding, so he's going to have to set up his microphones and speakers in record time. He does, however, have a floor plan of the hall. How can he make sure the pickup areas of the microphones and the broadcast area of the speakers don't overlap and cause feedback? He'll have time for some minimal trial and error, but he needs to have a general idea of where all the microphones and speakers will be during the reception. Can he use polar coordinates to help him place his equipment?

Polar Equations

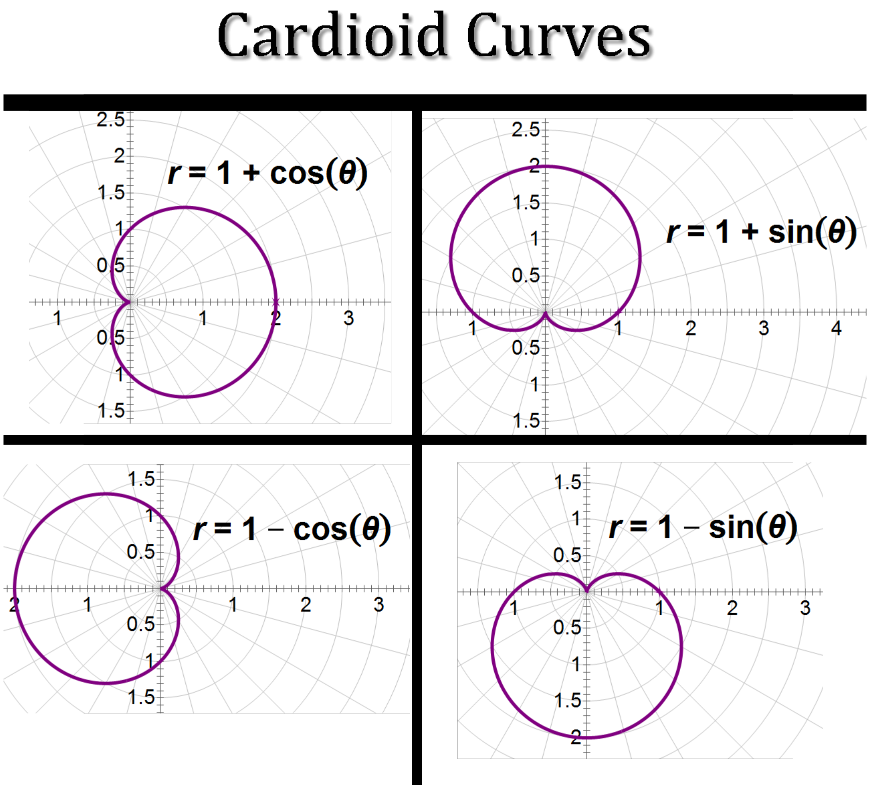
Why do people continue to use polar coordinates when modern computers are powerful and fast enough to solve extremely complicated problems in rectangular form? One reason is that many polar graphs are beautiful and intriguing. Polar graphs can help people see patterns they might otherwise overlook. Artists have even used polar graphs as the basis of their designs.

One of the simplest equations that forms a special polar curve is $r = a\theta$, where a is any real number and θ ranges from zero to infinity. Equations of this form create a shape known as an **Archimedean spiral**. As θ increases, the graph continues to spiral out like a perfect snail's shell. The graphs below (with the angles in degrees) demonstrate how changing the value of a alters the spiral. Note that each curve will continue to spiral forever.

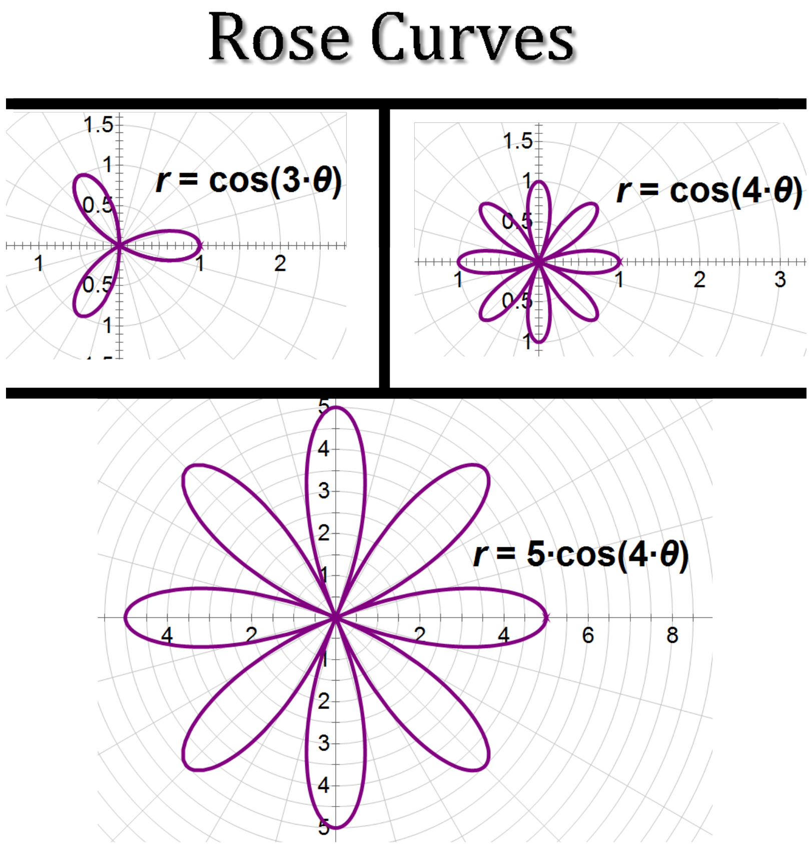


Another important polar curve is the **cardioid**. People who work with acoustics know that the cardioid is an accurate model for both the pickup range of certain types of microphones and the broadcast range for certain kinds of speakers.

Cardioids get their name from their heart-like shapes. Equations of the form $r = 1 + a\cos\theta$ produce cardioid curves. You can change the orientation of a cardioid, or of any other polar equation with cosine in its standard form, by replacing cosine with sine, negative cosine, or negative sine. Note that the graphs below include angles in radians.

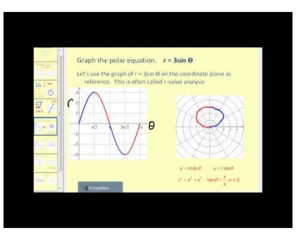


Rose curves are another interesting set of polar curves. For these, equations are of the form $r = a\cos n\theta$ or $r = a\sin n\theta$. If n is an integer, these equations will produce an n-petaled rose if n is odd, or a $2n$-petaled rose if n is even. If n is rational but not an integer, a rose-like shape may form, but with petals that overlap. The variable a represents the length of the petals of the rose. Once again, the angles are in radians.



Polar graphs can be created much like graphs of equations in the rectangular coordinate system. You can use a table of points, or you can use key information from the equations provided. Graphing software such as Desmos and GeoGebra as well as graphing calculators can help you produce both tables and graphs for polar equations.

The following video demonstrates how to graph basic polar equations:



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/183692

Examples

Example 1

Use graphing software, a calculator, or a plotting program to plot the following equation:

$$r = \frac{\theta}{2}.$$

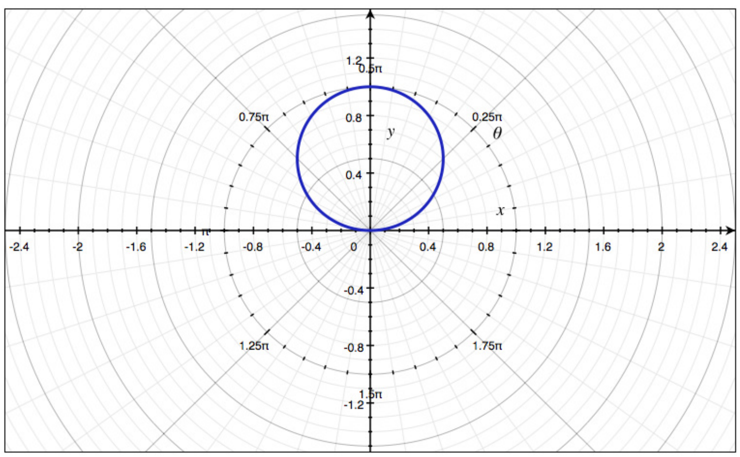
Solution:

Example 2

Graph by hand the following equation:

$$r = \sin \theta.$$

Solution:



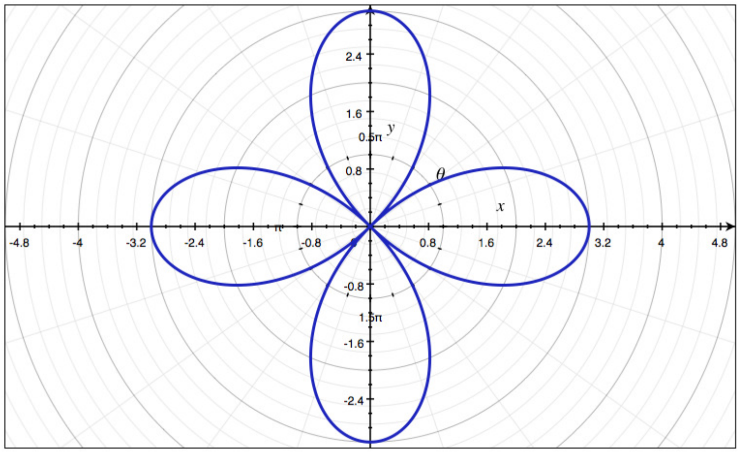
Depending on domain, the circle may be redrawn or retraced multiple times. The whole circle can be drawn when $0 \leq \theta \leq \pi$.

Example 3

Graph by hand the following equation:

$$r = 3 \cos 2\theta.$$

Solution:

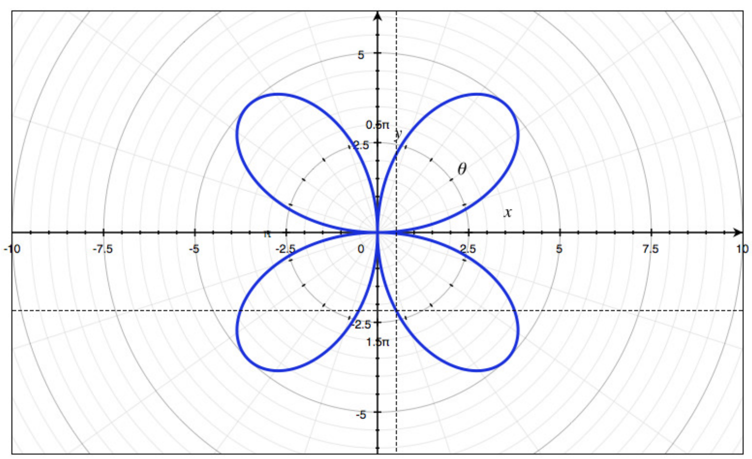


Example 4

Graph by hand the following equation:

$$r = 5 \sin 2\theta.$$

Solution:

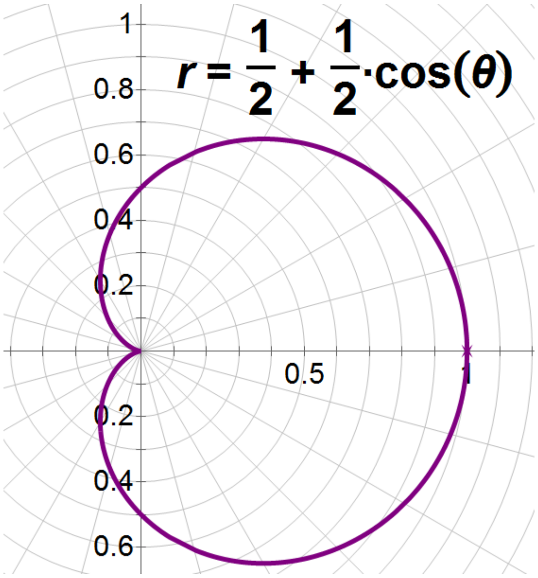


Example 5

Recall the problem from the Introduction: You were asked how Max can set up the sound system for his sister's wedding so the speakers don't overlap.

Solution:

Max can use cardioids to help him set up the sound system without causing feedback. Researching his microphones and speakers online, he finds out that the pickup pattern of his microphones can be graphed using the equation $r = \frac{1}{2} + \frac{1}{2}\cos\theta$. Since he has multiple microphones to place, he can graph the curve and use it with his floor plan to ensure that the mikes don't overlap, and to position the speakers in the dead zones behind the microphones, where they won't pick up any sound.



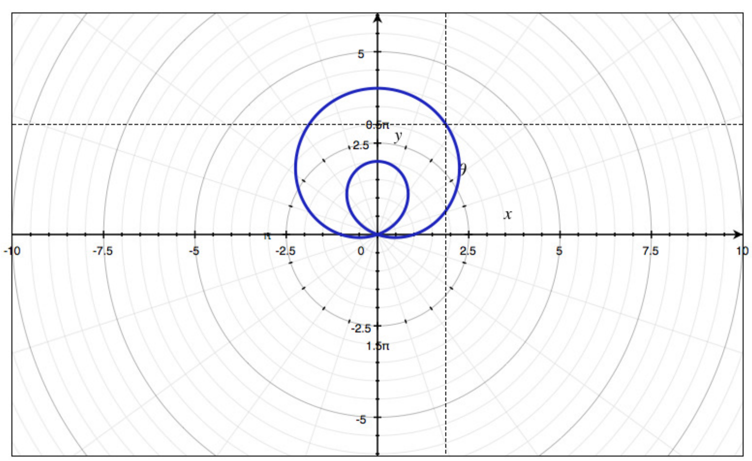
Since the microphones pick up along the polar axis between 0 and 1, he'll want to position speakers in the dead zone, where $r < 0$ and $\theta = 0$. He can also put speakers at other places in the dead zone, but cardioid microphones are least likely to pick up sound when they are πradians from their optimal pickup areas.

Example 6

Use graphing software, calculator, or plotting program to plot the following equation:

$$r = 1 + 3sin\theta.$$

Solution:

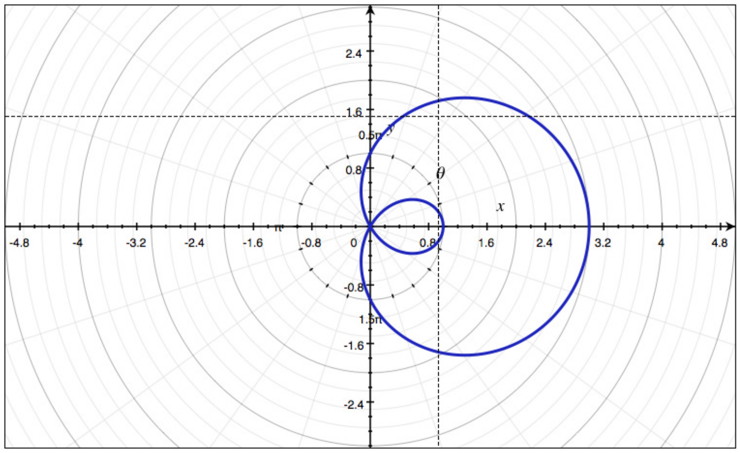


Example 7

Use graphing software, calculator, or plotting program to plot the following equation:

$$r = 1 + 2cos\theta.$$

Solution:



Summary

- Polar graphs can show important information and relationships.
- Tables and graphs can be created for polar equations using graphing software, calculators, or plotting programs.

Review

Plot the following equations:

1. $r = \sqrt{3}\cos\theta$
2. $r = 6\sin 3\theta$
3. $r = 2\sin\theta - 1$
4. $r = 2\cos\theta$
5. $r = 2\sin\theta$
6. $r = 4\cos\theta$
7. $r = 3\cos\theta - 1$
8. $r = 3\sin 2\theta$
9. $r = 3$
10. $r = 1.5$
11. $r = 1 - 4\cos\theta$
12. $r = \frac{1}{2} + \frac{1}{2}\sin(\theta)$
13. $r = -\theta$
14. $r = 5\theta$
15. $r = 1 - 2\sin\theta$

16. French mathematician Pierre de Fermat is credited with discovering the spiral defined by the equation $r^2 = \theta$. Graph this equation.

Review (Answers)

Please see the Appendix.

Resources

Graphing Basic Polar Equations

12.4 Polar and Cartesian Transformation

Learning Objectives

Learn to convert between polar and cartesian coordinates.

Introduction

In 1959, geographer Richard J. Chorley drew the conclusion that the shape of a drumlin (which is an elliptical, streamlined hill made of till found under glacial ice) could be modeled by $r = L\cos k\theta$.

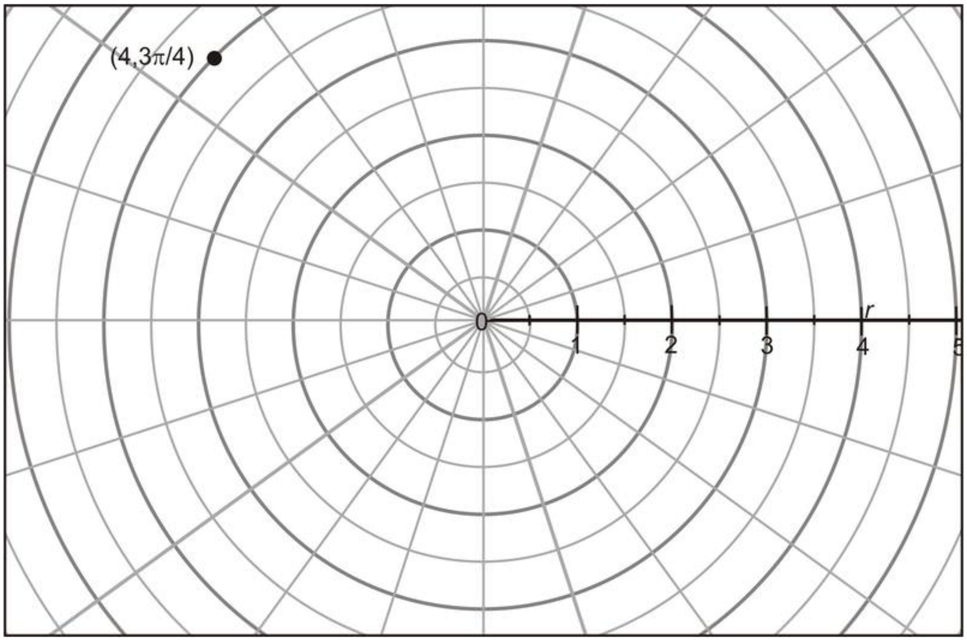
Coordinate points can be converted from rectangular form to polar form with a little algebra and some trigonometry.

Can the equation of a shape also be converted? How about a circle, for instance?

Polar Form to Rectangular Form

Sometimes you'll be given a problem with coordinates in polar form, but you need rectangular form.

To transform the polar point $\left(4, \frac{3\pi}{4}\right)$ into rectangular coordinates, first identify (r, θ). In this point, $r = 4$ and $\theta = \frac{3\pi}{4}$.



Second, draw a vertical line from the point to the polar axis (the horizontal axis). The distance from the pole to where the line you just drew intersects the polar axis is the x-value, and the length of the line segment from the point to the polar axis is the y-value.

These distances can be calculated using trigonometry:

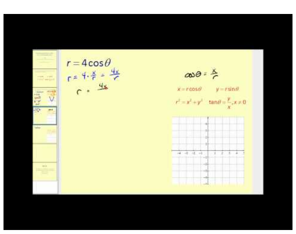
$$x = \cos\theta \qquad y = \sin\theta$$

$$x = 4\cos\frac{3\pi}{4} \quad \text{and} \quad y = 4\sin\frac{3\pi}{4}$$

$$x = -2\sqrt{2} \quad \text{and} \quad y = 2\sqrt{2}$$

In polar coordinates, $\left(4, \frac{3\pi}{4}\right)$ is equivalent to $(-2\sqrt{2}, 2\sqrt{2})$ in rectangular coordinates.

The following video provides several examples of how to convert polar equations to rectangular equations:



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URL: http://www.ck12.org/flx/render/embeddedobject/183702

Rectangular Form to Polar Form

Going from rectangular coordinates to polar coordinates is also possible, but the process takes a bit more work. Suppose we want to find the polar coordinates of the rectangular point (2, 2). To begin doing this operation, we have to determine the distance that the point (2, 2) is from the origin (the radius r). The radius can be found by:

$$r = \sqrt{x^2+y^2}$$

$$r = \sqrt{2^2+2^2}$$

$$r = \sqrt{8} = 2\sqrt{2}$$

The angle that is created by the line segment between the point and the origin can be found by the following procedure:

$$\tan\theta = \frac{y}{x}$$

$$\tan\theta = \frac{2}{2}$$

$$\tan\theta = 1$$

$$\theta = \tan^{-1} 1$$

$$\theta = \frac{\pi}{4}$$

Since this point is in the 1st quadrant (both the x- and y-coordinates are positive), the angle must be 45° or $\frac{\pi}{4}$ radians. It is also possible that when $\tan\theta = 1$, the angle can be in the 3rd quadrant, or $\frac{5\pi}{4}$ radians. But this angle will not satisfy the conditions of the problem, since a 3rd- quadrant angle must have both x and y negative.

Note: When using $\tan\theta = \frac{y}{x}$ to find the measure of θ, you should first consider the quotient $\tan\theta = \left|\frac{y}{x}\right|$, and find the 1st-quadrant angle that satisfies this condition. This angle will be called the reference angle, denoted θ_{ref}. Find the actual angle by analyzing which quadrant the angle must be in, given the signs of x and y.

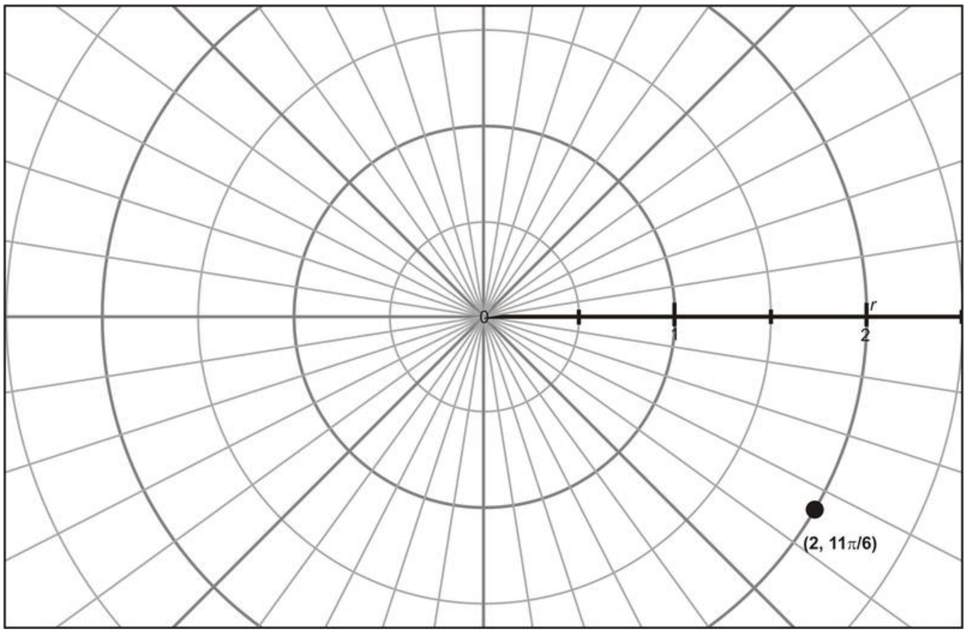
Play, Learn, and Explore Polar Coordinates: www.ck12.org/a/2173455

Examples

Example 1

Transform the polar coordinates $\left(2, \frac{11\pi}{6}\right)$ to rectangular coordinates.

Solution:



$$r = 2 \quad \text{and} \quad \theta = \frac{11\pi}{6}$$

Using the trigonometric conversion equations:

$$\begin{aligned} x = r\cos\theta \quad &\text{and} \quad y = r\sin\theta \\ x = 2\cos\frac{11\pi}{6} \quad &\text{and} \quad y = 2\sin\frac{11\pi}{6} \\ x = 3\sqrt{2} \quad &\text{and} \quad y = -1 \end{aligned}$$

Therefore, $\left(2, \frac{11\pi}{6}\right)$ is equivalent to $(3\sqrt{2}, -1)$.

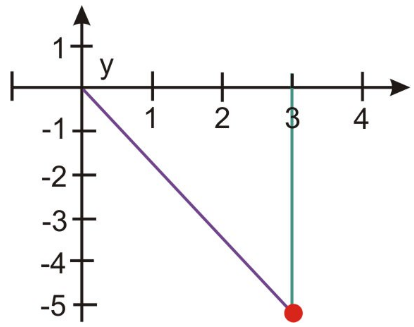
Example 2

Find the polar coordinate for the rectangular coordinate $(3, -3\sqrt{3})$.

Solution:

To calculate r, draw a right triangle and calculate the distance the point is from the origin.

$$x = 3 \quad \text{and} \quad y = -3\sqrt{3}$$



$$\begin{aligned} r &= \sqrt{3^2 + (-3\sqrt{3})^2} \\ &= \sqrt{9+27} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$

The angle created by the line segment between the point and the positive x-axis is:

$$\begin{aligned} \tan\theta_{\text{ref}} &= \left|\frac{(-3\sqrt{3})}{3}\right| \\ \tan\theta_{\text{ref}} &= \sqrt{3} \\ \theta_{\text{ref}} &= \tan^{-1}\sqrt{3} \\ \theta_{\text{ref}} &= \frac{\pi}{3} \end{aligned}$$

$\theta_{ref} = \frac{\pi}{3}$ in the 1st quadrant. Now determine the corresponding 4th-quadrant angle. $\theta = \frac{5\pi}{3}$ is a 4th-quadrant angle. Therefore, the rectangular coordinate $(3, -3\sqrt{3})$ is equivalent to the polar coordinate $\left(6, \frac{5\pi}{3}\right)$.

Example 3

Convert the rectangular coordinates below to polar coordinates.

1) $(3, 3\sqrt{3})$

Solution:

$(6, 60^\circ)$ **or** $\left(6, \frac{\pi}{3}\right)$

2) $(-2, 2)$

Solution:

$(2\sqrt{2}, 135^\circ)$ **or** $\left(2\sqrt{2}, \frac{3\pi}{4}\right)$

Example 4

Convert the polar coordinates below to rectangular coordinates.

1) $\left(4, \frac{2\pi}{3}\right)$

Solution:

$(-2, 2\sqrt{3})$

2) $\left(-1, \frac{5\pi}{6}\right)$

Solution:

$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

Example 5

Recall the question from the Introduction: Can the equation of a shape also be converted? How about a circle, for instance?

Solution:

$x^2 + y^2 = k^2$ is the equation of a circle with a radius of k in rectangular coordinates.

The equation of a circle is extremely simple in polar form. In fact, a circle on a polar graph is analogous to a horizontal line on a rectangular graph.

You can transform this equation to polar form by substituting the polar values for x, y. Recall $x = r\cos\theta$ *and* $y = r\sin\theta$.

$$\begin{aligned}(r\cos\theta)^2 + (r\sin\theta)^2 &= k^2\\ r^2\cos^2\theta + r^2\sin^2\theta &= k^2\end{aligned}$$

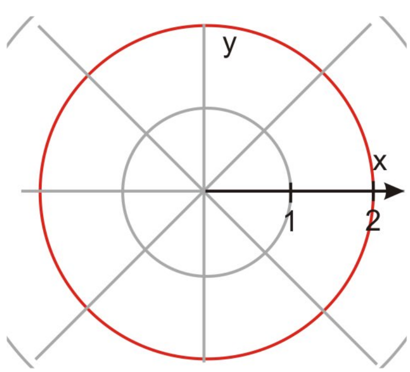
Factor the common factor r^2 on the left side of the equation:

$$r^2\left(\cos^2\theta + \sin^2\theta\right) = k^2$$

Recall the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$.

$$r^2 = k^2$$

Therefore, $r = \pm k$ is an equation for a circle in polar. When r is equal to a constant, the polar graph is a circle.



Example 6

1) Convert $(3, -45^\circ)$ to rectangular coordinates.

Solution:

Using the trigonometric conversion equations:

$$x = r \cdot \cos\theta \quad \text{and} \quad y = r \cdot \sin\theta$$

$$x = 2.121 \quad \text{and} \quad y = -2.121$$

Therefore, $(2.121, -2.121)$ in rectangular is equivalent to $(3, -45^\circ)$ in polar.

2) Convert $\left(-3, \frac{\pi}{4}\right)$ to rectangular coordinates.

Solution:

Using the trigonometric conversion equations:

$$\begin{aligned} x = r \cdot \cos\theta &\quad \text{and} \quad y = r \cdot \sin\theta \\ x = -3 \cdot \cos\left(\frac{\pi}{4}\right) &\quad \text{and} \quad y = -3 \cdot \sin\left(\frac{\pi}{4}\right) \\ x = -2.121 &\quad \text{and} \quad y = -2.121 \end{aligned}$$

Therefore, $(-2.121, -2.121)$ in rectangular is equivalent to $\left(-3, \frac{\pi}{4}\right)$ in polar.

Example 7

1) Express the polar equation in rectangular form: $r = 6\cos\theta$.

Solution:

Multiply both sides of the equation by r.

$$r^2 = 6r\cos\theta$$

Substitute $x^2 + y^2 = r^2$ and $x = r\cos\theta$.

$$x^2 + y^2 = 6x$$

Therefore, $x^2 + y^2 = 6x$ is the equivalent rectangular equation to the polar equation $r = 6\cos\theta$.

2) Express the polar equation in rectangular form: $r = 6$.

Solution:

Square both sides of the equation.

$$r^2 = 36$$

Substitute $x^2 + y^2 = r^2$.

$$x^2 + y^2 = 36$$

Therefore, $x^2 + y^2 = 36$ is the equivalent rectangular equation to the polar equation $r = 6$.

Summary

Relationships between polar and rectangular coordinates where $P = (r, \theta)$ are being converted to rectangular (x, y).

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = r^2$$
$$\tan\theta = \frac{y}{x}$$

Review

1. How is the point with polar coordinates $(5, \pi)$ represented in rectangular coordinates?

Plot each point below in polar coordinates (r, θ). Then write the rectangular coordinates (x, y) for the point.

2. $(3, 60^\circ)$
3. $\left(-10, \frac{\pi}{3}\right)$
4. $(15, \pi)$

Below, the rectangular coordinates (x, y) are given. For each question: a) Find two pairs of polar coordinates (r, θ), one with $r > 0$ and the other with r <0$r > 0$. b) Express θ in radians, and round to the nearest hundredth.

5. $(5, -5)$
6. $(0, 10)$
7. $(-8, 6)$

Transform each polar equation to an equation using rectangular coordinates. Identify the graph, and give a rough sketch or description of the sketch.

8. $\theta = \frac{\pi}{10}$
9. $r = 8$
10. $r\sin\theta = 7$
11. $r\cos\theta = -3$

Transform each rectangular equation to an equation using polar coordinates. Identify the graph, and give a rough sketch or description of the sketch.

12. $x^2 + y^2 - 2x = 0$
13. $y = \sqrt{3}x$
14. $y = -5$
15. $xy = 15$
16. The area of a drumlin at Glacier National Park in Montana is 4,380 square yards. If its length is 132 yards, find its polar equation.
17. If a drumlin is modeled by the equation $r = l\cos 5\theta$ and its area is 7,542 square meters, find its length.
18. The sonar screen of a submarine is a polar system. On the screen, the submarine is the origin of the graph. If the path of the ship in the polar system can be represented by $6 = 17r\cos(\theta - 25^\circ)$,find the linear equation of the movement of the ship.

Review (Answers)

Please see the Appendix.

12.5 Systems of Polar Equations

Learning Objectives

Learn to graph equations of polar graphs and find points of intersection.

Introduction

Graphing systems or simultaneous functions in the coordinate plane allows us to find solutions, or points of intersection, that can be helpful in determining important calculations, including the break-even point for a business or product.

The same is true when graphing equations in polar form and/or on a polar graph. When you graph the intersection of multiple polar equations, you treat them just as you would rectangular equations: Graph both and find the points that are true for both equations.

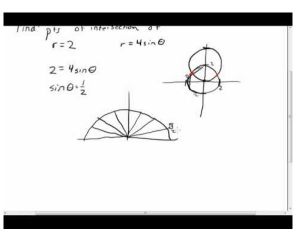
Systems of Polar Equations

Polar equations can be graphed using polar coordinates. Graphing two polar equations on the same axis may result in finding the point(s) of intersection.

One method of finding the point(s) of intersection for two polar graphs is by setting the equations equal to each other. Thus, the points of intersection are when $r_1 = r_2$. Then solve the resulting trigonometric equation.

All points on a polar graph are coordinates that make the equation valid. The coordinates of the point(s) of intersection when substituted into each equation will make both equations valid.

The following video explains how to find points of intersection on a polar graph:



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/55052

Examples

Example 1

Determine the point(s) of intersection for $r_1 = 3\sin\theta$ and $r_2 = \sqrt{3}\cos\theta$.

Solution:

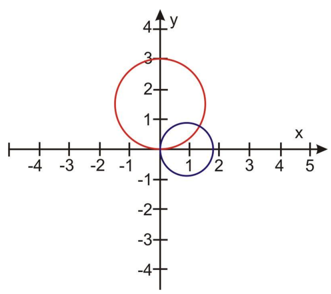
$$\begin{aligned}3\sin\theta &= \sqrt{3}\cos\theta\\ \frac{3\sin\theta}{3\cos\theta} &= \frac{\sqrt{3}\cos\theta}{3\cos\theta}\\ \tan\theta &= \frac{\sqrt{3}}{3}\\ \theta &= \tan^{-1}\frac{\sqrt{3}}{3}\\ \theta = \frac{\pi}{6} &\text{ or } \theta = \frac{7\pi}{6}\end{aligned}$$

Substitute $\frac{\pi}{6}$ into either one of the two polar equations to obtain $r = 1.5$.

Substitute $\frac{7\pi}{6}$ into either one of the two polar equations to obtain $r = -1.5$.

Note: The coordinates $\left(1.5, \frac{\pi}{6}\right)$ and $\left(-1.5, \frac{7\pi}{6}\right)$ represent the same polar point, so there is only one solution to this equation.

If we look at the graphs of r_1 and r_2, we can see there is another point of intersection:



When $\theta = 0$, $r_1 = 3\sin 0 = 0$,so r_1 goes through the point (0, 0).

When $\theta = \frac{\pi}{2}$, $r_2 = \sqrt{3}\cos\frac{\pi}{2} = 0$, so r_2 goes through the point $\left(0, \frac{\pi}{2}\right)$.

Therefore, both graphs also go through the pole $r = 0$.

The pole was NOT revealed as a point of intersection using the first step! (Why? Hint: How many ways are there to represent the pole in polar coordinates?) This shows that after you use algebraic methods to find intersections at points other than the pole, you should also check for intersections at the pole.

Example 2

Determine the point(s) of intersection for $r_1 = 1$ and $r_2 = 2\sin 2\theta$.

Solution:

$$\begin{aligned}1 &= 2\sin 2\theta\\ \frac{1}{2} &= \sin 2\theta\end{aligned}$$

Substitute $\alpha = 2\theta$ to solve.

$$\begin{aligned}\frac{1}{2} &= \sin\alpha\\ \alpha &= \frac{\pi}{6}, \frac{5\pi}{6}\end{aligned}$$

Since $\alpha = 2\theta$, solve for θ by dividing each answer by 2 .

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

Recall that θ usually has the range $0 \leq \theta \leq 2\pi$. However, since we solved for $0 \leq \alpha \leq 2\pi$,we actually need to consider values of θ with the range $0 \leq \theta \leq 4\pi$. Why? Recall that $\sin 2\theta$ has two cycles between 0 and 2π, add two more solutions:

$$\alpha = \frac{13\pi}{6}, \frac{17\pi}{6}$$

so

$$\theta = \frac{13\pi}{12}, \frac{17\pi}{12}.$$

Finally, we need to consider solutions when $r = -1$, because $r = 1$ and $r = -1$ yield the same polar graph.

$$-\frac{1}{2} = \sin\alpha$$
$$\alpha = \frac{7\pi}{6}, \frac{11\pi}{6}$$

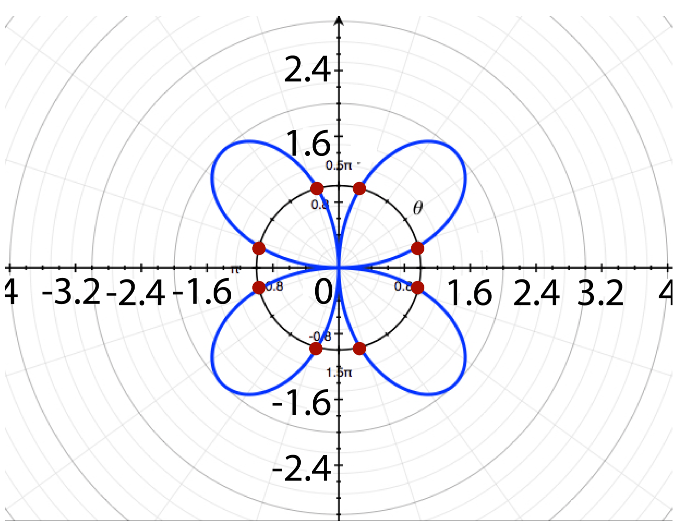
Then

$$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{21\pi}{12}.$$

In total, there are eight solutions to this set of equations.

Note: Recall that solving trigonometric equations where the angle is θ requires looking at all potential values between 0 and 2π. When the angle is 2θ, as it is in this case, be sure to look for all potential values between 0 and 4π.

The graph reveals eight points of intersection.



Example 3

Determine the point(s) of intersection for $r_1 = 2$ and $r_2 = \sec\theta$.

Solution:

Here we will use a table of values based on the unit circle for each function. Compare the values in the tables below to determine the intersection points for each quadrant. Recall that the period of $\sec\theta$ is 2π.

For the 1st quadrant

TABLE 12.1:

θ (angle)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r_1 (distance)	2	2	2	2	2
r_2	1	1.15	1.4	2	undefined

For the 2nd quadrant

TABLE 12.2:

θ (angle)	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r_1 (distance)	2	2	2	2
r_2	-2	-1.4	-1.15	-1

For the 3rd quadrant:

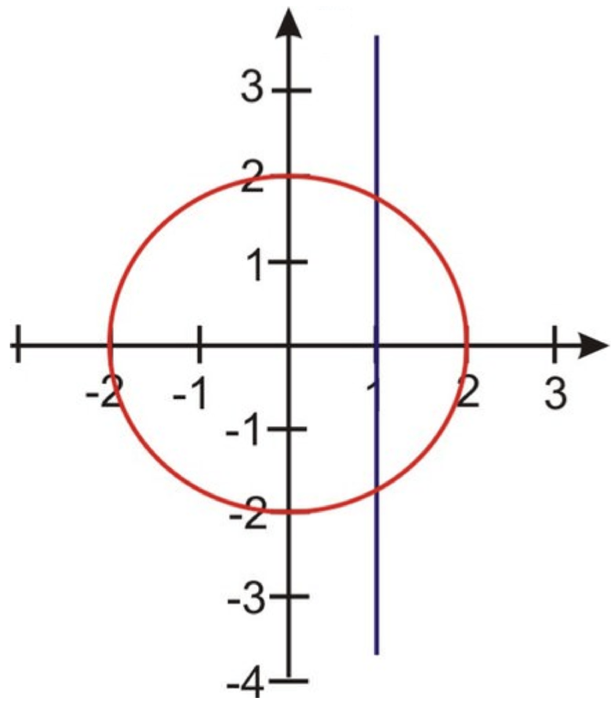
TABLE 12.3:

θ (angle)	$\frac{7\pi}{6}$	$\frac{6\pi}{5}$	$\frac{3\pi}{4}$	$\frac{3\pi}{2}$
r_1 (distance)	2	2	2	2
r_2	-1.15	-1.4	-2	undefined

For the 4th quadrant

TABLE 12.4:

θ (angle)	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r_1 (distance)	2	2	2	2
r_2	2	1.4	1.15	1



Note that the 3rd and 4th quadrant repeat 1st and 2nd quadrant values.

Observe in the table of values that $\left(2, \frac{\pi}{3}\right)$ and $\left(2, \frac{5\pi}{3}\right)$ are the two points of intersection. When looking at the graph, you can see that the curves have only two intersecting points.

Example 4

Determine the point(s) of intersection for $r = 2 + 4\sin\theta$ and $\theta = 60^\circ$.

Solution:

The equation $\theta = 60^\circ$ is a line making a 60° angle with the r-axis.

Make a table of values for $r = 2 + 4\sin\theta$.

For the 1st quadrant:

TABLE 12.5:

θ (angle)	0°	30°	45°	60°	90°
R (distance)	2	4	4.83	5.46	6

For the 2nd quadrant:

TABLE 12.6:

θ (angle)	120°	135°	150°	180°
R (distance)	5.46	4.83	4	2

For the 3rd quadrant:

TABLE 12.7:

θ (angle)	210°	225°	240°	270°
R (distance)	0	-.83	-1.46	-2

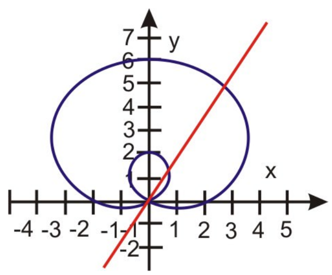
For the 4th quadrant

TABLE 12.8:

θ (angle)	300°	315°	330°	360°
R (distance)	-1.46	-.83	0	2

Notice there are two solutions in the table: (5.46, 60°) and (-1.46, 240°) = (1.46, 60°). Recall that when $r < 0$, a point (r, θ) is plotted by rotating 180^o or π.

Finally, we need to check the pole. $\theta = 60°$ is a straight line that passes through the pole. $r = 2 + 4\sin\theta$ passes through the pole when $\theta = 210°$ and $330°$. Thus, a 3rd point of intersection is at the pole, $r = 0$.



Example 5

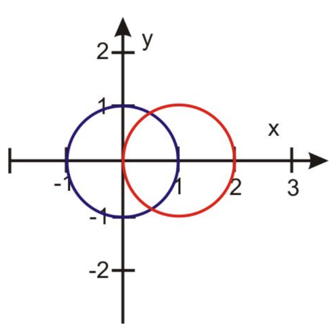
Determine the point(s) of intersection for $r_1 = 2\cos\theta$ and $r_2 = 1$.

Solution:

$$\begin{aligned} 2\cos\theta &= 1 \\ \cos\theta &= \frac{1}{2} \\ \theta &= \cos^{-1}\frac{1}{2} \\ \theta &= \frac{\pi}{3}, \frac{4\pi}{3} \end{aligned}$$

Thus, there are two unique solutions, $\theta = \frac{\pi}{3}, \frac{4\pi}{3}$. There are also two repeated solutions in this set. (Can you find them?)

Here is a graph showing the two solutions:



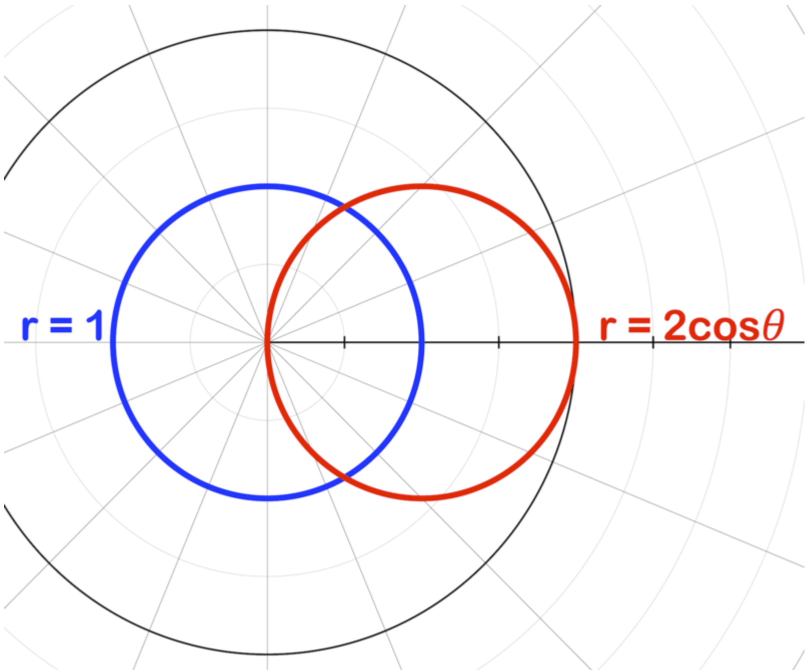
Summary

- Graphing simultaneous polar equations can yield solutions where the graphs meet.

- Setting two polar equations equal to each other and solving will determine the coordinate of a solution of the system.
- Solutions at the pole or repeat solutions can be found manually.

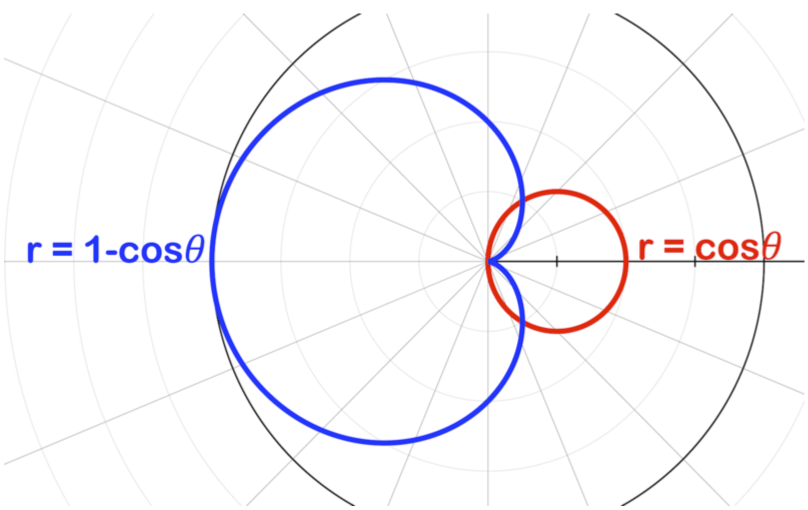
Review

The graphs of $r = 1$ and $r = 2\cos\theta$ are shown below.



1. How many times do they intersect?
2. In which quadrants do they intersect?
3. At what points do the intersections occur?

Answer questions 4-5 based on the image below and the following information: the intersection of the graphs of $r = \cos\theta$ and $r = 1 - \cos\theta$.



4. How many times do they intersect?

5. At what points do the intersections occur?

Determine the points of intersection of the following pairs of curves:

6. $r = 2$, $r = 2\cos\theta$
7. $r = \sin 2\theta$, $r = 2\sin\theta$
8. $r = 2 + 2\sin\theta$, $r = 2 - 2\cos\theta$
9. $r = 3\cos\theta$, $r = 2 - \cos\theta$

Determine the points of intersection for each system of equations below. Graph to verify your solution.

10. $r = \csc\theta$, $r = 2\sin\theta$
11. $r = \cos\theta$, $r = 1 + \sin\theta$
12. $r = \sin\theta$, $r = \sin 2\theta$
13. $r = -4\sin\theta$, $r = -4\cos\theta$
14. $r = 1 - 2\sin\theta$, $r = \sqrt{9\cos\theta}$
15. $r = 1 - \cos\theta$, $r = 4\cos 3\theta$

Review (Answers)

Please see the Appendix.

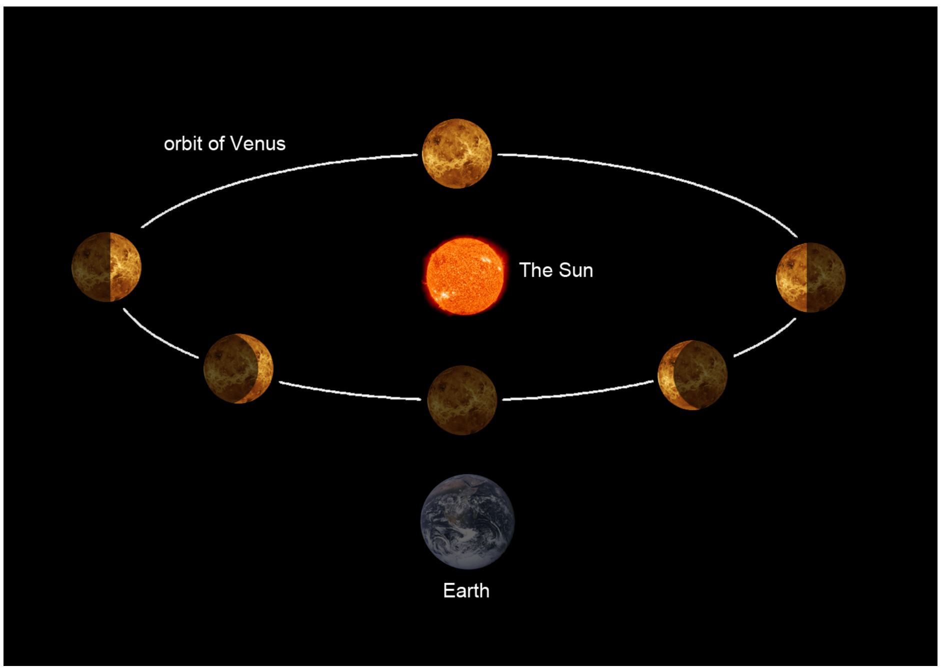
12.6 Polar Equations of Conics

Learning Objectives

Learn to find equations and graphs for various conics, including those whose major axis is at a slant.

Introduction

Janet is working on a physics problem that involves planets in orbit around a star. She tried to set up the problem in rectangular form, but the resulting equations were very messy. Janet needs a way to simplify her problem so she can manipulate the equations more easily. Can you help using polar form?



Janet is trying to model the paths of two planets and a comet around a star. By putting the equations of her orbiting objects into polar form, she can easily create a graph to model her problem and work with the equations in a simpler format. For instance, suppose she's tracking the paths of two planets and a comet around a star. All these objects will have the sun as one of the focal points of their orbits, so it's easy to represent their paths using polar equations.

Polar Equations of Conics

Recall that the eccentricity e is the measure of how much the conic section deviates from being circular. The eccentricity is equal to the ratio of the focal radius to the semi-major axis (half the length of the longer axis).

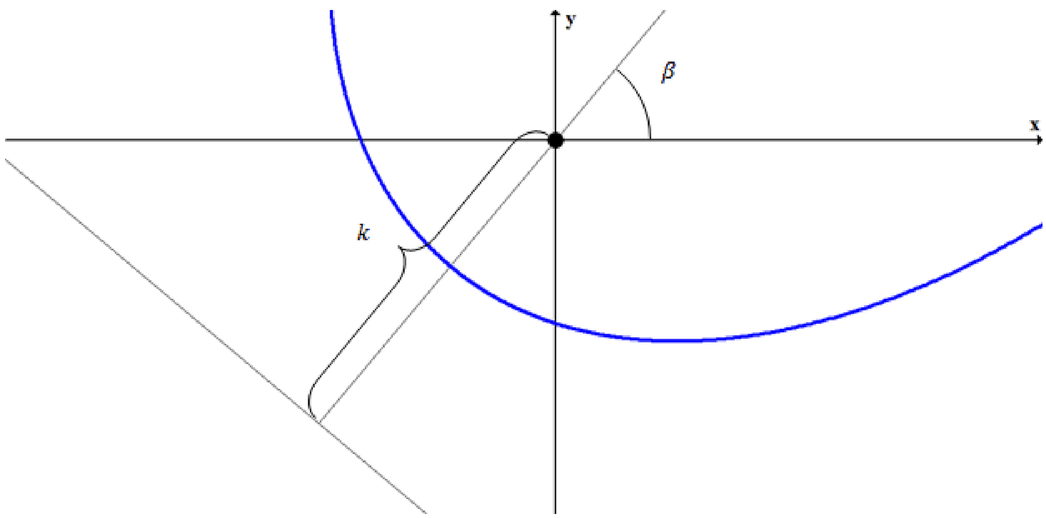
$$e = \frac{c}{a}$$

The value of the eccentricity defines the conic section.

- If $e = 0$, then the conic is a circle.
- If $0 < 1$, then the conic is an ellipse.
- If $e = 1$, then the conic is a parabola.
- If $e > 1$, then the conic is a hyperbola.

Ellipses, parabolas, and hyperbolas have a common general polar equation. Note there are other ways of representing these relations using cofunction and coterminal angles. However, this general form is easiest to use because each parameter can be immediately interpreted in a graph.

$$r = \frac{k \cdot e}{1 - e \cdot \cos(\theta - \beta)}$$



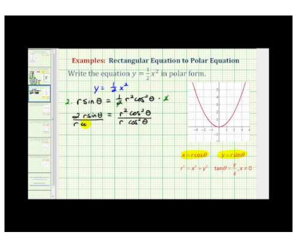
If the conic is an ellipse, the angle β indicates the angle towards the center. If the conic is a parabola, the angle β indicates the opening direction. If the conic is a hyperbola, the angle βindicates the angle away from the center. The constant k is the distance from the focus at the pole to the nearest directrix. This directrix lies in the opposite direction indicated by β.

The distance from the focus at the pole to the nearest directrix, k, can be calculated using these equations:

- Ellipses: $k = \frac{a^2}{c} - c$
- Hyperbolas: $k = c - \frac{a^2}{c}$

where a is the semi-major axis, and c is the focal radius.

The following video demonstrates how to write the polar equation for a parabola given the rectangular form.



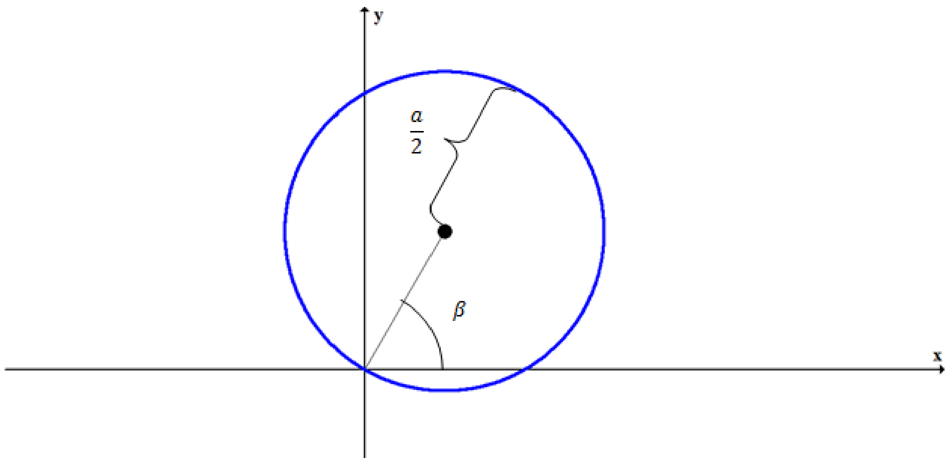
MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/62012

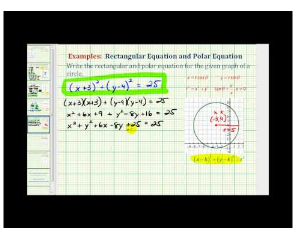
The general polar equation used with ellipses, parabolas, and hyperbolas does not apply to circles. Since $e = 0$ for circles, the equation would simplify to $r = 0$, which does not tell us much about the circle. Instead, the polar equation of a circle with radius $r = \frac{a}{2}$, or a equal to the diameter of the circle, passing through the origin, and having a center located at an angle β counterclockwise from the positive x-axis, is

$$r = a \cdot \cos(\theta - \beta).$$



Just as with the other conic sections, there are other ways of representing a circle like this using cofunction identities and coterminal angles.

The following video demonstrates how to determine the equation of a circle in rectangular form and polar form from the graph of a circle:



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Examples

Example 1

Identify the following conic sections:

1) $r = \frac{10}{5+5\cos\theta}$

Solution:

$$\begin{aligned} r &= \frac{10}{5-5\cos\theta}\cdot\frac{\frac{1}{5}}{\frac{1}{5}} \\ &= \frac{2}{1-\cos\theta} \end{aligned}$$

$e = 1$, so this conic is a parabola.

2) $r = \frac{5}{10-3\cos\theta}$

Solution:

$$\begin{aligned} r &= \frac{5}{10-3\cos\theta}\cdot\frac{\frac{1}{10}}{\frac{1}{10}} \\ &= \frac{\frac{1}{2}}{1-\frac{3}{10}\cos\theta} \end{aligned}$$

$e = \frac{3}{10}$, so this conic is an ellipse.

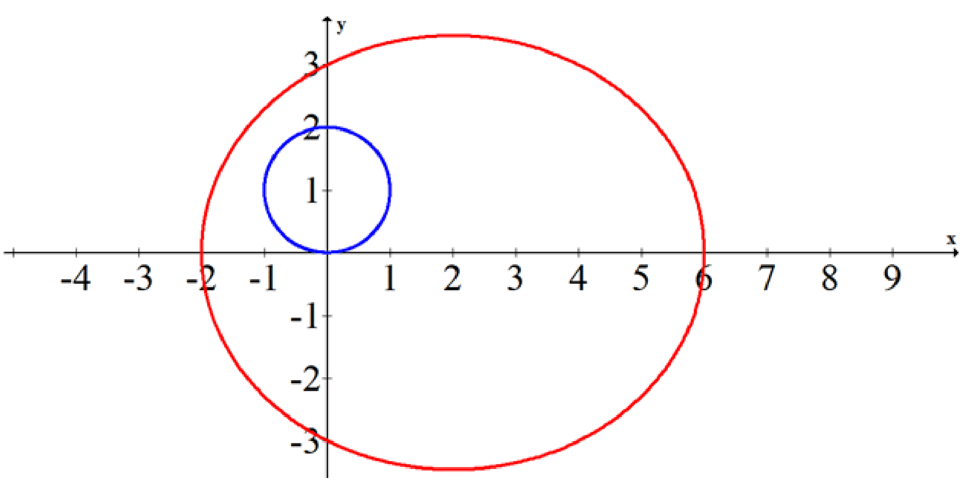
3) $r = \frac{4}{1-2\cos\theta}$

Solution:

$e = 2$, so this conic is a hyperbola.

Example 2

A great way to discover new types of graphs in polar coordinates is to experiment with graphing software such as Desmos or GeoGebra. Can you determine the two equations that would create the following graphs?



Solution:

The circle in blue has a center at 90° and a diameter of 2. Its equation is $r = 2\cos(\theta - 90°)$.

The red ellipse appears to have its center at (2, 0), with $a = 4$ and $c = 2$. This means the eccentricity is $e = \frac{c}{a} = \frac{2}{4} = \frac{1}{2}$.

To write the equation in polar form, you still need to find k.

$$k = \frac{a^2}{c} - c = \frac{4^2}{2} - 2 = 8 - 2 = 6$$

Thus, the equation for the ellipse is

$$r = \frac{6 \cdot \frac{1}{2}}{1 - \frac{1}{2} \cdot \cos(\theta)} = \frac{3}{1 - \frac{1}{2} \cdot \cos(\theta)}.$$

Example 3

Convert the following conic from polar form to rectangular form: $r = \frac{3}{2 - \cos\theta}$.

Solution:

$$\begin{aligned}
r &= \frac{3}{2 - \cos\theta} \\
r(2 - \cos\theta) &= 3 \\
2r - r \cdot \cos\theta &= 3 \\
2r &= 3 + r \cdot \cos\theta = 3 + y \\
4r^2 &= 9 + 6y + y^2 \\
4(x^2 + y^2) &= 9 + 6y + y^2 \\
4x^2 + 4y^2 &= 9 + 6y + y^2 \\
4x^2 + 3y^3 + 6y &= 9 \\
4x^2 + 3(y^2 + 2y + 1) &= 9 + 3 \\
4x^2 + 3(y + 1)^2 &= 12 \\
\frac{x^2}{3} + \frac{(y + 1)^2}{4} &= 1
\end{aligned}$$

Example 4

Recall the problem from the Introduction: How would you help Janet model the paths of two planets and a comet around a star using polar equations?

Solution:

Assume that one planet has an orbit described by the equation

$$r = \frac{\frac{1}{2}}{1 + \frac{1}{10}\cos\theta},$$

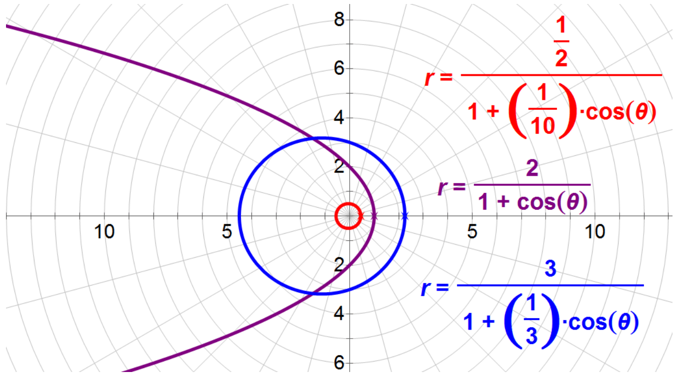
and the other planet has an orbit described by the equation

$$r = \frac{3}{1 + \frac{1}{3}\cos\theta}.$$

Let the comet's orbit be described by the equation

$$r = \frac{2}{1 + \cos\theta}.$$

By graphing the equations on her calculator or with graphing software such as Desmos or GeoGebra, Janet can quickly see that the comet will pass between the orbits of the inner planet and the outer planet on its journey past the star, and she can determine where the comet intersects the orbit of the 2nd planet.



To see exactly where the orbits will intersect, Janet can set the equation for the planet's orbit equal to the equation for the comet's path, and then solve for θ. So:

$$\begin{aligned}\frac{3}{1+\frac{1}{3}\cos\theta} &= \frac{2}{1+\cos\theta}\\ 3(1+\cos\theta) &= 2\left(1+\frac{1}{3}\cos\theta\right)\\ 3+3\cos\theta &= 2+\frac{2}{3}\cos\theta\\ \frac{7}{3}\cos\theta &= -1\\ \cos\theta &= -\frac{3}{7}\\ \theta &= 2.014,\ 2\pi-2.014.\end{aligned}$$

Substitute back into one of the original equations to find r.

$$r=\frac{2}{1+\cos\theta}=\frac{2}{1+\frac{-3}{7}}=\frac{2}{\frac{4}{7}}=\frac{7}{2}$$

So, the comet will intersect the outer planet's orbit at $\left(\frac{7}{2},2.014\right)$ and $\left(\frac{7}{2},4.269\right)$.

Example 5

Identify the center, foci, vertices, and equations of the directrix lines for the following conic:

$$r=\frac{20}{4-5\cdot\cos\left(\theta-\frac{3\pi}{4}\right)}.$$

Solution:

First, rewrite the denominator so that it matches the general form $1-e\cdot\cos(\theta-\beta)$.

$$\begin{aligned} r &= \frac{20}{4-5\cdot\cos\left(\theta-\frac{3\pi}{4}\right)}\cdot\frac{\frac{1}{4}}{\frac{1}{4}} \\ &= \frac{5}{1-\frac{5}{4}\cdot\cos\left(\theta-\frac{3\pi}{4}\right)} \\ &= \frac{4\cdot\frac{5}{4}}{1-\frac{5}{4}\cdot\cos\left(\theta-\frac{3\pi}{4}\right)} \\ e &= \frac{5}{4}, \quad k=4, \quad \beta=\frac{3\pi}{4}=135^\circ \end{aligned}$$

Use this information to solve for a and c.

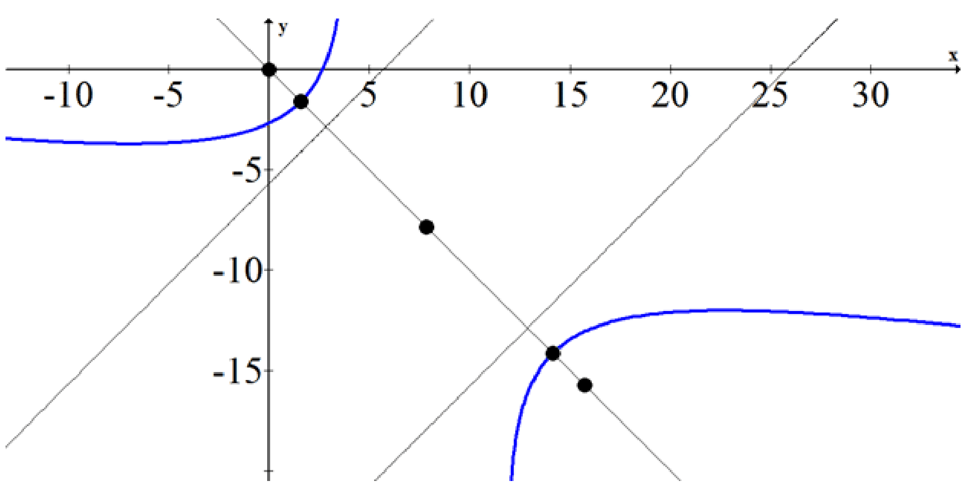
$$\begin{aligned} 4 &= c-\frac{a^2}{c} \\ \frac{5}{4} &= \frac{c}{a} \rightarrow \frac{4}{5}=\frac{a}{c} \rightarrow \frac{4c}{5}=a \\ 4 &= c-\left(\frac{4c}{5}\right)^2\cdot\frac{1}{c} \\ 4 &= c-\frac{16c^2}{25c} \\ 4 &= \frac{9c}{25} \\ \frac{100}{9} &= c \\ \frac{80}{9} &= a \end{aligned}$$

The center is the point $\left(\frac{100}{9},\frac{7\pi}{4}\right)$, which is much more convenient to write in polar coordinates. The closest directrix is the line $r=4\cdot\sec\left(\theta-\frac{7\pi}{4}\right)$. The other directrix is the line $r=\left(2\cdot\frac{100}{9}-4\right)\cdot\sec\left(\theta-\frac{7\pi}{4}\right)$. One focus is at the pole, the other focus is the point $\left(\frac{200}{9},\frac{7\pi}{4}\right)$. The vertices are at the center, plus or minus a in the same angle: $\left(\frac{100}{9}\pm\frac{80}{9},\frac{7\pi}{4}\right)$.

Example 6

Graph the conic from Example 2.

Solution:



Example 7

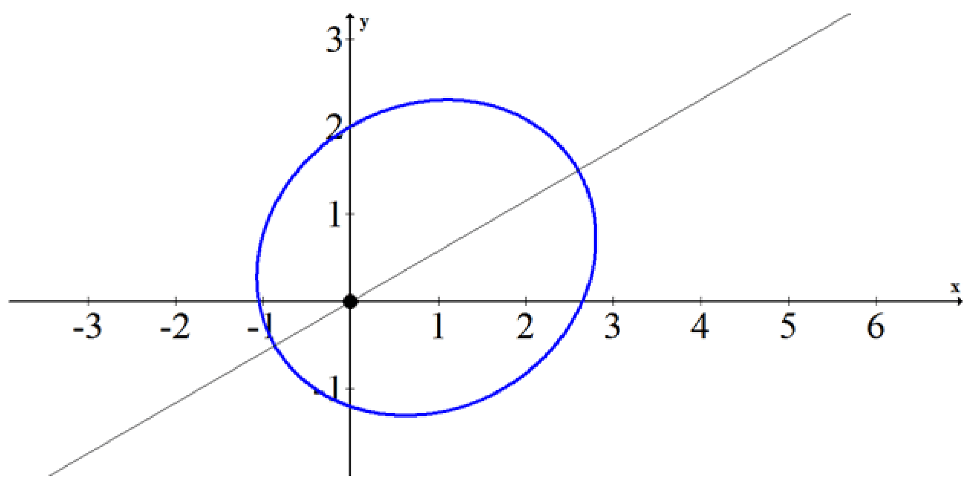
Graph the following conic: $r=\frac{3}{2-\cos(\theta-30^\circ)}$.

Solution:

$$r=\frac{3}{2-\cos(\theta-30^\circ)}$$

$$r=\frac{3}{2-\cos(\theta-30^\circ)}\cdot\frac{\frac{1}{2}}{\frac{1}{2}}=\frac{\frac{3}{2}}{1-\frac{1}{2}\cdot\cos(\theta-30^\circ)}=\frac{3\cdot\frac{1}{2}}{1-\frac{1}{2}\cdot\cos(\theta-30^\circ)}$$

$$k=3,\quad e=\frac{1}{2},\quad \beta=30^\circ$$



Summary

- The eccentricity e is the measure of how much the conic section deviates from being circular. The value of the eccentricity defines the conic section.
  - If $e=0$, then the conic is a circle.
  - If $0<1$, then the conic is an ellipse.
  - If $e=1$, then the conic is a parabola.
  - If $e>1$, then the conic is a hyperbola.
- Ellipses, parabolas, and hyperbolas have a common general polar equation:

$$r=\frac{k\cdot e}{1-e\cdot\cos(\theta-\beta)}.$$

- A circle has the equation $r=a\cdot\cos(\theta-\beta)$.

Review

Convert the conics below from polar form to rectangular form. Then identify the conic section described by the equation.

1. $r=\frac{5}{3-\cos\theta}$

2. $r=\frac{4}{2-\cos\theta}$

3. $r = \frac{2}{2-\cos\theta}$

4. $r = \frac{3}{2-4\cos\theta}$

5. $r = 5\cos(\theta)$

Graph the following conics:

6. $r = \frac{5}{4-2\cos(\theta-90^\circ)}$

7. $r = \frac{5}{3-7\cos(\theta-60^\circ)}$

8. $r = \frac{3}{3-3\cos(\theta-30^\circ)}$

9. $r = \frac{1}{2-\cos(\theta-60^\circ)}$

10. $r = \frac{3}{6-3\cos(\theta-45^\circ)}$

Translate the following conics to polar form:

11. $\frac{(x-1)^2}{4} + \frac{y^2}{3} = 1$

12. $(x-5)^2 + (y+12)^2 = 169$

13. $x^2 + (y+1)^2 = 1$

14. $(x-1)^2 + y^2 = 1$

15. $-3x^2 - 4x + y^2 - 1 = 0$

Calculate the value of the eccentricity e. Then identify the conic section described by the equation.

16. $r = \frac{4}{1-\cos\theta}$

17. $r = \frac{2}{4+2\cos\theta}$

18. $r = \frac{8}{6-6\cos\theta}$

19. $r = \frac{12}{2+4\cos\theta}$

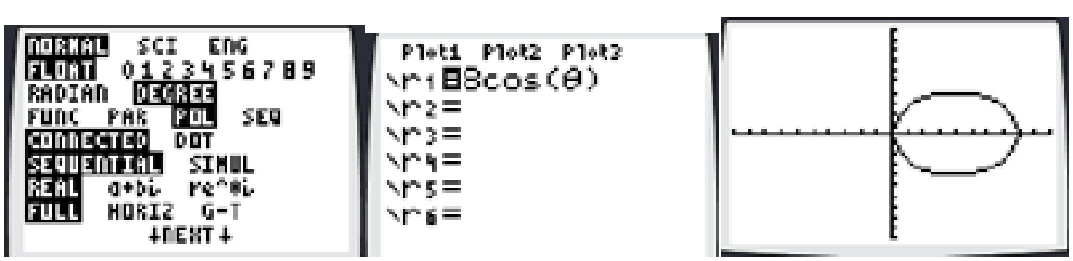
20. What are the differences between creating the graphs of conics on the polar grid as opposed to the rectangular grid?

Review (Answers)

Please see the Appendix.

Resource: A Note When Using TI-Calculators

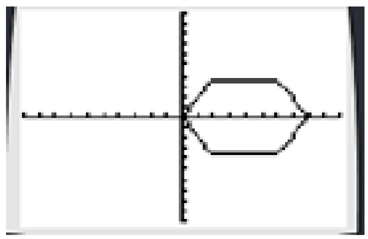
On the TI-84, the mode can be switched to polar in the mode menu. This changes the graphing features. You can choose to be in radians or degrees, and the graphs will look the same. When you graph a circle of the form $r = 8 \cdot \cos\theta$, you should see the following on your calculator:



When you go to the window setting, you should notice that in addition to X_{min}, X_{max}, there are new settings called $\theta_{min}, \theta_{max}$ and θ_{step}.

If θ_{min} and θ_{max} do not span an entire period, you may end up missing part of your polar graph.

The θ_{step} controls how accurate the graph should be. If you put θ_{step} at a low number like 0.1, the graph will plot extremely slowly because the calculator is doing 3,600 cosine calculations. On the other hand, if $\theta_{step} = 30$, then the calculator will do fewer calculations, producing a rough circle, but it will probably not be accurate enough for your purposes.



12.7 Polar Form of Complex Numbers

Learning Objectives

Learn how to convert complex numbers from rectangular form to polar form and trigonometric form. You will also explore the graphs of complex numbers on a polar graph.

Introduction

Complex numbers can be graphed on a polar graph in a few different ways. This section discusses how to write a complex number in standard form, polar form, and trigonometric form. Then, based on these forms, it covers how to graph a complex number on the rectangular and polar coordinate systems.

Polar Form of Complex Numbers

There are three common forms of complex numbers you will see when graphing:

1) The standard form of a complex number, $z = a + bi$, can be graphed using rectangular coordinates (a, b), where a represents the x-coordinate and b represents the y-coordinate.

2) The polar form, (r, θ), can also be used to graph a complex number. Recall that you can convert between rectangular and polar forms with

$$r = \sqrt{x^2 + y^2}$$

and

$$\tan \theta_{ref} = \left|\frac{y}{x}\right|.$$

Unfortunately, there is a problem with using a conversion from rectangular form to polar form:

$$a + bi \rightarrow (r, \theta)$$

or

$$-1 - i\sqrt{3} \rightarrow \left(2, \frac{4\pi}{3}\right).$$

Notice that we no longer see the imaginary number in this form.

3) The trigonometric form, which is often abbreviated as $r\ cis\ \theta$, comes from the substitutions$x = r\cos\theta$ and $y = r\sin\theta$. Suppose $z = -1 - i\sqrt{3}$. Note that in polar form, this complex number is $\left(2, \frac{4\pi}{3}\right)$.

$$\begin{aligned} z &= 2\cos\frac{4\pi}{3} + 2i\sin\frac{4\pi}{3} \\ &= 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) \end{aligned}$$

The complex number $z = -1 - \sqrt{3}i$ can be written as a rectangular point $(-1, -\sqrt{3})$, a polar point $\left(2, \frac{4\pi}{3}\right)$, or in trigonometric form $2\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$ or $2 \text{ cis } \left(\frac{4\pi}{3}\right)$.

Conversion

To convert from rectangular to polar form, the distance that the point (2, 2) is from the origin can be found by

$$r = \sqrt{x^2 + y^2}$$

or

$$r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.$$

The reference angle (i.e. the corresponding angle in the 1st quadrant) that the line segment between the point and the origin can be found by using

$$\tan \theta_{ref} = \left|\frac{y}{x}\right|.$$

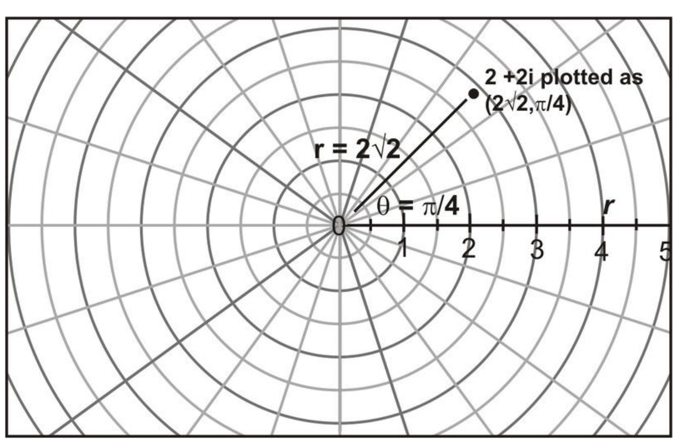
For $z = 2 + 2i$,

$$\tan \theta_{\text{ref}} = \frac{2}{2} = 1.$$

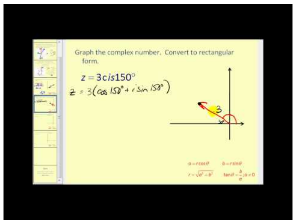
Since this point is in the 1st quadrant (both the x- and y-coordinates are positive), the angle must be 45° or $\frac{\pi}{4}$ radians. It is also possible that when $\tan \theta = 1$,the angle can be in the 3rd quadrant or $\frac{5\pi}{4}$ radians. However, this angle will not satisfy the conditions of the problem, since a 3rd-quadrant angle must have both x and y as negatives.

Note: When using $\tan \theta = \frac{y}{x}$, you should consider the quotient $\left|\frac{y}{x}\right|$ and find the 1st-quadrant angle that satisfies this condition. This angle will be called the **reference angle,** denoted θ_{ref}. Find the actual angle by analyzing which quadrant the angle must be in, given the signs of x and y.

The complex number $2 + 2i$ or (2, 2) in rectangular form has polar coordinates $\left(2\sqrt{2}, \frac{\pi}{4}\right)$.



The following video demonstrates how to write complex numbers in trigonometric form:



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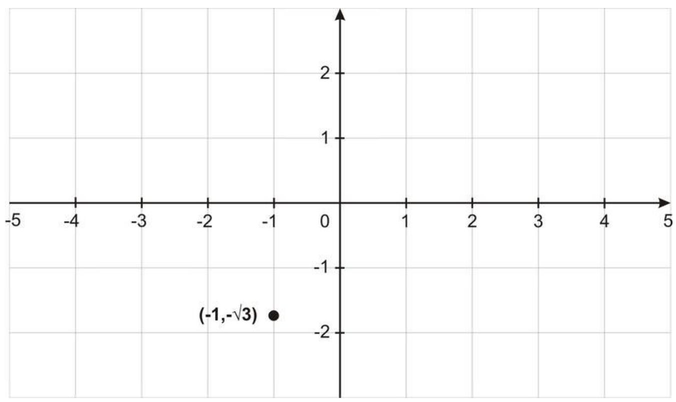
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/53135

Example 1

Convert into polar form: $z = -1 - i\sqrt{3}$.

Here is what it looks like in the rectangular coordinate system:



Solution:

To convert $z = -1 - i\sqrt{3}$ to polar, first determine r and θ.

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{(-1)^2 + (-\sqrt{3})^2} \\ &= \sqrt{1+3} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

$$\tan\theta_{\text{ref}} = \left|\frac{-\sqrt{3}}{-1}\right| = \sqrt{3}$$

$$\theta_{\text{ref}} = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

Since this angle is in the 3rd quadrant, $\theta = \frac{4\pi}{3}$.

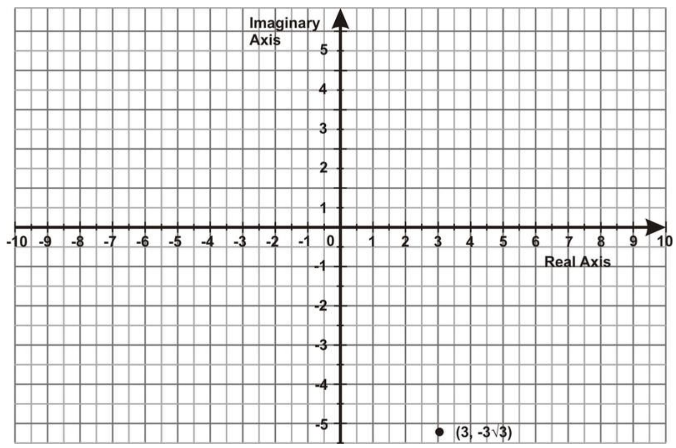
Example 2

Find the polar coordinates that represent the complex number $z = 3 - 3\sqrt{3}i$.

Solution:

a = 3 and b = $-3\sqrt{3}$, so the rectangular coordinates of the point are $\left(3, -3\sqrt{3}\right)$.

Now, draw a right triangle in standard form. Find the distance the point is from the origin, and the angle the line segment that represents this distance makes with the +x axis:



We know a = 3, $b = -3\sqrt{3}$.

$$\begin{aligned} r &= \sqrt{3^2 + (-3\sqrt{3})^2} \\ &= \sqrt{9+27} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$

$$\begin{aligned} \tan\theta_{\text{ref}} &= \left|\frac{-3\sqrt{3}}{3}\right| = \sqrt{3} \\ \theta_{\text{ref}} &= \tan^{-1}\sqrt{3} = \frac{\pi}{3} \end{aligned}$$

Since it is a 4th-quadrant angle, $\theta = \frac{5\pi}{3}$.

The rectangular point $(3, -3\sqrt{3}i)$ is equivalent to the polar point $\left(6, \frac{5\pi}{3}\right)$.

In trigonometric form, $(3, -3\sqrt{3}i)$ is $6\left(cos\ \frac{5\pi}{3} + i\ sin\ \frac{5\pi}{3}\right)$.

Example 3

Plot the complex number $z = 12 + 9i$.

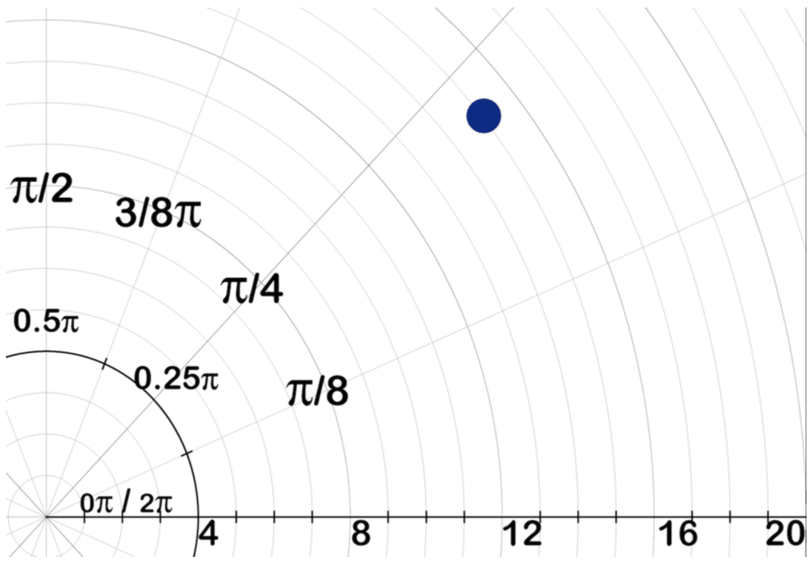
Solution:

To plot $z = 12 + 9i$ on a polar graph, first determine r and θ.

$$\begin{aligned} r &= \sqrt{12^2+9^2} \\ &= \sqrt{144+81} \\ &= \sqrt{225} \\ &= 15 \end{aligned}$$

$$\begin{aligned} \tan\theta_{\text{ref}} &= \left|\frac{9}{12}\right| \\ \theta_{\text{ref}} & \end{aligned}$$

$z = 12+9i$ looks like the image below when plotted on a polar plane.



Example 4

In what quadrant does $z = -3+2i$ occur when graphed?

Solution:

The point $z = -3+2i$ occurs 3 units to the left and 2 units up, placing it in quadrant 2.

Example 5

What are the coordinates of $z = -3+2i$ in polar form and trigonometric form?

Solution:

To convert $z = -3+2i$, first determine r and θ.

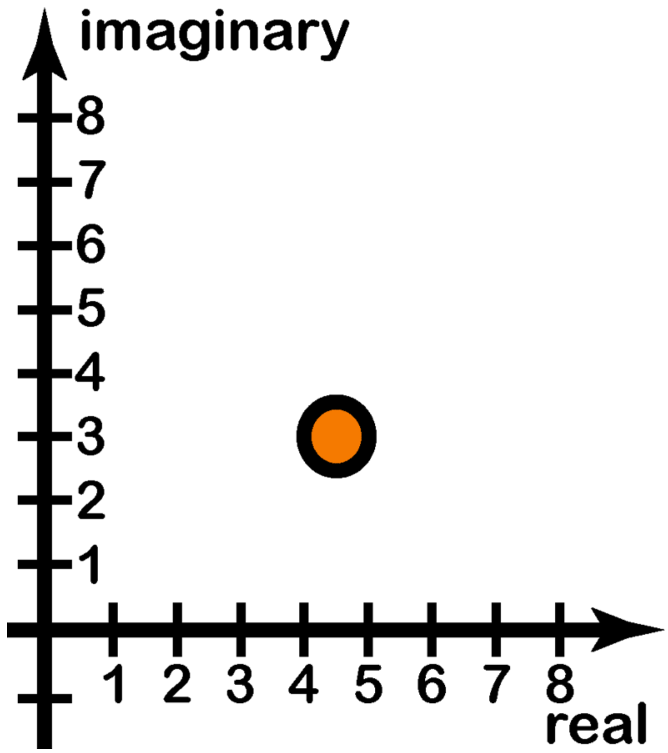
$$r = \sqrt{(-3)^2+2^2} = \sqrt{13}$$

$$\begin{aligned} \tan\theta_{\text{ref}} &= \left|\frac{2}{-3}\right| = \frac{2}{3} \\ \theta_{\text{ref}} &= \tan^{-1}\frac{2}{3} \approx 33.7^\circ \end{aligned}$$

Therefore, $\left(\sqrt{13}, 33.7^\circ\right)$ is the coordinate in polar form, and $\sqrt{13}\ cis\ (33.7^\circ)$ is the coordinate in trigonometric form.

Example 6

What would be the polar coordinates of the point graphed below?



Solution:

The rectangular coordinates are (4.5, 3), so the complex number is $z = 4.5 + 3i$.

$$r = \sqrt{4.5^2 + 3^2} \approx 5.4$$

$$\tan \theta_{\text{ref}} = \left|\frac{3}{4.5}\right|$$
$$\theta_{\text{ref}} = \tan^{-1}\frac{3}{4.5} \approx 33.7^\circ$$

Therefore, $(5.4, 33.7^\circ)$ is the coordinate in polar form, and $5.4\ cis\ (33.7^\circ)$ is the coordinate in trigonometric form.

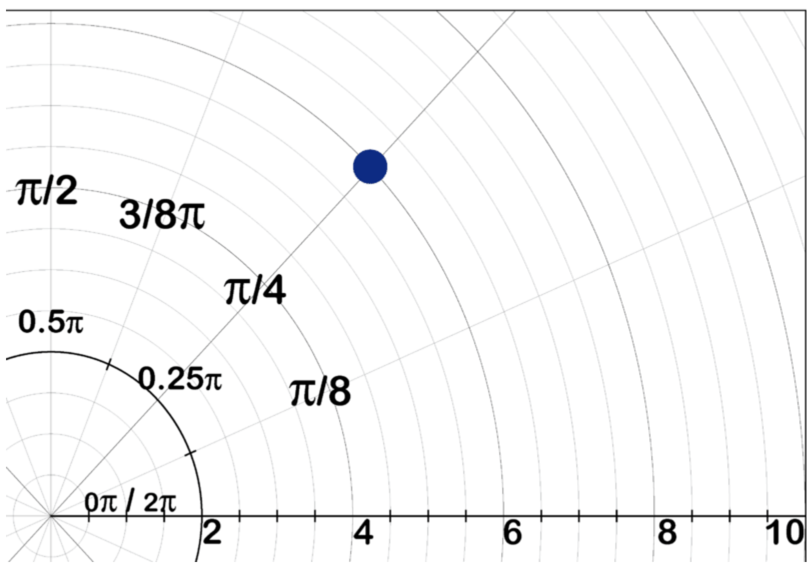
Summary

- In the standard form of $z = a + bi$, a complex number z can be graphed using rectangular coordinates (a, b), where a represents the x-coordinate, while b represents the y-coordinate.
- Use x and y to convert between rectangular and polar forms with $r = \sqrt{x^2 + y^2}$ and $\tan \theta_{ref} = \left|\frac{y}{x}\right|$.

Review

Plot each complex number below in the complex plane. Convert its polar form (r, θ), where θ is in degrees.

1. $1+i$
2. $(1+i)i$
3. $(-2)(3i)$
4. $1+i$
5. $1-i$
6. $(1+i)(1-i)$
7. $1+i\sqrt{3}$
8. $\sqrt{3}-i$
9. $(1+i\sqrt{3})(\sqrt{3}-i)$
10. What are the rectangular coordinates for the point graphed below?



Convert to rectangular form:

11. $15(cos120^o + isin120^o)$
12. $12\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)$
13. For the complex number in standard form $x+iy$, find: a) polar form, and b) trigonometric form. (Hint: Recall that $x = rcos\theta$ and $y = rsin\theta$.)
14. Find the zeros of $f(x) = x^2 - 4x + 8$. Graph them on the complex plane.

Convert to trigonometric form:

15. $1+i^6$
16. $\frac{\sqrt{3}}{2} + \frac{1}{2}i^{10}$
17. $-3-2i$
18. $2\sqrt{3}-2i$

Review (Answers)

Please see the Appendix.

Resource: Graphing Calculator Instructions

Convert the following complex numbers into polar form, using technology:

a) $\sqrt{3}-i$
b) $9\sqrt{3}+9i$

Solution

On the TI-84, go to **[ANGLE]** (or **[2nd]** function) **[APPS]**. Scroll down to 5 or "R-Pr(" and press **[Enter]**. Next, enter the rectangular coordinates and close the parentheses. Press **[Enter];** the "r" value appears. Scroll down to 6R-Pθ, and the polar angle appears in decimal radian form.
Note: Also under the **[ANGLE]** menu, commands 7 and 8 allow transformation from polar form to rectangular form.

12.8 Product and Quotient Theorems

Learning Objectives

Learn how to multiply and divide complex numbers in polar form. You will also explore the process of checking answers using rectangular form and using a graphing calculator.

Introduction

Complex numbers are found in real-world calculations involving quantum mechanics, signal analysis, fluid dynamics, control theory, and many other fields.

In electrical engineering, complex numbers are used for calculations involving *impedance* (the resistance to electric flow in a circuit). Electrical engineers are familiar with the formula

$$V = V_0 e^{j\omega t} = V_0 \left(\cos \omega t + j \sin \omega t\right).$$

By comparing it to the similar expression below, which is explored in this lesson, can you identify the variable j?

$$r_2(cos\theta_2 + isin\theta_2)$$

Product Theorem

Since complex numbers can be transformed to polar form, the operation of multiplication is also possible when complex numbers are in polar form.

Given two complex numbers, $z_1 = r_1(\cos\, \theta_1 + i\, \sin\, \theta_1)$ and $z_2 = r_2(\cos\, \theta_2 + i\, \sin\, \theta_2)$,

multiply the two complex numbers and apply the commutative property of multiplication.

$$\begin{aligned} z_1 \cdot z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \end{aligned}$$

Next, apply the distributive property of multiplication over addition.

$$z_1 \cdot z_2 = r_1 r_2(\cos\theta_1 \cdot \cos\theta_2 + i\cos\theta_1 \cdot \sin\theta_2 + i\sin\theta_1 \cdot \cos\theta_2 + i^2 \sin\theta_1 \cdot \sin\theta_2)$$

Recall $i^2 = -1$.

$$\begin{aligned} z_1 \cdot z_2 &= r_1 r_2(\cos\theta_1\ \cos\theta_2\ + i\cos\theta_1 \sin\theta_2 + i\sin\theta_1 \cos\theta_2\ - \sin\theta_1 \sin\theta_2) \\ &= r_1 r_2(\cos\theta_1\ \cos\theta_2\ - \sin\theta_1 \sin\theta_2 + i\cos\theta_1 \sin\theta_2 + i\sin\theta_1 \cos\theta_2\ ) \\ &= r_1 r_2\left[(\cos\theta_1\ \cos\theta_2\ - \sin\theta_1 \sin\theta_2) + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2\ )\right] \end{aligned}$$

Recall

$$cis\ \theta = \cos\theta + i\sin\theta.$$

Also, note that the product of two complex numbers in trigonometric form is

$$(r_1\ \ cis\ \ \theta_1)(r_2\ \ cis\ \ \theta_2) = r_1 r_2\ \ cis\ \ (\theta_1 + \theta_2).$$

Product of Two Complex Numbers

$$z_1 \cdot z_2 = r_1 r_2\ cis\ (\theta_1 + \theta_2)$$

Quotient Theorem

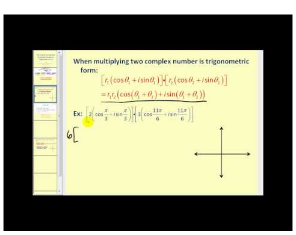
Dividing complex numbers in polar form can be shown using a similar proof that was used to show multiplication of complex numbers. Given two complex numbers, $z_1 = r_1(\cos\ \theta_1 + i\ \sin\ \theta_1)$ and $z_2 = r_2(\cos\ \theta_2 + i\ \sin\ \theta_2)$, then $\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot\ cis\ (\theta_1 - \theta_2)$.

Quotient Theorem

For $z_1 = r_1(cos_{,1} + isin_{,1})$ and $z_1 = r_1(cos_{,1} + isin_{,1})$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} cis\ (\theta_1 - \theta_2).$$

The following video demonstrates how to multiply and divide complex numbers in trigonometric form:



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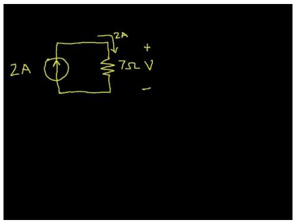
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/53136

The following video demonstrates examples of using Ohm's Law to compute voltage, currents, and power in resistors:



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/183688

Examples

Example 1

Multiply $z_1 = 2+2i$ and $z_2 = 1-\sqrt{3}i$.

Solution:

For z_1,

$$r_1 = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$$

and

$$\tan\theta_1 = \left|\frac{2}{2}\right| = 1$$
$$\theta_1 = \frac{\pi}{4}.$$

Note that θ_1 is in the 1st quadrant since a and b >0.

For z_2,

$$r_2 = \sqrt{1^2+(-\sqrt{3})^2} = \sqrt{4} = 2$$

and

$$\tan\theta_{\text{ref}} = \left|\frac{-\sqrt{3}}{1}\right| = \sqrt{3}$$
$$\theta_{\text{ref}} = \frac{\pi}{3}.$$

Note that θ_2 is in the 4th quadrant, since $a > 0$ and b <0, so $\theta_2 = \frac{5\pi}{3}$.

Use the formula $z_1 \cdot z_2 = r_1 \cdot r_2 cis(\theta_1+\theta_2)$.

$$z_1 \cdot z_2 = 2\sqrt{2}\cdot 2\ cis\ \left[\frac{\pi}{4}+\frac{5\pi}{3}\right] = 4\sqrt{2}\ cis\ \left[\frac{23\pi}{12}\right]$$

Then, $z_1 \cdot z_2 = 4\sqrt{2}\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)$.

If the problem is done using rectangular units,

$$z_1 \cdot z_2 = (2+2i)(1-\sqrt{3}i) = 2-2\sqrt{3}+2i-2\sqrt{3}i = (2+2\sqrt{3})-(2+2\sqrt{3})i.$$

Both results gives $5.46-1.46i$ in decimal form.

Example 2

Using polar multiplication, determine the product: $(6-2\sqrt{3}i)(4+4\sqrt{3}i)$.

Solution:

$$r_1 = \sqrt{(6)^2-(2\sqrt{3})^2} = \sqrt{48} = 4\sqrt{3}$$
$$\tan\theta_{\text{ref}} = \left|\frac{2\sqrt{3}}{6}\right| = \frac{\sqrt{3}}{3}$$
$$\theta_{\text{ref}} = \frac{\pi}{6}$$

Since a >0 and b <0, θ_1 is in the 4th quadrant: $\theta_1 = \frac{11\pi}{6}$.

$$r_2 = \sqrt{(4)^2+(4\sqrt{3})^2} = \sqrt{64} = 8$$
$$\tan\theta_{\text{ref}} = \left|\frac{4\sqrt{3}}{4}\right| = \sqrt{3}$$
$$\theta_{\text{ref}} = \frac{\pi}{3}$$

Since a, b >0, θ_2 is in the 1st quadrant: $\theta_2 = \frac{\pi}{3}$.

Using polar multiplication,

$$z_1 \cdot z_2 = 4\sqrt{3}\cdot 8\ cis\ \left(\frac{11\pi}{6}+\frac{\pi}{3}\right) = 32\sqrt{3}\ cis\ \frac{13\pi}{6}.$$

Since $\theta > 2\pi$, subtract 2π from the argument:

$$z_1 \cdot z_2 = 32\sqrt{3}\ cis\ \frac{\pi}{6}.$$

Example 3

Use polar division for $z_1 = 5-5i$ and $z_2 = -2\sqrt{3}-2i$.

Solution:

$$r_1 = \sqrt{5^2+(-5)^2} = 5\sqrt{2} \quad \text{and} \quad \tan\theta_{\text{ref}} = \left|\frac{-5}{5}\right| = 1 \rightarrow \theta_1 = \frac{7\pi}{4}$$

$$r_2 = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = 4 \quad \text{and} \quad \tan\theta_{ref} = \left|\frac{-2}{-2\sqrt{3}}\right| = \frac{\sqrt{3}}{3} \rightarrow \theta_2 = \frac{7\pi}{6}$$

Use the formula $\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \text{cis}\,[\theta_1 - \theta_2]$.

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{5\sqrt{2}}{4} \cdot cis\left(\frac{7\pi}{4} - \frac{7\pi}{6}\right) \\ &= \frac{5\sqrt{2}}{4} \cdot cis\,\frac{7\pi}{12}\end{aligned}$$

Example 4

Recall the question from the Introduction: Can you identify the variable j when comparing the electrical engineering formula

$$V = V_0 e^{j\omega t} = V_0(\cos\omega t + j\sin\omega t)$$

with the similar expression

$$r(\cos\theta + i\sin\theta)?$$

Solution:

In electrical calculations, the letter I is commonly used to denote *current.* Therefore, imaginary numbers are identified with a j.

Example 5

Find the product: $\left(7, \frac{\pi}{6}\right) \cdot \left(5, \frac{-\pi}{4}\right)$.

Solution:

$$\begin{aligned} r_1 \cdot r_2 &\rightarrow 7 \cdot 5 = 35 \\ \theta_1 + \theta_2 &\rightarrow \frac{\pi}{6} + \frac{-\pi}{4} = -\frac{\pi}{12}\end{aligned}$$

Therefore, the product is 35 $cis\left(\frac{-\pi}{12}\right)$.

Example 6

Find the quotient $\frac{1+2i}{2-i}$.

Solution:

$$r_1 = \sqrt{(1)^2 + (2)^2} = \sqrt{5} \qquad r_2 = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$$

$$\tan\theta_1 = \left|\frac{2}{1}\right| = 2 \rightarrow \theta_1 = 1.107 \text{ radians}$$

Since $a, b > 0$, θ_1 is in the 1st quadrant.

$$\tan\theta_{\text{ref}} = \left|\frac{-1}{2}\right| = \frac{1}{2} \rightarrow \theta_{\text{ref}} = 0.464 \text{ radians}$$

Since $a > 0$ and $b < 0$, θ_2 is in the 4th quadrant, so $\theta_2 = 5.820$ radians.

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{\sqrt{5}}{\sqrt{5}}\, cis\,(1.107 - 5.820)\\ &= cis\,(-4.713)\\ &= \cos(-4.713) + i\sin(-4.713)\\ &= \cos(1.570) + i\sin(1.570)\end{aligned}$$

Summary

- The **Product theorem** can be used to multiply complex numbers:

$$z_1 \cdot z_2 = r_1 \cdot r_2 cis\,(\theta_1 + \theta_2).$$

- The **Quotient theorem** can be used to divide complex numbers:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} cis\,(\theta_1 - \theta_2).$$

Review

1. Find the product using polar form: $(2+2i)(\sqrt{3}-i)$.
2. $2cis\ 40^\circ \cdot 4cis\ 20^\circ$
3. $\frac{2cis(80)}{6cis(200)}$
4. By how many degrees would the radius from the origin to (a,b) need to be rotated to coincide with the radius from the pole to the graph of the product of $a+bi$ and $1+i\sqrt{3}$?

Let $z_1 = \left(7, \frac{\pi}{6}\right)$ and $z_2 = \left(5, \frac{-\pi}{4}\right)$. Find:

5. $z_1 \cdot z_2$
6. $\left(\frac{z_1}{z_2}\right)$
7. $\left(\frac{z_2}{z_1}\right)$

Let $z_1 = \left(8, \frac{\pi}{3}\right)$ and $z_2 = \left(5, \frac{\pi}{6}\right)$. Find:

8. $z_1 \cdot z_2$
9. $\left(\frac{z_1}{z_2}\right)$
10. $\left(\frac{z_2}{z_1}\right)$
11. $(z_1)^2$

12. $(z_2)^3$

Find the products and put your answer in complex number form:

13. Find the product using poorm:$(2+2i)(\sqrt{3}-i)$.
14. $2(\cos 40^\circ + i\sin 40^\circ)\cdot 4(\cos 20^\circ + i\sin 20^\circ)$
15. $2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)\cdot 2\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

Find the quotients and put your answer in complex number form:

16. $2(\cos 80^\circ + i\sin 80^\circ)\div 6(\cos 200^\circ + i\sin 200^\circ)$
17. $3\ cis\ 130^\circ \div 4\ cis\ 270^\circ$

Apply your knowledge. Remember Ohm's Law: $E = I\cdot Z$.

18. The voltage of a circuit can be represented by $160(\cos 330^\circ + i\sin 330^\circ)$ohms. Find the polar form of the current if the polar form of the impedance is $35(\cos 300^\circ + i\sin 300^\circ)$ ohms.
19. What is the voltage of a circuit with an impedance of $14(\cos(-10^\circ)+i\sin(-10^\circ))$ and current of $6(\cos(47^\circ)+i\sin(47^\circ))$?

Review (Answers)

Please see the Appendix.

12.9 Powers and Roots of Complex Numbers

Learning Objectives

Learn to apply "De Moivre's Theorem," which will allow you to calculate powers and roots of complex numbers.

Introduction

Manually calculating (or simplifying) a statement such as: $(14-17i)^5$ or $\sqrt[4]{(3-2i)}$ in its present rectangular form would be a very intensive process at best.

Fortunately, in this lesson you will find there is an alternative: De Moivre's Theorem. De Moivre's Theorem is really the only practical method for finding the powers or roots of a complex number, but there is a catch.

What must be done to a complex number before De Moivre's Theorem can be utilized?

Powers of Complex Numbers

How do we raise a complex number to a power? Let's start with an example:

$$(-4-4i)^3 = (-4-4i)\cdot(-4-4i)\cdot(-4-4i).$$

In rectangular form, this can get very complex. What about in $r\ cis\ \theta$ form?

$$(-4-4i)^3 = (4\sqrt{2})^3\ cis\ \frac{15\pi}{4}$$

The problem becomes

$$4\sqrt{2}\ cis\ \frac{5\pi}{4}\cdot 4\sqrt{2}\ cis\ \frac{5\pi}{4}\cdot 4\sqrt{2}\ cis\ \frac{5\pi}{4},$$

and using our multiplication rule from the previous section,

$$(-4-4i)^3 = (4\sqrt{2})^3\ cis\ \frac{15\pi}{4}.$$

Notice, $(a+bi)^3 = r^3\ cis\ 3\theta$.

In other words, raise r to the same degree that the complex number is raised, and then multiply that by *cis* of the angle multiplied by the number of the degree.

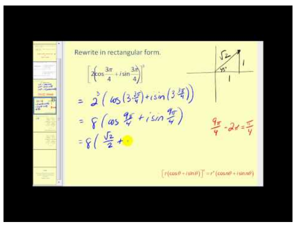
Reflecting on the example above, we can identify **De Moivre's Theorem**:

De Moivre's Theorem

Let $z = r(\cos\theta + i\sin\theta)$ be a complex number in trigonometric form. If n is a positive integer, then

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

The following video demonstrates how to use De Moivre's Theorem to raise complex numbers in trigonometric form to any power:



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Roots of Complex Numbers

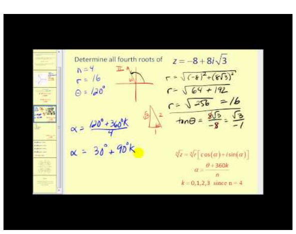
When a new operation is presented in mathematics, the inverse operation often follows. That is generally because the inverse operation is often quite useful in applying the operation and using the new mathematics to solve equations or find new results. This is no exception: The inverse operation of finding a power for a number is to find a root of the same number.

a) Recall from algebra that any root can be written as $x^{1/n}$.

b) Given that the formula for De Moivre's Theorem also works for fractional powers, the same formula can be used for finding roots:

$$z^{1/n} = (a+bi)^{1/n} = r^{1/n} cis\left(\frac{\theta}{n}\right).$$

The following video demonstrates how to determine the nth root of a complex number:



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URL: http://www.ck12.org/flx/render/embeddedobject/53130

Examples

Example 1

Find the value of $(1+\sqrt{3}i)^4$.

Solution:

$$r=\sqrt{(1)^2+(\sqrt{3})^2}=2 \qquad \tan\theta_{\text{ref}}=\left|\frac{\sqrt{3}}{1}\right| \to \theta=\frac{\pi}{3}$$

Since $a, b>0$, θ is in the 1st quadrant.

Use De Moivre's Theorem:

$$\begin{aligned}(\sqrt{3}+i)^7 &= r^7\ cis\ 7\theta\\ &= 2^7\ cis\left(7\cdot\frac{\pi}{3}\right)\\ &= 128\left(\cos\frac{7\pi}{3}+i\sin\frac{7\pi}{3}\right)\\ &= 128\left(\frac{-\sqrt{3}}{2}+\frac{-1}{2}i\right)\\ &= -64\sqrt{3}-64i.\end{aligned}$$

Example 2

Find $\sqrt{1+i}$.

Solution:

In polar form:

$$\sqrt{1+i}=(1+i)^{\frac{1}{2}}=\left(\sqrt{2}\ cis\ \frac{\pi}{4}\right)^{\frac{1}{2}}$$

In trigonometric form:

$$\sqrt{1+i}=\left(\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right)^{\frac{1}{2}}$$

Use De Moivre's Theorem:

$$\begin{aligned}\sqrt{1+i} &= (2^{1/2})^{1/2}\left(\cos\left(\frac{1}{2}\cdot\frac{\pi}{4}\right)+i\sin\left(\frac{1}{2}\cdot\frac{\pi}{4}\right)\right)\\ &= 2^{1/4}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)\\ &\approx 1.189\,(0.924+0.383i)\\ &\approx 1.099+0.455i\end{aligned}$$

Example 3

Find the value of x for the equation $x^3 = \left(1-\sqrt{3}i\right)$.

Solution:

For $a = 1$ and $b = -\sqrt{3}$, $r = 2$ and $\theta = \frac{5\pi}{3}$.

If $z = 1-\sqrt{3}i$, then $z = 2\ cis\ \frac{5\pi}{3}$.

$$\begin{aligned} x &= \left(1-\sqrt{3}i\right)^{\frac{1}{3}} \\ &= \left(2\ cis\ \frac{5\pi}{3}\right)^{\frac{1}{3}} \end{aligned}$$

Use De Moivre's Theorem to find the first solution:

$$x_1 = 2^{\frac{1}{3}} cis\left(\frac{5\pi/3}{3}\right) = 2^{\frac{1}{3}} cis\left(\frac{5\pi}{9}\right).$$

$n = 3$, which means that the three solutions are $\frac{2\pi}{3}$ radians apart, so

$$x_2 = 2^{\frac{1}{3}} cis\left(\frac{5\pi}{9}+\frac{2\pi}{3}\right)$$

and

$$x_3 = 2^{\frac{1}{3}} cis\left(\frac{5\pi}{9}+\frac{2\pi}{3}+\frac{2\pi}{3}\right).$$

Note: It is not necessary to add $\frac{2\pi}{3}$ again. Adding $\frac{2\pi}{3}$ three times equals 2π. That would result in rotating around a full circle and starting where it all began—that is, the 1st solution.

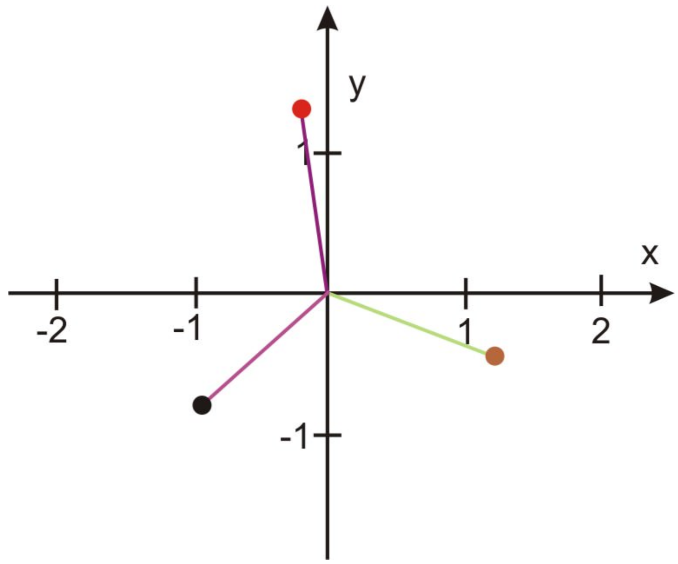
The three solutions are:

$$x_1 = 2^{\frac{1}{3}} cis\left(\frac{5\pi}{9}\right)$$

$$x_2 = 2^{\frac{1}{3}} cis\left(\frac{11\pi}{9}\right)$$

$$x_3 = 2^{\frac{1}{3}} cis\left(\frac{17\pi}{9}\right)$$

Each of these solutions, when graphed, will be $\frac{2\pi}{3}$ apart.



Example 4

Recall the question from the Introduction: What **must** be done to a complex number before De Moivre's Theorem can be utilized?

Solution:

A complex number operation written in rectangular form, such as $(13-4i)^3$, must be converted to polar form to utilize De Moivre's Theorem.

Example 5

What are the two square roots of i?

Solution:

$$z = \sqrt{0+i} \rightarrow r = 1 \text{ and } \theta = \frac{\pi}{2}$$

$$z = \left[1 \cdot cis\ \frac{\pi}{2}\right]^{1/2}$$

Using De Moivre's Theorem:

$$z_1 = \left[1 \cdot cis \frac{\pi}{4}\right] \quad \text{and} \quad z_2 = \left[1 \cdot cis \frac{5\pi}{4}\right]$$
$$z_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \quad \text{and} \quad z_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$$

Example 6

Calculate $\sqrt[4]{(1+0i)}$. What are the four 4th roots of 1?

Solution:

$$\sqrt[4]{(1+0i)} \rightarrow r = 1, \theta = 0, \text{ and } z^{1/4} = (1 \cdot cis\ 0)^{1/4}$$

Using De Moivre's Theorem:

$$z_1 = 1^{1/4}\left(\cos\frac{0}{4} + i\ \sin\frac{0}{4}\right) = 1$$
$$z_2 = 1^{1/4}\left(\cos\left(0+\frac{\pi}{2}\right) + i\sin\left(0+\frac{\pi}{2}\right)\right)$$

Since there are four roots, dividing 2π by 4 yields:

$$z_3 = 1^{1/4}\left(\cos\left(0+\frac{2\pi}{2}\right) + i\sin\left(0+\frac{2\pi}{2}\right)\right) = -1.$$

Finally,

$$z_4 = 1^{1/4}\left(\cos\left(0+\frac{3\pi}{2}\right) + i\sin\left(0+\frac{3\pi}{2}\right)\right) = -i.$$

The four 4th roots of 1 are 1, *i*, -1, and -*i*.

Example 7

Calculate $(\sqrt{3}+i)^7$.

Solution:

$$(\sqrt{3}+i)^7 \to r = 2, \theta = \frac{\pi}{6}, \text{ and } z = \left(2\cdot cis\ \frac{\pi}{6}\right)^{\frac{1}{2}}$$

$$\begin{aligned}(\sqrt{3}+i)^7 &= 2^7\left(\cos(7\cdot\frac{\pi}{3}) + i\sin(7\cdot\frac{\pi}{3})\right)\\ &= 128\left(\cos\frac{7\pi}{3} + i\sin(\frac{7\pi}{3})\right)\\ &= 128\left(\frac{-\sqrt{3}}{2} + \frac{-1}{2}i\right)\\ &= -64\sqrt{3} - 64i\end{aligned}$$

Summary

- **De Moivre's Theorem**: $z^n = r^n(\cos(n_\text{,}) + i\sin(n_\text{,}))$
- De Moivre's Theorem for finding roots: $z^{1/n} = (a+bi)^{1/n} = r^{1/n}cis\left(\frac{\theta}{n}\right)$

Review

Perform the indicated operation on these complex numbers:

1. Divide $\frac{2+3i}{1-i}$.

2. Multiply $(-6-i)(-6+i)$.
3. Multiply $\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^2$.
4. Find the product using polar form: $(2+2i)(\sqrt{3}-i)$.
5. Multiply $2(\cos 40^\circ + i \sin 40^\circ) \cdot 4(\cos 20^\circ + i \sin 20^\circ)$.
6. Multiply $2\left(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8}\right) \cdot 2\left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}\right)$.
7. Divide $2(\cos 80^\circ + i \sin 80^\circ) \div 6(\cos 200^\circ + i \sin 200^\circ)$.
8. Divide $3 \text{ cis}(130^\circ) \div 4 \text{ cis}(270^\circ)$.

Use De Moivre's Theorem.

9. $[3(\cos 80^\circ + i \sin 80^\circ)]^3$
10. $\left[\sqrt{2}\left(\cos \frac{5\pi}{16} + i \sin \frac{5\pi}{16}\right)\right]^4$
11. $\left(\sqrt{3}-i\right)^6$
12. Identify the three complex cube roots of $1+i$
13. Identify the four complex 4th roots of $-16i$
14. Identify the five complex 5th roots of i.

Review (Answers)

Please see the Appendix.

12.10 Parameters and Parameter Elimination

Learning Objectives

Learn to represent equations and graphs with parametric equations.

Introduction

In a parametric equation, the variables x and y are not dependent on one another. Instead, both variables are dependent on a 3rd variable, t. Usually t will stand for time. A real-world example of the relationship between x, y, and t is the height, weight, and age of a baby.

Both the height and weight of a baby depend on time, but there is also clearly a positive relationship between just the height and weight. By focusing on the relationship between the height and weight and letting time hide in the background, you create a parametric relationship between the three variables.

What other types of real-world situations are modeled with parametric equations?

Parametric Equations

Note for Using TI Calculators

In your graphing calculator there is a parametric mode. Once you put your calculator into parametric mode, on the graphing screen you will no longer see $y =$ ___. Instead, you will see:

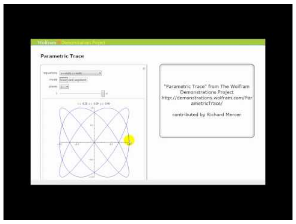
Notice how, for plot one, the calculator is asking for two equations based on variable T:

$$x_{1T} = f(t)$$
$$y_{1T} = g(t)$$

To transform a parametric equation into a rectangular equation, you need to do a process called "eliminating the parameter." To do this, you must solve the $x = f(t)$ equation and substitute this value of t into the y equation or vice versa. This will produce a rectangular equation of y based on x.

There are two major benefits of graphing in parametric form. First, it is straightforward to graph a portion of the parametric equations using the T_{min}, T_{max}, and T_{step} in the window setting. Second, parametric form enables you to graph projectiles in motion and see the effects of time.

The following video defines parametric equations and shows how to graph a parametric equation by hand:



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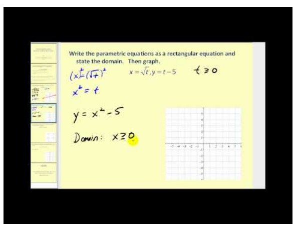
URL: http://www.ck12.org/flx/render/embeddedobject/62014

The following video explains how to write a parametric equation in rectangular form:



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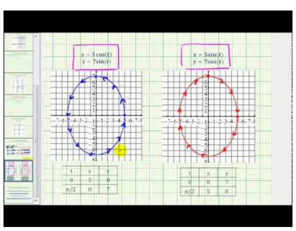
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/62016

The following video explains how to write the equation of an ellipse given in Cartesian form as parametric equations:



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URL: http://www.ck12.org/flx/render/embeddedobject/62018

Examples

Example 1

Eliminate the parameter in the following equations:

$$x = 6t - 2$$
$$y = 5t^2 - 6t$$

Solution:

Solve for t in the 1st equation.

$$x = 6t - 2 \rightarrow \frac{x+2}{6} = t$$

Substitute this value for t into the 2nd equation.

$$y = 5\left(\frac{x+2}{6}\right)^2 - 6\left(\frac{x+2}{6}\right)$$

Example 2

For the given parametric equation, graph over each interval of t.

$$x = t^2 - 4$$
$$y = 2t$$

1) $-2 \le t \le 0$

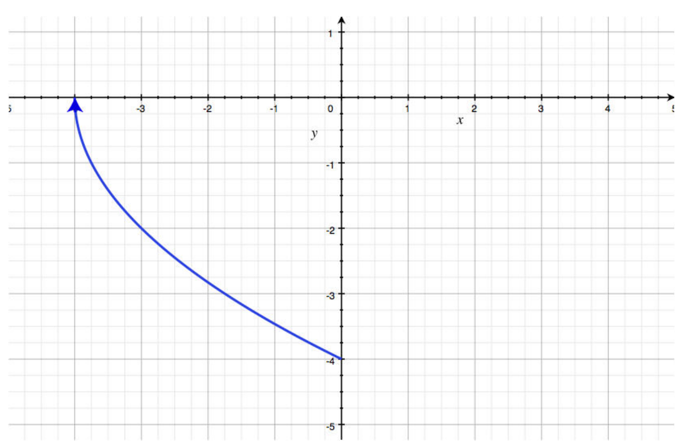
2) $0 \le t \le 5$

3) $-3 \le t \le 2$

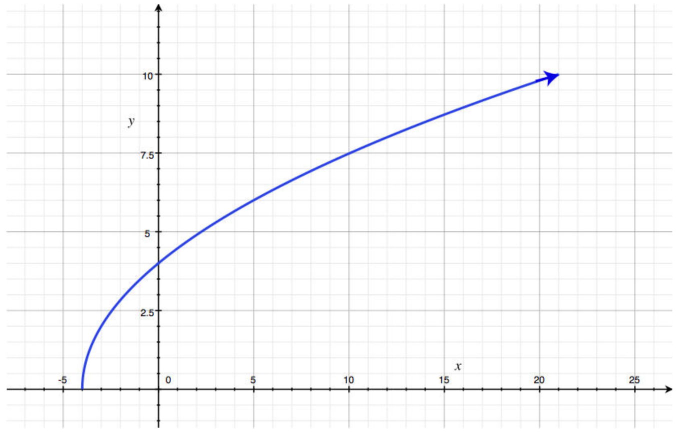
Solutions:

Start by finding the coordinates where t indicates the graph will start and end. You can then use any other points to fill in the graph.

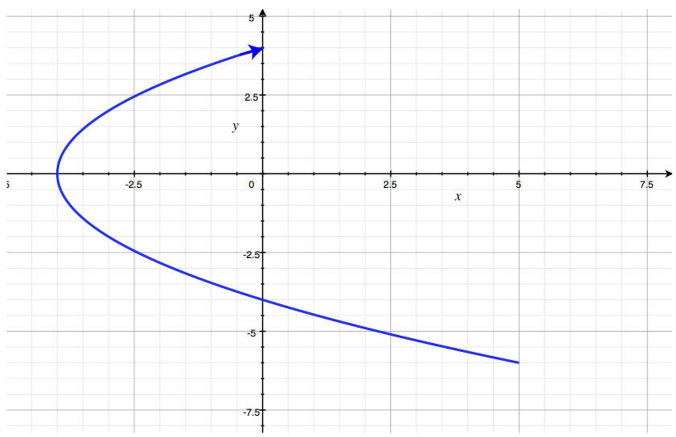
1) For $-2 \le t \le 0,\ t = -2$, and $t = 0$, indicate that the points (0, -4) and (-4, 0) are the endpoints of the graph.



2) $0 \le t \le 5$



3) $-3 \leq t \leq 2$



Example 3

Eliminate the parameter and graph the following parametric curve with the domain $t \in [0, 2\pi]$:

$$x = 3\sin t$$
$$y = 3\cos t$$

Solution:

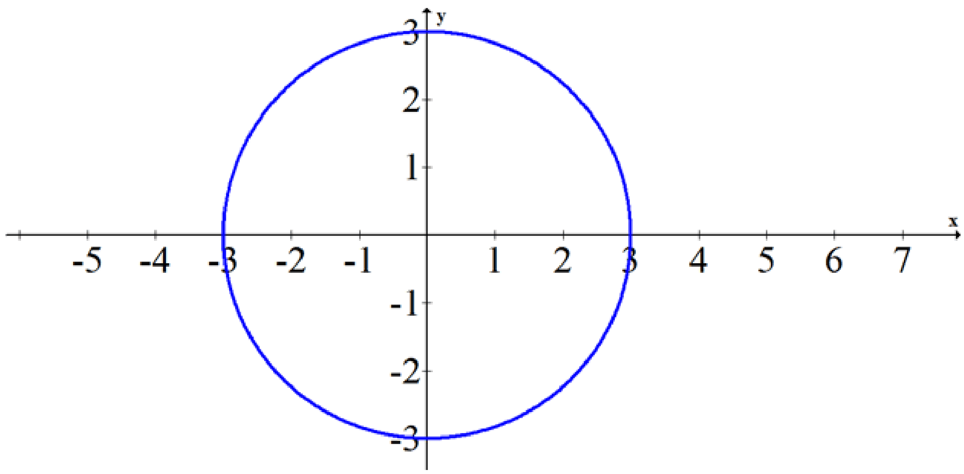
When parametric equations involve trigonometric functions, you can use the Pythagorean Identity $\sin^2 t + \cos^2 t = 1$.

In this problem, $\sin t = \frac{x}{3}$ (from the 1st equation) and $\cos t = \frac{y}{3}$ (from the 2nd equation).

Substitute these values into the Pythagorean Identity, and you have

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$
$$x^2 + y^2 = 9.$$

This is a circle centered at the origin with radius 3.



Example 4

Recall the question from the Introduction: What other types of real-world situations are modeled with parametric equations?

Solution:

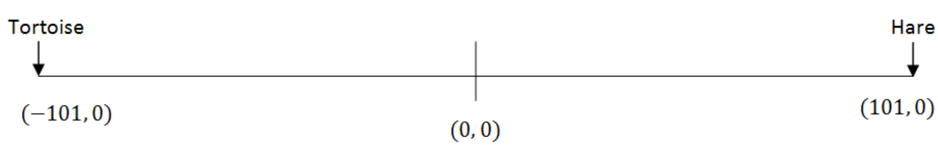
Parametric equations are often used when only a portion of a graph is useful. By limiting the domain of t, you can graph the precise interval of the function you want. Parametric equations are also useful when two different variables jointly depend on a 3rd variable, and you wish to look at the relationship between the two dependent variables. This is very common in statistics where an underlying variable may actually be the cause of a problem, and the observer can only examine the relationship between the outcomes they see. In the physical world, parametric equations are exceptional at graphing position over time, because the horizontal and vertical vectors of objects in free motion are each dependent on time, yet independent of one another.

Example 5

A tortoise and a hare start 202 feet apart and race to a flag halfway between them. The hare decides to take a nap and give the tortoise a 21-second head start. The hare runs at 9.8 feet per second, and the tortoise hustles along at 3.2 feet per second. Who wins this epic race and by how much? Use your calculator to model the race.

Solution:

First, draw a picture and then represent each character with a set of parametric equations.



The tortoise's position is (-101, 0) at $t = 0$ and (-97.8, 0) at $t = 1$. You can deduce that the equation modeling the tortoise's position is:

$$x_1 = -101 + 3.2 \cdot t$$
$$y_1 = 0$$

The hare's position is (101, 0) at $t = 21$ and (91.2, 0) at $t = 22$. Note it does not make sense to make equations modeling the hare's position before 21 seconds have elapsed because the hare is napping and not moving. You can set up an equation to solve for the hare's theoretical starting position had he been running the whole time.

$$x_2 = b - 9.8t$$
$$101 = b - 9.8 \cdot 21$$
$$306.8 = b$$

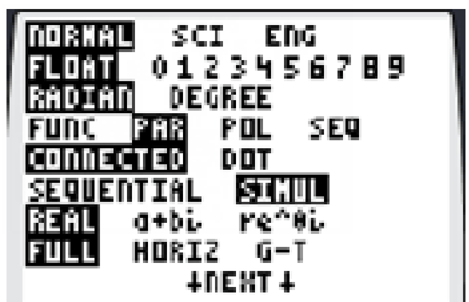
The hare's position equation after $t = 21$ can be modeled by:

$$x_2 = 306.8 - 9.8 \cdot t$$
$$y_2 = 0$$

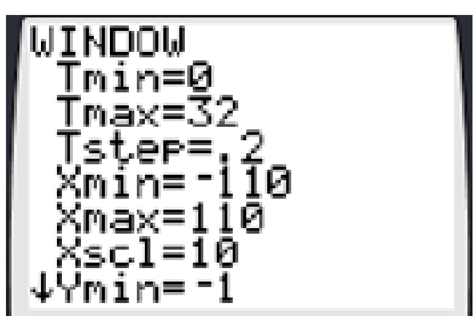
The tortoise crosses $x = 0$ when $t \approx 31.6$. The hare crosses $x = 0$ when $t \approx 31.3$. The hare wins by about a foot.

There are many settings you should know for parametric equations that bring questions like this to life. The TI-84 has features that allow you to see the race happen.

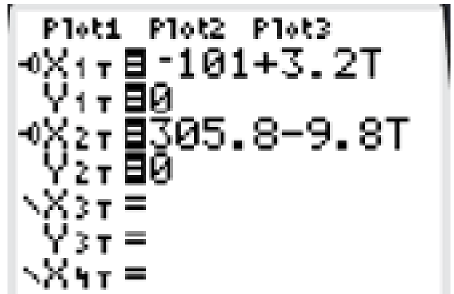
First, set the mode to simultaneous graphing. This will show both the tortoise's and hare's positions at the same time.



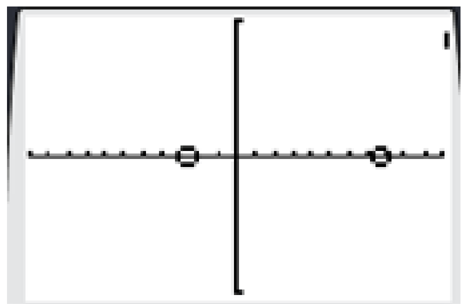
Next, change the graphing window so that t varies between 0 and 32 seconds. The T_{step} determines how often the calculator will calculate points. The larger the T_{step}, the faster and less accurately the graph will plot. Also, change the x to vary between -110 and 110 so you can see the positions of both characters.



Input the parametric equations. Toggle to the left of the x and change the cursor from a line to a line with a bubble at the end. This shows their positions more clearly.



Now when you graph, you can watch the race unfold as the two position graphs race towards each other.



Summary

- **Eliminating the parameter** is a phrase that means to turn a parametric equation that has $x = f(t)$ and $y = g(t)$ into just a relationship between y and x.
- **Parametric form** refers to a relationship that includes $x = f(t)$ and $y = g(t)$.
- **Parameterization** means to write or describe in parametric form.

Review

Eliminate the parameter in the following sets of parametric equations:

1. $x = 3t - 1;\ y = 4t^2 - 2t$

2. $x = 3t^2 + 6t;\ y = 2t - 1$

3. $x = t + 2;\ y = t^2 + 4t + 4$

4. $x = t - 5;\ y = t^3 + 1$

5. $x = t + 4;\ y = t^2 - 5$

For the parametric equation $x = t, y = t^2 + 1$, graph over each interval of t.

6. $-2 \le t \le -1$

7. $-1 \le t \le 0$

8. $-1 \le t \le 1$

9. $-2 \le t \le 2$

10. $-5 \le t \le 5$

11. Eliminate the parameter and graph the following parametric curve: $x = \sin t,\ y = -4 + 3\cos t$.

12. Eliminate the parameter and graph the following parametric curve: $x = 1 + 2\cos t,\ y = 1 + 2\sin t$.

13. Using the previous problem as a model, find a parameterization for the circle with center (2, 4) and radius 3.

14. Find a parameterization for the ellipse $\frac{x^2}{4}+\frac{y^2}{25}=1$. Use the fact that $\cos^2 t+\sin^2 t=1$. Check your answer with your calculator.

15. Find a parameterization for the ellipse $\frac{(x-4)^2}{9}+\frac{(y+1)^2}{36}=1$. Check your answer with your calculator.

Review (Answers)

Please see the Appendix.

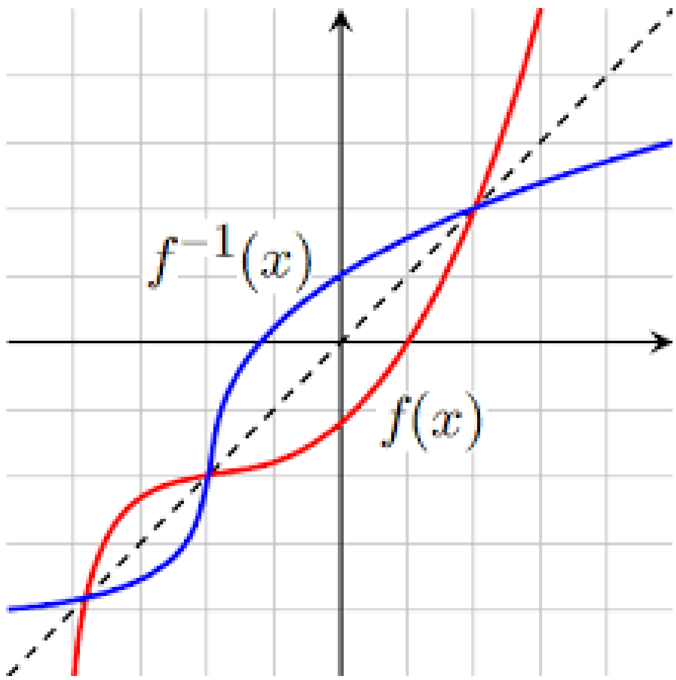
12.11 Parametric Inverses

Learning Objectives

Learn to find and graph inverses of parametric functions and relations.

Introduction

A graph and its inverse are reflections of each other across the line $y = x$.



To find an inverse algebraically, you can substitute x for y and then solve for y. Parametric equations are based on a 3rd variable, so we need to explore how to find the inverse of parametric equations.

Is the inverse of a function always a function?

Parametric Inverses

To find the **inverse** of a parametric equation, you must switch the function for x with the function for y. This will switch all the points from (x,y) to (y,x), and also have the effect of visually reflecting the graph over the line $y = x$.

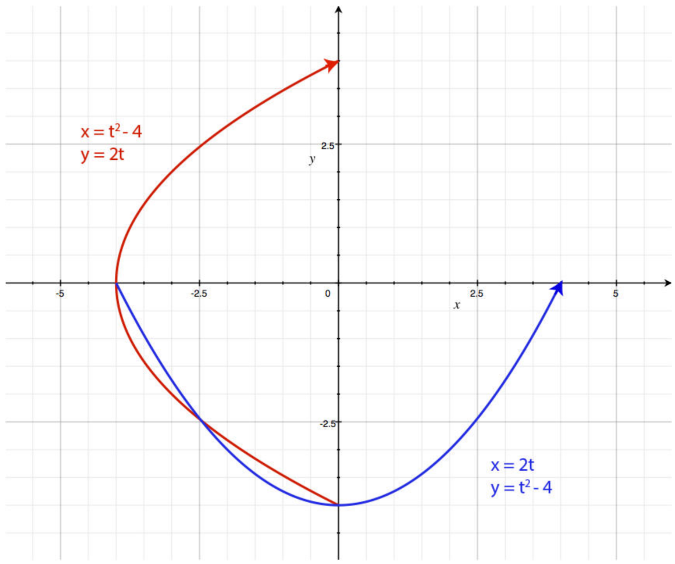
Similar to the inverses of regular functions, the inverses of parametric equations are often restricted so they are also functions. Take the following parametric equations:

$$x = 2t$$
$$y = t^2 - 4$$

To find and graph the inverse of the parametric function on the domain $-2 < t < 2$, first switch the x and y functions and graph.

$$x = t^2 - 4$$
$$y = 2t$$

The original function is shown in blue and the inverse is shown in red.



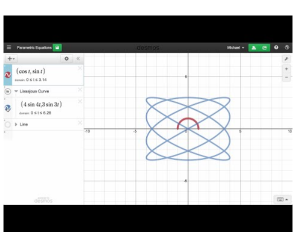
Play, Learn, and Explore Parametric Inverses: www.ck12.org/a/2205042

The following video shows how to graph parametric equations using the TI-84:

MEDIA
Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/189450

In Desmos, graphing a parametric equation is as easy as plotting an ordered pair. Instead of numerical coordinates, you'll need to use expressions in terms of t, such as $(\cos t, \sin t)$.

The following video demonstrates how to graph parametric equations using Desmos:



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URL: http://www.ck12.org/flx/render/embeddedobject/194974

Examples

Example 1

Does the coordinate point (4, 8) satisfy the following function or its inverse?

$$x = 2t^2 - 2$$
$$y = t^2 - 1$$

Solution:

Substitute the point into the original function and solve for t.

$$\begin{aligned} x &= 2t^2 - 2 \qquad \text{and} & amp;y &= t^2 - 1 \\ 4 &= 2t^2 - 2 & 8 &= t^2 - 1 \\ 6 &= 2t^2 & 9 &= t^2 \\ 3 &= t^2 & \pm 3 &= t \\ \pm\sqrt{3} &= t \end{aligned}$$

Since the value solved for t differs, the point does not satisfy the original function.

Substitute the point into the inverse and solve for t.

$$\begin{aligned} x &= t^2 - 1 \qquad \text{and} & amp;y &= 2t^2 - 2 \\ 4 &= t^2 - 1 & 8 &= 2t^2 - 2 \\ 5 &= t^2 & 10 &= 2t^2 \\ \pm\sqrt{5} &= t & 5 &= t^2 \\ & & \pm\sqrt{5} &= t \end{aligned}$$

Since the value solved for t is the same, the point satisfies the inverse function.

Example 2

Parameterize the following function and then graph the function and its inverse:

$$y = x^2 + x - 4$$

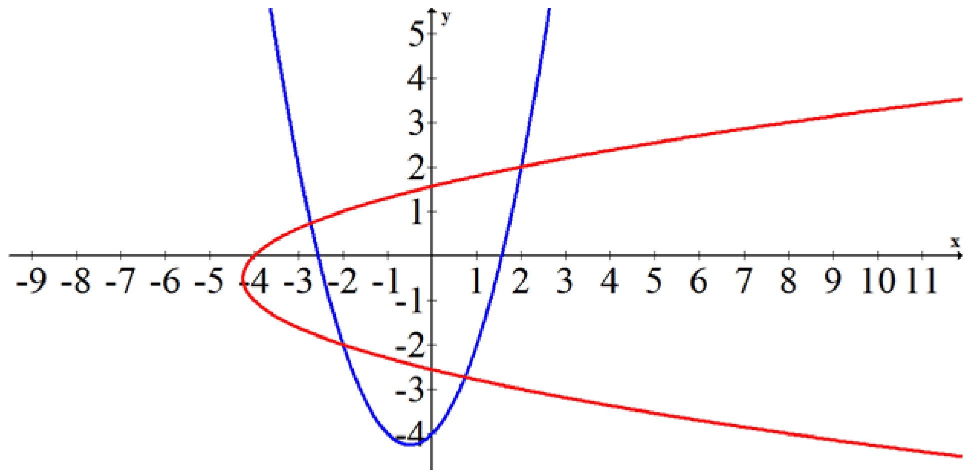
Solution:

For the original function, the parameterization is

$$x = t$$
$$y = t^2 + t - 4.$$

The inverse is

$$x = t^2 + t - 4$$
$$y = t.$$



Example 3

Find the points of intersection of the function and its inverse from Example 2.

Solution:

The parameterized function is

$$x_1 = t$$
$$y_1 = t^2 + t - 4.$$

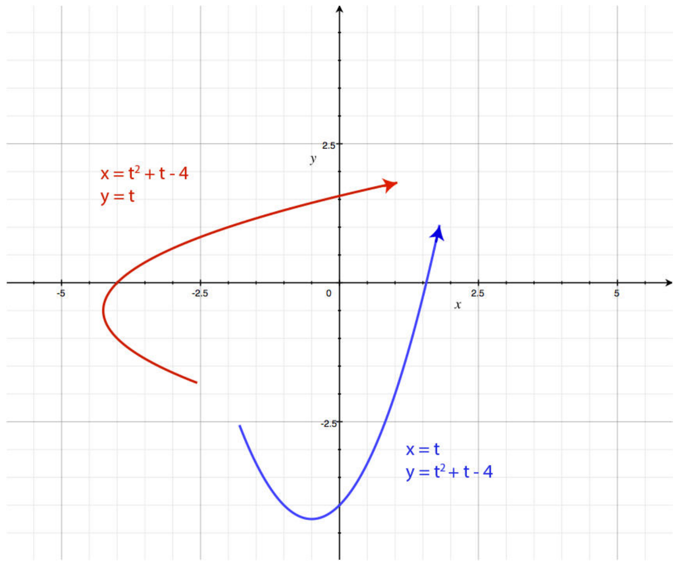
The inverse is

$$x_2 = t^2 + t - 4$$
$$y_2 = t.$$

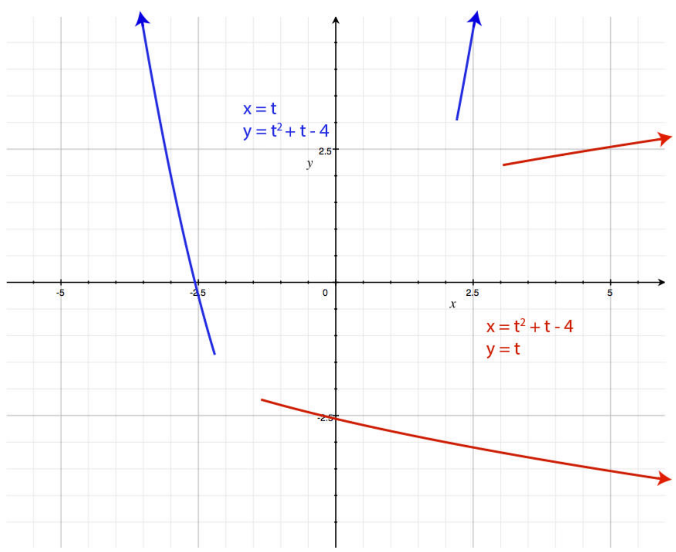
An intersection for two sets of parametric equations happens when the points exist at the same x, y, and t. To find where these intersect, set $x_1 = x_2$ and $y_1 = y_2$ and solve.

$$t = t^2 + t - 4$$
$$t^2 = 4$$
$$t = \pm 2$$

You can tell from the graph in Example 2 that there seem to be four points of intersection. Since t can mean time, the question of intersection is more complicated than simply overlapping. It means that the points are at the same $x-$ and $y-$coordinate at the same time. Note what the graphs look like when $-1.8 < t < 1.8$.



Note what the graphs look like when $t > 2.2$ or $t < -2.2$.



Notice how when these partial graphs are examined, there is no intersection at anything besides $t = \pm 2$ and the points (2, 2) and (-2, -2). While the paths of the graphs intersect in four places, they intersect at the same time only twice.

Example 4

Recall the question from the Introduction: Is the inverse of a function always a function?

Solution:

The inverse of a function is not always a function. To see whether the inverse of a function will be a function, you must perform the horizontal line test on the original function. If the function passes the horizontal line test, then the inverse will be a function. If the function does not pass the horizontal line test, then the inverse produces a relation rather than a function. To find the inverse of a parametric equation, switch the x function with the y function.

Example 5

Does the coordinate point (-2, 6) satisfy the following function or its inverse?

$$x = t^2 - 10$$
$$y = \frac{t}{2} - 4$$

Solution:

Substitute the point into the original function and solve for t.

$$-2 = t^2 - 10 \quad \text{and} \quad 6 = \frac{t}{2} - 4$$
$$8 = t^2 \qquad 10 = \frac{t}{2}$$
$$\pm 2\sqrt{2} = t \qquad 20 = t$$

Since the value solved for t differs, the point does not satisfy the original function.

Substitute the point into the inverse and solve for t.

$$6 = t^2 - 10 \quad \text{and} \quad -2 = \frac{t}{2} - 6$$
$$16 = t^2 \qquad 4 = \frac{t}{2}$$
$$\pm 4 = t \qquad 8 = t$$

Since the value solved for t differs, the point does not satisfy the inverse function either.

Example 6

Identify where the following parametric function intersects with its inverse.

$$x = 4t$$
$$y = t^2 - 16$$

Solution:

The inverse is

$$x_2 = t^2 - 16$$
$$y_2 = 4t.$$

Solve for t when $x_1 = x_2$ and $y_1 = y_2$.

$$4t = t^2 - 16$$
$$0 = t^2 - 4t - 16$$
$$t = \frac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot (-16)}}{2} = \frac{4 \pm \sqrt{80}}{2} = \frac{4 \pm 4\sqrt{5}}{2} = 2 \pm 2\sqrt{5}$$

The points that correspond to these two times are

$$x = 4(2 + 2\sqrt{5}),\ y = (2 + 2\sqrt{5})^2 - 16$$
$$x = 4(2 - 2\sqrt{5}),\ y = (2 - 2\sqrt{5})^2 - 16.$$

Summary

- Two functions are **inverses** if for every point (a,b) on the 1st function, there exists a point (b,a) on the 2nd function.
- An **intersection** for two sets of parametric equations happens when the points exist at the same x, y, and t.

Review

Use the function $x = t-4;\ y = t^2+2$ for numbers 1-3 below.

1. Find the inverse of the function.

2. Does the coordinate point (-2, 6) satisfy the function or its inverse?

3. Does the coordinate point (0, 1) satisfy the function or its inverse?

Use the relation $x = t^2;\ y = 4-t$ for 4-6 below.

4. Find the inverse of the relation.

5. Does the coordinate point (4, 0) satisfy the relation or its inverse?

6. Does the coordinate point (0, 4) satisfy the relation or its inverse?

Use the function $x = 2t+1;\ y = t^2-3$ for 7-9.

7. Find the inverse of the function.

8. Does the coordinate point (1, 5) satisfy the function or its inverse?

9. Does the coordinate point (9, 13) satisfy the function or its inverse?

Use the function $x = 3t+14;\ y = t^2-2t$ for 10-11.

10. Find the inverse of the function.

11. Identify where the parametric function intersects with its inverse.

Use the relation $x = t^2;\ y = 4t-4$ for 12-13.

12. Find the inverse of the relation.

13. Identify where the relation intersects with its inverse.

14. Parameterize $f(x) = x^2+x-6$ and then graph the function and its inverse.

15. Parameterize $f(x) = x^2+3x+2$ and then graph the function and its inverse.

Review (Answers)

Please see the Appendix.

12.12 Applications of Parametric Equations

Learning Objectives

Learn how to apply parametric functions to solve problems you would see in real life situations.

Introduction

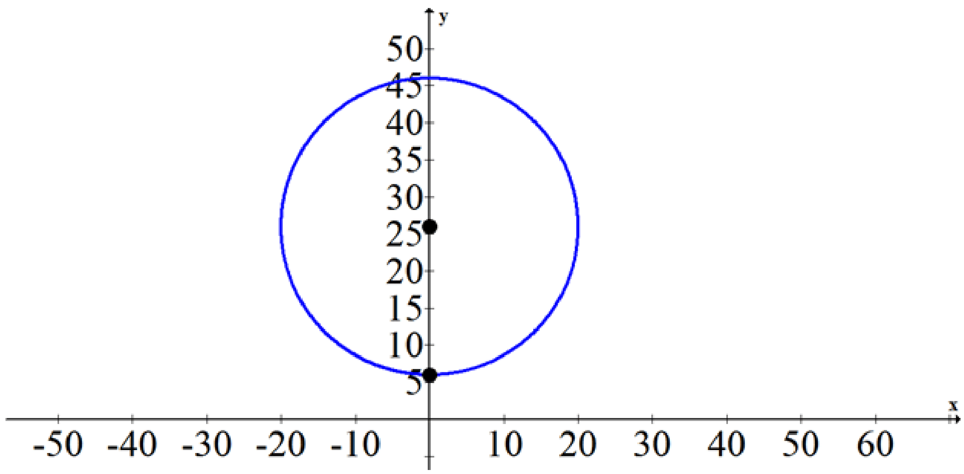
A regular function has the ability to graph the height of an object over time. Parametric equations allow you to actually graph the complete position of an object over time. For example, parametric equations allow you to make a graph that represents the position of a point on a Ferris wheel. All the details like height off the ground, direction, and speed of spin can be modeled using the parametric equations.

What is the position, equation, and graph of a point on a Ferris wheel that starts at a low point of 6 feet off the ground, spins counterclockwise to a height of 46 feet off the ground, then goes back down to 6 feet in 60 seconds?

Applications of Parametric Equations

Two types of parametric equations are typical in real-life situations: The 1st is circular motion as was described in the concept problem. The 2nd is projectile motion.

Parametric equations that describe circular motion will have x and y as periodic functions of sine and cosine. Either x will be a sine function and y will be a cosine function, or the other way around. The best way to come up with parametric equations for these situations is to first draw a picture of the circle you are trying to represent.



Next, it is important to note the starting point, center point, and direction. You should already have the graphs of sine and cosine memorized, so that when you see a pattern in words or as a graph, you can identify what you see as + sine, - sine, + cosine, and - cosine. In this example, the vertical component starts at a low point of 6, travels to a middle point of 26, up to a height of 46, and then back down. This is a - cosine pattern. The amplitude of the - cosine graph is 20, and the vertical shift is 26. Lastly, the period is 60. You can use the period to help you find b.

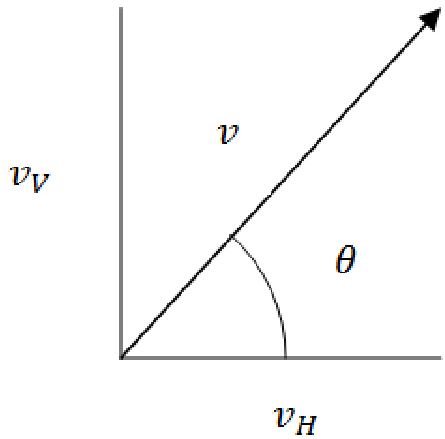
$$60 = \frac{2\pi}{b}$$
$$b = \frac{\pi}{30}$$

Thus, the vertical parameterization is

$$y = -20\cos\left(\frac{\pi}{30}t\right) + 26.$$

Try to find the horizontal parameterization on your own. The solution will be discussed when we revisit this problem later.

Projectile motion has a vertical component that is quadratic and a horizontal component that is linear. This is because there are three parameters that influence the position of an object in flight: starting height, initial velocity, and force of gravity. The horizontal component is independent of the vertical component. This means that gravity doesn't affect it and the starting horizontal velocity will remain the horizontal velocity for the entire flight of the object.



Note that gravity, g, has a force of about $-32\ ft/s^2$ or $-9.81\ m/s^2$. The examples and practice questions in this concept will use feet.

If an object is launched from the origin at a velocity of v, then it has horizontal and vertical components that can be found using basic trigonometry.

$$\sin\theta = \frac{v_V}{v} \rightarrow v \cdot \sin\theta = v_V$$
$$\cos\theta = \frac{v_H}{v} \rightarrow v \cdot \cos\theta = v_H$$

The horizontal component is basically finished. The only adjustments that would have to be made are if the starting location is not at the origin, wind is added, or the projectile travels to the left instead of the right. (See Example 1, below.)

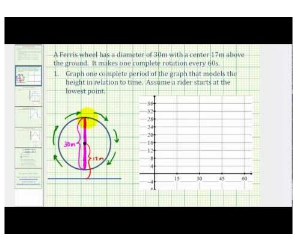
$$x = t \cdot v \cdot \cos\theta + h, \text{ where } h \text{ is the initial } x\text{-position}$$

The vertical component also needs to include gravity and the starting height. The general equation for the vertical component is

$$y = \frac{1}{2} \cdot g \cdot t^2 + t \cdot v \cdot \sin\theta + k.$$

The constant g represents the acceleration due to gravity, t represents time, v represents initial velocity, and k represents starting height. You will explore this equation further in calculus and physics. Note that in this concept, most answers will be found and confirmed using technology such as your graphing calculator.

The video below explains how to determine the equation that models the height of a person on a Ferris wheel, and the times when a person is at a specific height.

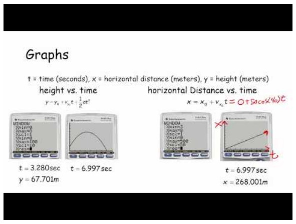


MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/189449

The following video demonstrates applications of parametric equations:



MEDIA

Click image to the left or use the URL below.

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Examples

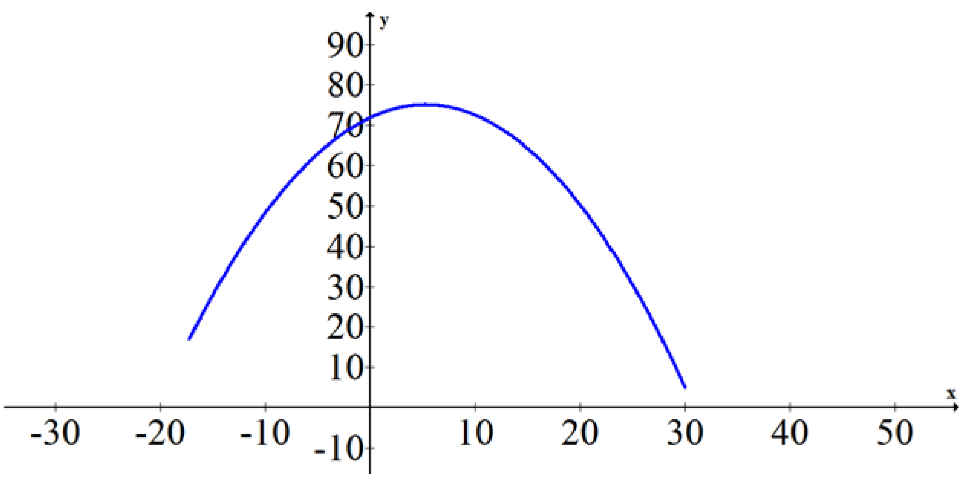
Example 1

A ball is thrown from the point (30, 5) at an angle of $\frac{4\pi}{9}$ to the left, at an initial velocity of 68 ft/s. Model the position of the ball over time using parametric equations. Use your graphing calculator to graph your equations for the 1st four seconds that the ball is in the air.

Solution:

The horizontal component is $x = -t \cdot 68 \cdot \cos(\frac{4\pi}{9}) + 30$. Note the negative sign because the object is traveling to the left, and the +30 because the object starts at (30, 5).

The vertical component is $y = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot 68 \cdot \sin(\frac{4\pi}{9}) + 5$. Note that $g = -32$ because gravity has a force of $-32\ ft/s^2$, and the +5 because the object starts at (30, 5).



Example 2

When does the ball from Example 1 reach its maximum, and when does the ball hit the ground? How far did the person throw the ball?

Solution:

To find when the function reaches its maximum, you can find the vertex of the parabola. Use your calculator to approximate the maximum after you have graphed it. Depending on how small you make your T_{step}, you should find the maximum height to be about 75 feet.

To find out when the ball hits the ground, you can set the vertical component equal to zero and solve the quadratic equation. You can also use the table feature on your calculator to determine when the graph goes from having a positive vertical value to a negative vertical value. The benefit for using the table is that it simultaneously tells you the x-value of the zero.

After about 4.2588 seconds, the ball hits the ground at (-20.29, 0). This means the person threw the ball from (30, 5) to (-20.29, 0), a horizontal distance of just over 50 feet.

Example 3

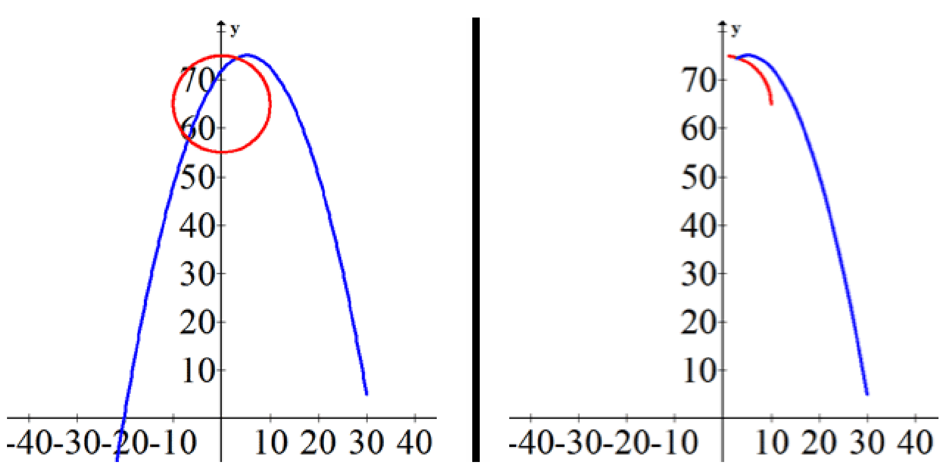
Kieran is on a Ferris wheel, and his position is modeled by these parametric equations:

$$x_K = 10 \cdot \cos\left(\frac{\pi}{5}t\right)$$
$$y_K = 10 \cdot \sin\left(\frac{\pi}{5}t\right) + 65$$

Jason throws the ball modeled by the equation in Example 1 towards Kieran, who can catch the ball if it gets within three feet. Does Kieran catch the ball?

Solution:

This question is designed to demonstrate the power of your calculator. If you simply model the two equations simultaneously and ignore time, you will see several points of intersection. This graph is shown below on the left. These intersection points are not interesting because they represent where Kieran and the ball are at the same place, but at different moments in time.



When the T_{max} is adjusted to 2.3 so that each graph represents the time from 0 to 2.3, you get a better sense that at about 2.3 seconds, the two points are close. This graph is shown above on the right.

You can now use your calculator to help you determine if the distance between Kieran and the ball actually does go below 3 feet. Start by plotting the ball's position in your calculator as x_1 and y_1, and Kieran's position as x_2 and y_2. Then, plot a new parametric equation that compares the distance between these two points over time. You can put this under x_3 and y_3. Note that you can find the x_1, x_2, y_1, y_2 entries in the **VARS** under **Y-VARS** and the parametric menu.

$$x_3 = t$$
$$y_3 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Now when you graph, you should change your window settings and let t vary between 0 and 4, the x window show between 0 and 4, and the y window show between 0 and 5. This way it should be clear if the distance truly does get below 3 feet.

Depending on how accurate your T_{step} is, you should find that the distance drops below 3 feet. Kieran does indeed catch the ball.

Example 4

Recall the question from the Introduction: What is the position equation and graph of a point on a Ferris wheel that starts at a low point of 6 feet off the ground, spins counterclockwise to a height of 46 feet off the ground, then goes back down to 6 feet in 60 seconds?

Solution:

The parametric equations for the point on the wheel are:

$$x = 20\sin\left(\frac{\pi}{30}t\right)$$
$$y = -20\cos\left(\frac{\pi}{30}t\right) + 26$$

The horizontal parameterization is found by noticing that the x values start at 0, go up to 20, go back to 0, then down to -20, and finally back to 0. This is a + sine pattern with amplitude 20. The period is the same as with the vertical component.

Example 5

At what velocity does a football need to be thrown at a 45° angle in order to make it all the way across a football field? Assume the person at the other end of the field is catching at the same height as the person throwing the ball.

Solution:

A football field is 100 yards or 300 feet. The parametric equations for a football thrown from (300, 0) back to the origin at speed v are:

$$x = -t \cdot v \cdot \cos\left(\frac{\pi}{4}\right) + 300$$
$$y = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot v \cdot sin\left(\frac{\pi}{4}\right)$$

Substituting the point (0, 0) in for (x, y) produces a system of two equations with two variables v, t.

$$0 = -t \cdot v \cdot \cos\left(\frac{\pi}{4}\right) + 300$$
$$0 = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot v \cdot \sin\left(\frac{\pi}{4}\right)$$

You can solve this system many different ways, including graphing, the quadratic formula, etc.

$$t = \frac{5\sqrt{3}}{2} \approx 4.3 \text{ seconds}$$
$$v = 40\sqrt{6} \approx 97.98 \text{ ft/s}$$

In order for someone to throw a football at a 45° angle all the way across a football field, they would need to throw at about 98 ft/s, which is about 66.8 mph.

$$\frac{98 \text{ ft}}{1 \text{ s}} \cdot \frac{3{,}600 \text{ s}}{1 \text{ hr}} \cdot \frac{1 \text{ mile}}{5280 \text{ ft}} \approx \frac{66.8 \text{ miles}}{1 \text{ hr}}$$

Example 6

Suppose Danny stands at the point (300, 0) and launches a football at 67 mph at an angle of 45° towards Johnny, who is at the origin. Suppose Johnny also throws a football towards Danny at 60 mph at an angle of 50°, at the exact same moment. There is a 4 mph breeze in Johnny's favor. Do the balls collide in midair?

Solution:

Calculate the velocity of each person and of the wind in feet per second:

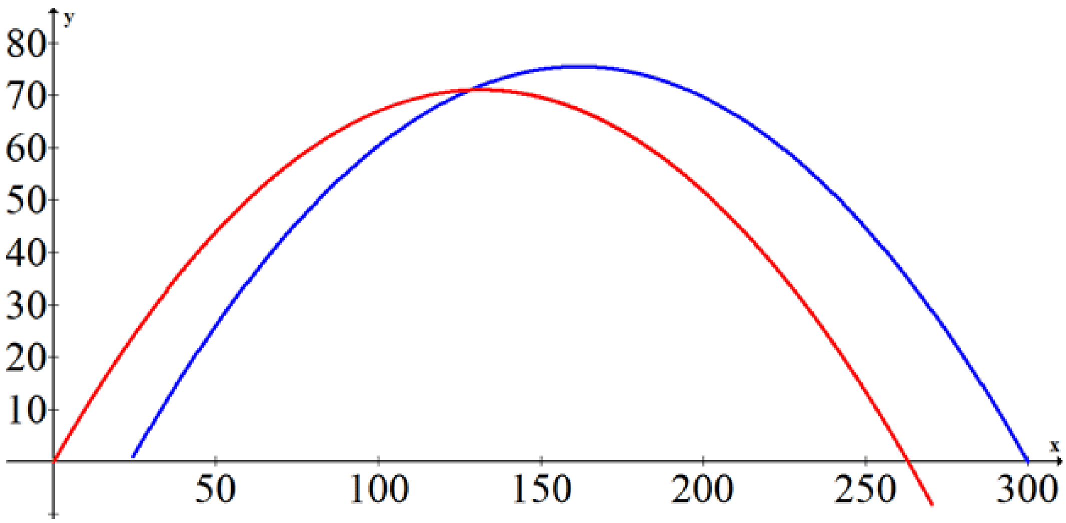
$$\frac{67 \text{ miles}}{1 \text{ hr}} \cdot \frac{1 \text{ hr}}{3{,}600 \text{ s}} \cdot \frac{5{,}280 \text{ ft}}{1 \text{ mile}} \approx 98.27 \text{ ft/s}$$
$$\frac{60 \text{ miles}}{1 \text{ hr}} \cdot \frac{1 \text{ hr}}{3{,}600 \text{ s}} \cdot \frac{5{,}280 \text{ ft}}{1 \text{ mile}} = 88 \text{ ft/s}$$
$$\frac{4 \text{ miles}}{1 \text{ hr}} \cdot \frac{1 \text{ hr}}{3{,}600 \text{ s}} \cdot \frac{5{,}280 \text{ ft}}{1 \text{ mile}} \approx 5.87 \text{ ft/s}$$

The location of Danny's ball can be described with the parametric equations below (in radians). Note that the wind simply adds a linear term.

$$x_1 = -t \cdot 98.27 \cdot \cos\left(\frac{\pi}{4}\right) + 300 + 5.87t$$
$$y_1 = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot 98.27 \cdot \sin\left(\frac{\pi}{4}\right)$$

The location of Johnny's ball can be described with the following parametric equations (also in radians):

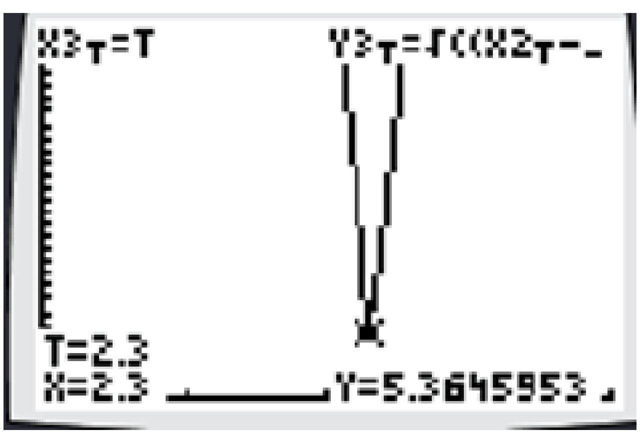
$$x_2 = t \cdot 88 \cdot \cos\left(\frac{5\pi}{18}\right) + 5.87t$$
$$y_2 = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot 88 \cdot \sin\left(\frac{5\pi}{18}\right)$$



You might observe that the two graphs overlap at about 125 ft, but this is unimportant because they are probably at that location at different times. Plot the distance over time and see how close the footballs actually get in midair.

$$x_3 = t$$
$$y_3 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The TI-83 may run slower than the TI-84 when you do this calculation, depending on how small your T_{step} is. The resulting graph is not a parabola, and the calculator cannot find a minimum in the way it can in other circumstances. Still, you can trace and determine that the footballs get 5.36 ft or closer in midair.



Example 7

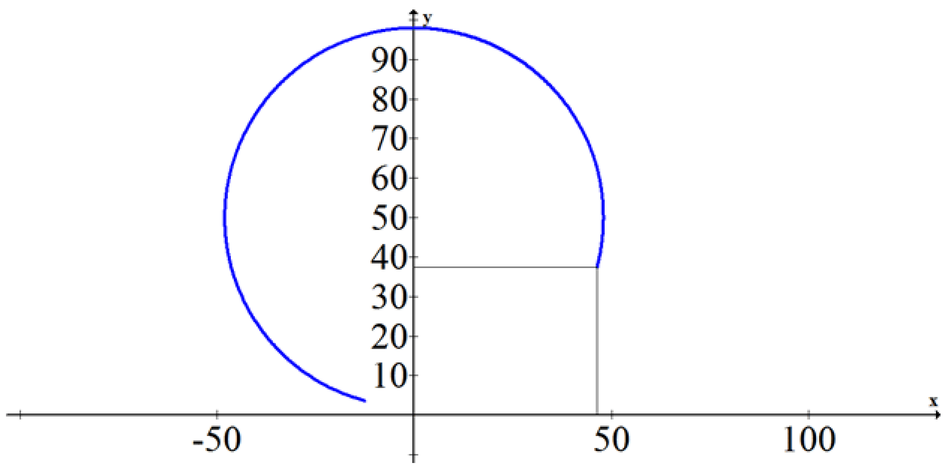
Nikki got on a Ferris wheel 10 seconds ago. She started 2 ft off the ground at the lowest point of the wheel, and will make a complete cycle in 4 minutes. The ride reaches a maximum height of 98 ft and spins clockwise. Write parametric equations that model Nikki's position over time. Where will Nikki be 3 minutes from now?

Solution:

Don't let the 10-second difference confuse you. To deal with the time difference, use $\left(t+\frac{1}{6}\right)$ instead of t in each equation. When $t=0$, 10 seconds $\left(\frac{1}{6}\text{ of a minute}\right)$ have already elapsed.

$$x=-48\cdot\sin\left(\frac{\pi}{2}\left(t+\frac{1}{6}\right)\right)$$
$$y=-48\cdot\cos\left(\frac{\pi}{2}\left(t+\frac{1}{6}\right)\right)+50$$

At $t=3, x\approx 46.36$ and $y\approx 37.58$.



Summary

- A calculator can reference **internal variables** like x_1, y_1 that have already been set in the calculator's memory to form new variables like x_3, y_3.
- The **horizontal and vertical components** of parametric equations are the $x=$ and $y=$ functions respectively.

Review

Candice gets on a Ferris wheel at its lowest point, 3 feet off the ground. The Ferris wheel spins clockwise to a maximum height of 103 feet, making a complete cycle in 5 minutes.

1. Write a set of parametric equations to model Candice's position.

2. Where will Candice be in 2 minutes?

3. Where will Candice be in 4 minutes?

One minute ago, Guillermo got on a Ferris wheel at its lowest point, 3 feet off the ground. The Ferris wheel spins clockwise to a maximum height of 83 feet, making a complete cycle in 6 minutes.

4. Write a set of parametric equations to model Guillermo's position.

5. Where will Guillermo be in 2 minutes?

6. Where will Guillermo be in 4 minutes?

Kim throws a ball from (0, 5) to the right at 50 mph at a 45° angle.

7. Write a set of parametric equations to model the position of the ball.

8. Where will the ball be in 2 seconds?

9. How far does the ball get before it lands?

David throws a ball from (0, 7) to the right at 70 mph at a 60° angle. There is a 6-mph wind in David's favor.

10. Write a set of parametric equations to model the position of the ball.

11. Where will the ball be in 2 seconds?

12. How far does the ball get before it lands?

Suppose Riley stands at the point (250, 0) and launches a football at 72 mph at an angle of 60° towards Kristy who is at the origin. Suppose Kristy also throws a football towards Riley at 65 mph at an angle of 45° at the exact same moment. There is a 6-mph breeze in Kristy's favor.

13. Write a set of parametric equations to model the position of Riley's ball.

14. Write a set of parametric equations to model the position of Kristy's ball.

15. Graph both functions and explain how you know that the footballs don't collide even though the two graphs intersect.

Review (Answers)

Please see the Appendix.

12.13 Project: Polar Coordinates and Parametric Equations

A **cycloid** is the locus of a point on a circle as the circle "rolls" along a flat surface.

A **hypocycloid** is the locus of a point on the circumference of a circle that rolls without slipping inside the circumference of another circle.

For this project, we will investigate the parametric equations of the hypocycloid and the impact of changing the ratio of the radii of the two circles. We'll define our variables as follows:

Our hypocycloid is a curve traced by a fixed point P on a circle C of a radius b, as C rolls on the inside of a circle with center O and radius a. The initial position of P is $(a, 0)$, and the parameter θ is the angle created by segments $\overline{OA}$ and the diameter of circle C extended to meet O.

Step 1: Determine the parametric equations of the hypocycloid.

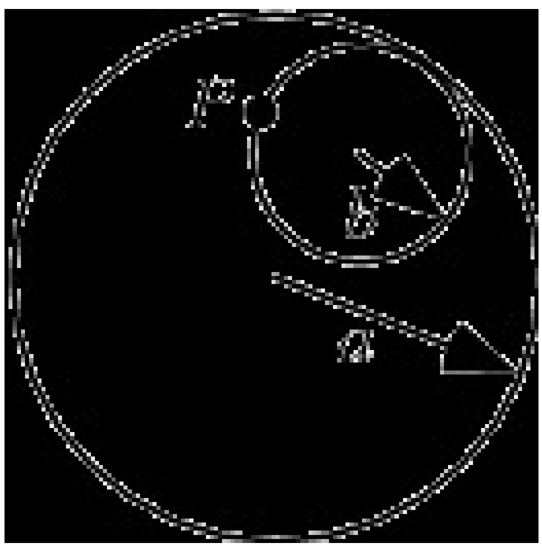
Step 2: Determine how the value of a impacts the graph.

Step 3: Determine the impact of the graph when $b = 1$ and $a = \frac{n}{d}$, where n and d have no common factor. What happens when $n = d + 1$?

Step 4: Determine the impact on the graph if a is not rational.

Step 5: If the circle C rolls outside of the circle, the curve is called an epicycloid. Determine the parametric equations for the epicycloid.

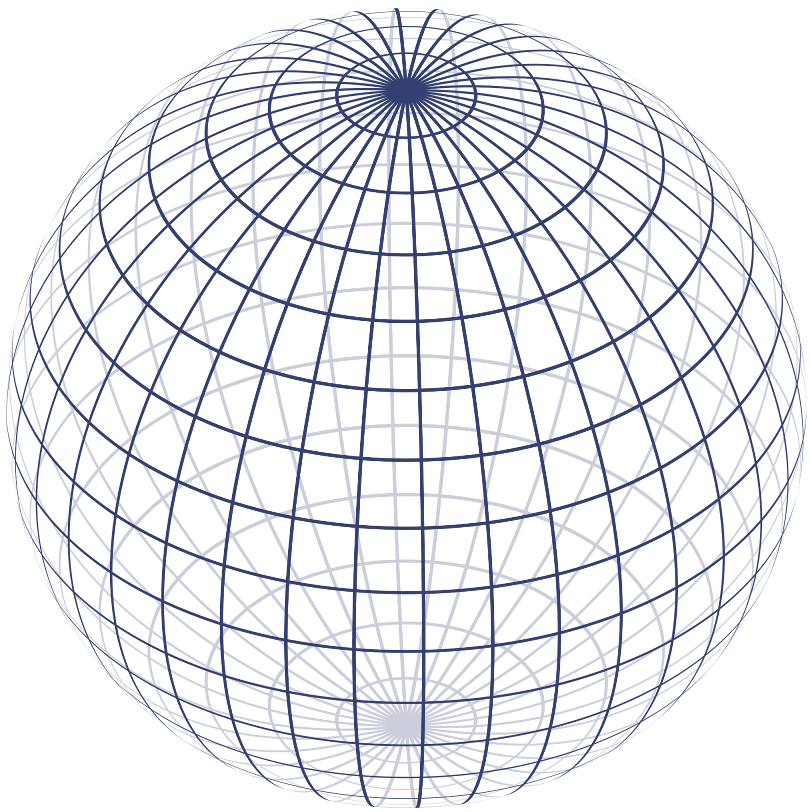
Step 6: Explain how the graph of epicycloids are impacted by changes in the radii.



12.14 Summary: Polar Coordinates and Parametric Equations

In this chapter you explored polar forms of graphs and equations, as well as parameters and parametric equations.

Polar graphs are useful, we discovered, for many applications in science—and you can just imagine their applications for mapping the globe.



Polar coordinates can be converted from rectangular coordinates, and equations can also be converted to and from polar form. Systems of polar equations can be solved just as systems in the rectangular plane. Conics can be converted for use in the polar plane as well. Complex numbers can be converted to polar form and graphed accordingly.

Operations with polar coordinates include the Product and Quotient theorems, as well as power and root applications, including De Moivre's Theorem. Applications of these theorems provide insight into the nature of divisors and roots of larger and larger numbers.

Parametric equations involve multiple independent variables within sets of equations that are dependent on each other. These equations can help us model physical situations including movement, allowing us to add direction or orientation to our calculations.

Chapter Summary

- Points in the polar coordinate system are given in the form (r,θ). The r-axis is referring to the radius r. To plot a specific point, first find the point that is r units from the origin on the r-axis. Then rotate counterclockwise by the given angle, commonly represented as "θ."

- We can use the formula $R = \frac{d\pi}{180}$, where R is the number of radians and d is the number of degrees, to convert from radians to degrees or degrees to radians.
- We can use graphing software, calculators, or plotting programs to plot polar equations.
- We can convert coordinates from rectangular form to polar form.
- We can also convert coordinates from polar form to rectangular form.
- We can solve a system of polar equations by plotting the equations on the same set of axes and determining their points of intersection.
- We can also solve a system of polar equations by setting the equations equal to each other and solving the resulting trigonometric equation.
- Conic sections—including circles, ellipses, parabolas, and hyperbolas—have a common general polar equation.
- In the standard form of $z = a + bi$, a complex number z can be graphed using rectangular coordinates (a, b), where 'a' represents the x-coordinate, while 'b' represents the y-coordinate.
- Use x and y to convert between rectangular and polar forms with $r = \sqrt{x^2 + y^2}$ and $\tan \theta_{ref} = \left|\frac{y}{x}\right|$.
- The trigonometric form of $z = r(\cos\theta + i\sin\theta)$,which is often abbreviated as $rcis\theta$.
- The Product Theorem states $(r_1\ cis\ \theta_1)(r_2\ cis\ \theta_2) = r_1 r_2\ cis\ (\theta_1 + \theta_2)$.

- The Quotient Theorem states that for $z_1 = r_1(cos\theta_1 + isin\theta_1)$ and $z_1 = r_1(cos\theta_1 + isin\theta_1)$, $\frac{z_1}{z_2} = \frac{r_1}{r_2} cis\ (\theta_1 - \theta_2)$.

- De Moivre's Theorem tells us that for $z = r(\cos\theta + i\sin\theta)$, a complex number in $rcis\theta$ form, if n is a positive integer, $z^n = r^n(cos(n\theta) + isin(n\theta))$.
- Given that the formula for De Moivre's Theorem also works for fractional powers, the same formula can be used for finding roots: $z^{1/n} = (a+bi)^{1/n} = r^{1/n} cis\left(\frac{\theta}{n}\right)$.

- "Eliminating the parameter" is a phrase that means to turn a parametric equation that has $x = f(t)$ and $y = g(t)$ into just a relationship between y and x.
- Parametric form refers to a relationship that includes $x = f(t)$ and $y = g(t)$.
- Parameterization also means writing or describing in parametric form.
- Two functions are inverses if for every point (a,b) on the 1st function, there exists a point (b,a) on the 2nd function.
- An intersection for two sets of parametric equations happens when the points exist at the same x, y, and t.
- Circular motion and projectile motion are two applications of parametric equations.

Review

Try the following cumulative review problems to practice the concepts we studied in this chapter:

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/195916

12.15 References

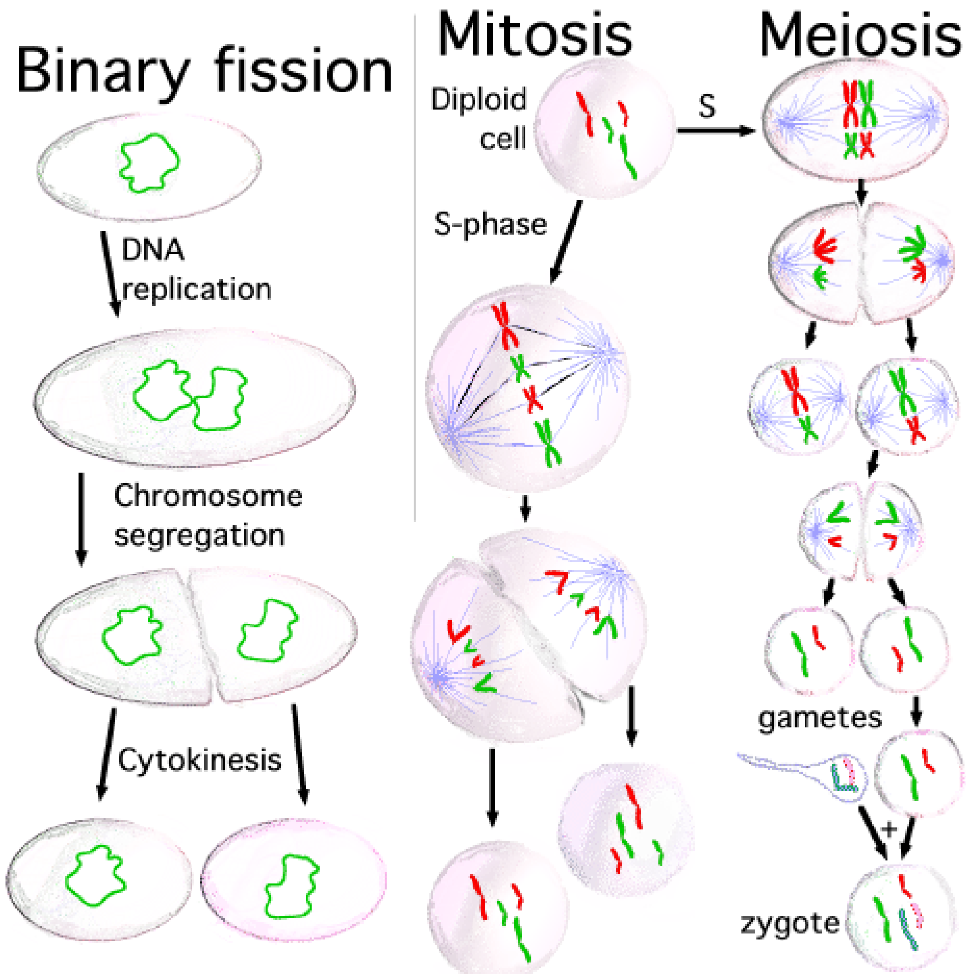
1. By JonathanD Jonathan J. Dickau [Public domain], via Wikimedia Commons. https://commons.wikimedia.org/wiki/File%3AGalaxyOfGalaxies.jpg .
2. Unknown. https://es.wikipedia.org/wiki/Archivo:Lahore-PIA-747-TakeOff-80375.JPG .
3. CK-12 Foundation. .
4. CK-12 Foundation. .
5. Massachusetts Dept. of Environmental Protection. https://www.flickr.com/photos/massdep/4472895387/in/photolist-7PfKXZ-UhtQTD-UhtSXZ-VjvsCF-UVVbX1-6jUJqS-68McDj-6TYvuN-6TYozN-az5Ck6-az8gqE-6TYdzs-6jUGF1-6TXXkb-az5CGz-6TTVQi-az5C6z-6TTYbT-6jUGVS-6jQxen-6TU9ZK-az5Dp4-8hLpDn-UVVadu-az8ga3-6jUHV9-az8h8u-az8hnQ-6jUGzb-6RbWLA-6jUH8h-az8fSA-az8fYC-az5Dwc-az5DgX-6RbSBo-6RbTX9-6jQxQx-6jQvBR-6TU2Sz-6TTXDK-6TYuwf-fgGB6F-6TUpvV-6TTWHx-az5D8V-8hsQ7z-Vw42hz-UhtQcZ-UhtTEa .
6. CK-12 Foundation. .
7. NASA. https://commons.wikimedia.org/wiki/File:The_Phases_of_Venus.jpg .
8. CK-12 Foundation. .
9. CK-12 Foundation. .
10. CK-12 Foundation. .
11. CK-12 Foundation. .
12. CK-12 Foundation. .
13. CK-12 Foundation. .
14. Boris23. https://commons.wikimedia.org/wiki/File:Roue_De_Paris_(Geleen).jpg .

CHAPTER 13 Sequences and Series

Chapter Outline

13.1 Introduction: Sequences and Series

Patterns, sequences, summing numbers, and counting can be used to help us understand exciting phenomena. These topics have numerous applications in finance and the sciences. Some applications of these topics include cell division, population growth, interest, and modeling the motion of objects.



13.2 Recursion

Learning Objectives

Learn to define patterns recursively and use recursion to solve problems.

Introduction

The Fibonacci sequence is a famous recursive sequence named after the mathematician Leonardo Fibonacci. In the 12th century, Fibonacci developed an interesting problem involving rabbits. He began with a fictional pair (one male and one female) of baby rabbits. Fibonacci assumed that it would take one month for a baby rabbit to become an adult rabbit and reproduce. He also assumed that each pair of adult rabbits would produce a pair (one male and one female) of baby rabbits each month. Finally, he assumed that the rabbits would never die. Under these imaginary conditions, Fibonacci pondered how many rabbits would be present in one year. In solving this problem, he stumbled across the pattern now referred to as the Fibonacci sequence:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ ...$$

How can we represent the Fibonacci sequence using recursion?

Recursion

A sequence is a list of numbers separated by commas. A sequence can be finite or infinite. If the sequence is infinite, the 1st few terms are followed by an ellipsis $(\ldots)$, indicating that the pattern continues forever.

An infinite sequence: $1, 2, 3, 4, 5, \ldots$

A finite sequence: $2, 4, 6, 8$

In general, you describe a sequence with subscripts that are used to index the terms. The k^{th} term in the sequence is a_k.

$$a_1, a_2, a_3, a_4, \ldots, a_k, \ldots$$

When you look at a pattern, there are many ways to describe it. You can describe patterns explicitly by stating how each term a_k is obtained from the term number k. You can also describe patterns recursively by stating how each new term a_k is calculated based on the previous term a_{k-1}. **Recursion** defines an entire sequence based on the 1st term (or the 1st few terms) and the pattern between consecutive terms.

When people see a pattern they seek to articulate how consecutive terms are related to one another. Patterns can be described with phrases like the ones below:

TABLE 13.1:

Pattern	Recursive Description
$3, 6, 12, 24, \ldots$	"Each term is twice as big as the previous term."
$3, 6, 9, 12, \ldots$	"Each term is three more than the previous term."

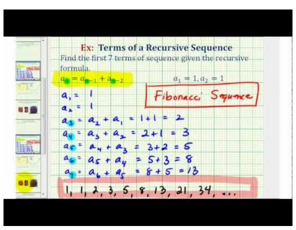
Each phrase demonstrates **recursive thinking**, in that it defines each term as a function of the previous term.

General Form of a Recursive Term

$$a_k = f(a_{k-1}),\ a_1 \text{ is the 1st term of the sequence.}$$

In some cases, a recursive formula can be a function of the previous two or three terms. Keep in mind that the downside of a recursively-defined sequence is that it is impossible to immediately know the 100th term without knowing the 99th term, which in turn might require knowing the 98th term, and so on.

The following video provides two examples of how to find the terms in a sequence given a_n, which is a recursive sequence formula:



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URL: http://www.ck12.org/flx/render/embeddedobject/62257

Play, Learn, and Explore with Recursive Formulas: Fibonacci

Examples

Example 1

For the Fibonacci sequence, determine the 1st 11 terms and the sum of these terms.

Solution:

$$0+1+1+2+3+5+8+13+21+34+55 = 143$$

Example 2

Write a recursive definition that fits the following sequence:

$$3, 7, 11, 15, 18, \ldots$$

Solution:

To write a recursive definition for a sequence, you must define the pattern and state the 1st term. With this information, others would be able to replicate your sequence without having seen it for themselves.

$$a_1 = 3$$
$$a_k = a_{k-1} + 4$$

Example 3

What are the 1st nine terms of the sequence defined by $a_1 = 1$ and $a_k = \frac{1}{a_{k-1}} + 1$?

Solution:

$$1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{3}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}$$

Example 4

Recall the problem from the Introduction: How can we represent the Fibonacci sequence using recursion?

Solution:

The Fibonacci sequence is represented by the recursive definition:

$$a_1 = 0$$
$$a_2 = 1$$
$$a_k = a_{k-2} + a_{k-1}$$

Example 5

The Lucas sequence is like the Fibonacci sequence, except that the starting numbers are 2 and 1 instead of 1 and 0. What are the 1st 10 terms of the Lucas sequence?

Solution:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76$$

Example 6

Write the next five terms of the following sequence: $a_1 = -4$, $a_2 = -4$, and $a_n = 2a_{n-1} + a_{n-2}$.

Solution:

$$a_3 = 2(-4) + (-4) = -12$$
$$a_4 = 2(-12) + (-4) = -28$$
$$a_5 = 2(-28) + (-12) = -68$$
$$a_6 = 2(-68) + (-28) = -164$$
$$a_7 = 2(-164) + (-68) = -396$$

So our answer is: -12, -28, -68, -164, and -396.

Example 7

Consider the following pattern-generating rule:

If the last number is odd, multiply it by 3 and add 1.

If the last number is even, divide the number by 2.

Repeat.

Try a few different starting numbers and see if you can state what you think always happens.

Solution:

You can choose any starting positive integer you like. Here are the sequences that start with 7 and 15.

$7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, \ldots$

$15, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, \ldots$

You could make the conjecture that any starting number will eventually lead to the repeating sequence 4, 2, 1.

This problem is called the Collatz conjecture, and is an unproven statement in mathematics. People have used computers to try all the numbers up to 5×2^{60}, and many mathematicians believe it to be true, but since all natural numbers are infinite in number, this test does not constitute a proof.

Summary

- If the sequence is **infinite**, the 1st few terms are followed by an ellipsis (...), indicating that the pattern continues forever.
- **Recursion** defines an entire sequence based on the 1st term (or the 1st few terms) and the pattern between consecutive terms.
- General Form of Recursive Term: $a_k = f(a_{k-1})$, $a_1$is the 1st term of the sequence.

Review

Write a recursive definition for each of the following sequences:

1. $3, 7, 11, 15, 19, \ldots$

2. $3, 9, 27, 81, \ldots$

3. $3, 6, 9, 12, 15, \ldots$

4. $3, 6, 12, 24, 48, \ldots$

5. $1, 4, 16, 64, \ldots$

6. Find the 1st six terms of the following sequence:

$b_1 = 2$, $b_2 = 8$, and $b_k = 6b_{k-1} - 4b_{k-2}$

7. Find the 1st six terms of the following sequence:

$c_1 = 4$, $c_2 = 18$, and $c_k = 2c_{k-1} + 5c_{k-2}$

Suppose the Fibonacci sequence started with 2 and 5:

8. List the 1st 10 terms of the new sequence.

9. Find the sum of the 1st 10 terms of the new sequence.

Write a recursive definition for each of the following sequences:

10. $1, 4, 13, 40, \ldots$

11. $1, 5, 17, 53, \ldots$
12. $2, 11, 56, 281, \ldots$
13. $2, 3, 6, 18, 108, \ldots$
14. $4, 6, 11, 18, 30, \ldots$
15. $7, 13, 40, 106, 292, \ldots$

Review (Answers)

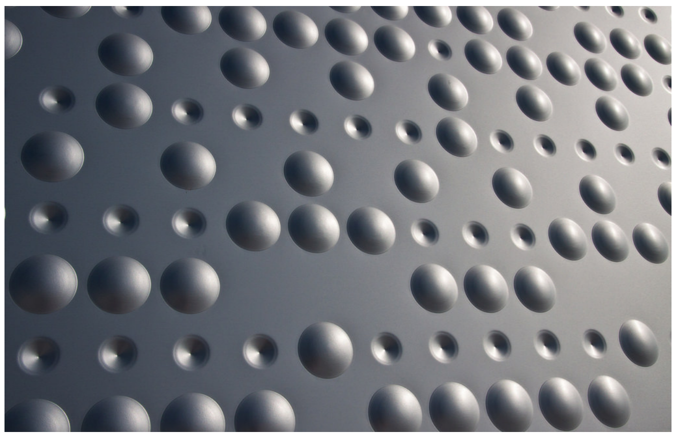
Please see the Appendix.

13.3 Arithmetic and Geometric Sequences

Learning Objectives

Learn to identify different types of sequences and use sequences to make predictions.

Introduction

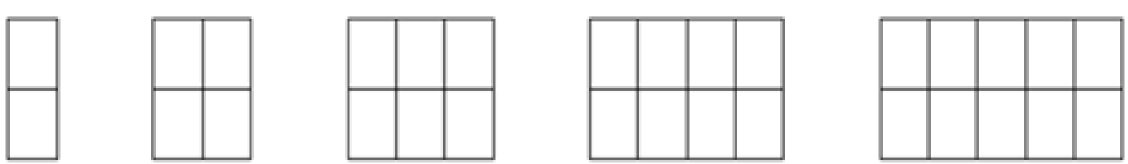


The image below represents a pattern. The pattern is an arithmetic sequence. Can you identify the pattern and write it as a general equation?

In this concept, you will learn to recognize, extend, and graph arithmetic sequences .

Arithmetic Sequence

Consider the following images:



You probably saw a pattern right away. If there were another set of boxes, you could probably guess how many there would be.

If you saw this same pattern in terms of number of rectangles, it would look like this:

2, 4, 6, 8, 10.

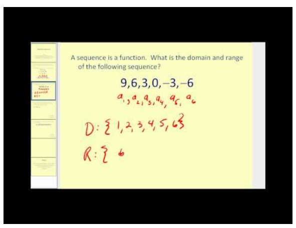
This set of numbers is called a sequence. A sequence is a list of numbers with a common pattern. The common pattern in an arithmetic sequence is that the same number is added to each number to produce the next number.

Arithmetic sequences are defined by an initial value a_1 and a common difference d.

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= a_1 + d \\ a_3 &= a_1 + 2d \\ a_4 &= a_1 + 3d \\ &\vdots \\ a_n &= a_1 + (n-1)d \end{aligned}$$

Play, Learn, and Explore with Arithmetic Sequences: Paying Off a Loan

The following video introduces arithmetic sequences:



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URL: http://www.ck12.org/flx/render/embeddedobject/62261

Geometric Sequence

The common pattern in a geometric sequence is that the same number is multiplied or divided to each number to produce the next number.

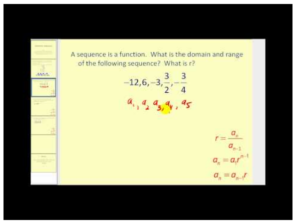
Geometric sequences are defined by an initial value a_1 and a common ratio r.

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= a_1 \cdot r \\ a_3 &= a_1 \cdot r^2 \\ a_4 &= a_1 \cdot r^3 \\ &\vdots \\ a_n &= a_1 \cdot r^{n-1} \end{aligned}$$

If a sequence does not have a common ratio or a common difference, it is neither an arithmetic nor a geometric sequence. You should still try to figure out the pattern and come up with a formula that describes it.

Play, Learn, and Explore with Geometric Sequences: Bacteria Colony

The following video introduces geometric sequences:



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Examples

Example 1

For each of the following three sequences, determine if it is arithmetic, geometric, or neither:

1) $0.135, 0.189, 0.243, 0.297, \ldots$

Solution:

The sequence is arithmetic because the common difference is 0.054.

2) $\frac{2}{9}, \frac{1}{6}, \frac{1}{8}, \ldots$

Solution:

The sequence is geometric because the common ratio is $\frac{3}{4}$.

3) $0.54, 1.08, 3.24, \ldots$

Solution:

The sequence is not arithmetic because the differences between consecutive terms are 0.54 and 2.16, which are not common. The sequence is not geometric because the ratios between consecutive terms are 2 and 3, which are not common.

Example 2

For the following sequence, determine the common ratio or difference, the next three terms, and the 2,013th term.

$$\frac{2}{3}, \frac{5}{3}, \frac{8}{3}, \frac{11}{3}, \ldots$$

Solution:

The sequence is arithmetic because the difference is exactly 1 between consecutive terms. The next three terms are $\frac{14}{3}, \frac{17}{3}, \frac{20}{3}$. An equation for this sequence would be

$$a_n = \frac{2}{3} + (n-1) \cdot 1.$$

Therefore, the 2,013th term requires 2,012 times the common difference added to the first term.

$$a_{2,013} = \frac{2}{3} + 2{,}012 \cdot 1 = \frac{2}{3} + \frac{6{,}036}{3} = \frac{6{,}038}{3}$$

Example 3

For the following sequence, determine the common ratio or difference and the next three terms:

$$\frac{2}{3}, \frac{4}{9}, \frac{6}{27}, \frac{8}{81}, \frac{10}{243}, \ldots$$

Solution:

This sequence is neither arithmetic nor geometric. The differences between the first few terms are $-\frac{2}{9}, -\frac{2}{9}, -\frac{10}{81}, -\frac{14}{243}$. While there was a common difference at first, this difference did not hold through the sequence. Be sure to always check the sequence in multiple places to make sure the common difference holds up throughout.

The sequence is also not geometric because the ratios between the first few terms are $\frac{2}{3}, \frac{1}{2}, \frac{4}{9}$. These ratios are not common.

Even though you cannot get a common ratio or a common difference, it is still possible to produce the next three terms in the sequence by noticing that the numerator is an arithmetic sequence with starting term of 2 and a common difference of 2. The denominators are a geometric sequence with an initial term of 3 and a common ratio of 3. The next three terms are

$$\frac{12}{3^6}, \frac{14}{3^7}, \frac{16}{3^8}.$$

Example 4

Recall the problem from the Introduction: Can you identify the pattern and write it as a general equation?

Solution:

Each time, 2 dots are added to the previous image. Thus, this sequence is an arithmetic sequence with a common difference of 2. The equation for this sequence is then

$$a_n = 3 + 2(n-1).$$

Example 5

What is the 10th term in the following sequence?

$$-12, 6, -3, \frac{3}{2}, \ldots$$

Solution:

The sequence is geometric and the common ratio is $-\frac{1}{2}$. The equation is $a_n = -12 \cdot \left(-\frac{1}{2}\right)^{n-1}$. The 10th term is

$$-12 \cdot \left(-\frac{1}{2}\right)^9 = \frac{3}{128}.$$

Example 6

What is the 10th term in the following sequence?

$$-1, \frac{2}{3}, \frac{7}{3}, 4, \frac{17}{3}, \ldots$$

Solution:

The pattern might not be immediately recognizable, but try ignoring the $\frac{1}{3}$ in each number to see the pattern a different way.

$$-3, 2, 7, 12, 17, \ldots$$

You should see the common difference of 5. This means the common difference from the original sequence is $\frac{5}{3}$. The equation is $a_n = -1 + (n-1)\left(\frac{5}{3}\right)$. The 10th term is

$$-1 + 9 \cdot \left(\frac{5}{3}\right) = -1 + 3 \cdot 5 = -1 + 15 = 14.$$

Example 7

Find an equation that defines the a_k term for the following sequence:

$0, 3, 8, 15, 24, 35, \ldots$

Solution:

The sequence is neither arithmetic nor geometric. It will help to find the pattern by examining the common differences, and then the common differences of the common differences. This numerical process is connected to ideas in calculus.

0, 3, 8, 15, 24, 35

3, 5, 7, 9, 11

2, 2, 2, 2

Notice that when you examine the common differences of the common differences, the pattern becomes increasingly clear. Since it took *two* layers to find a constant function, this pattern is *quadratic* and very similar to the perfect squares.

$$1, 4, 9, 16, 25, 36, \ldots$$

The a_k term can be described as $a_k = k^2 - 1$.

Summary

- A **sequence** is a list of numbers separated by commas.
- The common pattern in an **arithmetic sequence** is that the same number is added or subtracted to each number to produce the next number. This is called the **common difference.**
- The common pattern in a **geometric sequence** is that the same number is multiplied or divided to each number to produce the next number. This is called the **common ratio.**

Review

Use the sequence $1, 5, 9, 13, \ldots$ for questions 1-3.

1. Find the next three terms in the sequence.

2. Find an equation that defines the a_k term of the sequence.

3. Find the 150th term of the sequence.

Use the sequence $12, 4, \frac{4}{3}, \frac{4}{9}, \ldots$ for questions 4-6.

4. Find the next three terms in the sequence.

5. Find an equation that defines the a_k term of the sequence.

6. Find the 17th term of the sequence.

Use the sequence $10, -2, \frac{2}{5}, -\frac{2}{25}, \ldots$ for questions 7-9.

7. Find the next three terms in the sequence.

8. Find an equation that defines the a_k term of the sequence.

9. Find the 12th term of the sequence.

Use the sequence $\frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \ldots$ for questions 10-12.

10. Find the next three terms in the sequence.

11. Find an equation that defines the a_k term of the sequence.

12. Find the 314th term of the sequence.

13. Find an equation that defines the a_k term of the sequence $4, 11, 30, 67, \ldots$

14. Explain the connections between arithmetic sequences and linear functions.

15. Explain the connections between geometric sequences and exponential functions.

16. An ant colony invades the caramels in a candy store. The 1st day they eat $\frac{1}{4}$ of a caramel, the 2nd day $\frac{1}{2}$ of a caramel, and the 3rd day $\frac{3}{4}$.

a) What is the difference between each day?

b) How many do you think they'll eat on the 4th, 5th, and 6th days?

17. On the way home from school on the day of a trip downtown, a bunch of students stopped off at an arcade. Sam and Henry began to play one of their favorite games, which features aliens. "In this video game," Sasha stated convincingly, "an alien splits into two aliens, who then split into two more aliens every 10 minutes." Henry challenged Sasha, "So tell me how many aliens there would be after they split 10 times." What is the answer to Henry's challenge?

18. The amount of memory that computer chips can hold in the same amount of space doubles every year. In 1992 they could hold 1MB. Determine the amount of memory on the same size chip in 2007.

Review (Answers)

Please see the Appendix.

13.4 Sigma Notation

Learning Objectives

Learn how to represent the sum of sequences of numbers using sigma notation.

Introduction

Sayber and Tuscany sell ice pops during the summer for pocket money. One particular weekend, they purchased a package of 30 ice pops from the store.

Usually they just offer the ice pops for free, one per customer, and accept tips. This time, Tuscany wonders if they would make more money by charging $0.50 per ice pop. At the same time, Sayber wonders if he might be able to increase his tips by encouraging customers to "outbid" each other. The two children decide to each take 15 ice pops and see who makes the most.

How can you calculate how much money each of them makes, assuming Sayber gets a $0.10 tip from the 1st customer and is able to convince each successive customer to double the previous person's tip?

Series and Sigma Notation

Writing the sum of long lists of numbers that have a specific pattern is not very efficient. Summation notation allows you to use the pattern and the number of terms to represent the same sum in a much more concise way.

A **series** is a sum of a sequence. The Greek capital letter sigma $\sum$ is used for summation notation because it stands for the letter S, as in sum.

Consider the following general sequence, and note that the subscript for each term is an index telling you the term number:

$$a_1, a_2, a_3, a_4, \ldots, a_n$$

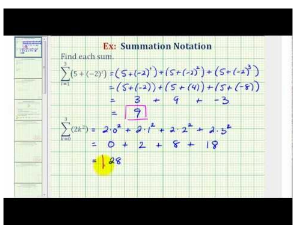
When you write the sum of this sequence in a series, it can be represented as a sum of each individual term or abbreviated using a capital sigma.

Summation Notation Using Sigma for the Sum of a Series

$$a_1 + a_2 + a_3 + a_4 + \ldots + a_n = \sum_{i=1}^{n} a_i$$

The three parts of sigma notation that you need to be able to read are the argument, the lower index, and the upper index. The argument, a_i, tells you what terms are added together. The lower index, $i = 1$, tells you where to start, and the upper index, 5, tells you where to end.

The following video explains how to find a sum when given in summation/sigma notation:



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URL: http://www.ck12.org/flx/render/embeddedobject/62273

Play, Learn, and Explore with Sigma Notation: Cracking the Code

Examples

Example 1

Write out all the terms of the series.

$$\sum_{k=4}^{8} 2k$$

Solution:

$$\sum_{k=4}^{8} 2k = 2 \cdot 4 + 2 \cdot 5 + 2 \cdot 6 + 2 \cdot 7 + 2 \cdot 8$$

Example 2

Write the sum in sigma notation: $2+3+4+5+6+7+8+9+10$.

Solution:

$$2+3+4+5+6+7+8+9+10=\sum_{i=2}^{10} i$$

Example 3

Write the sum in sigma notation:

$$1+4+9+16+25+\cdots+144$$

Solution:

The hardest part when first using sigma representation is determining how each pattern generalizes to the k^{th} term. Once you know the k^{th} term, you know the argument of the sigma. For the sequence creating the series below, $a_k = k^2$. Therefore, the argument of the sigma is i^2.

$$1+4+9+16+25+\cdots+144 = 1^2+2^2+3^2+4^2+\cdots 12^2 = \sum_{i=1}^{12} i^2$$

Example 4

Recall the problem from the Introduction: How can you calculate how much money Sayber and Tuscany each make, assuming Sayber gets a \$0.10 tip from the 1st customer and is able to convince each successive customer to double the previous person's tip?

Solution:

Tuscany's income can be expressed as

$$\sum_{n=1}^{10} 0.50 \rightarrow 0.50 \cdot 10 = \$5.$$

Sayber's income can be expressed as

$$\sum_{n=1}^{10} 0.10 \cdot 2^n \rightarrow 0.10(2^0)+0.10(2^1)+0.10(2^2)...+0.10(2^{10}) \rightarrow \$102.30.$$

Example 5

Write out all the terms of the sigma notation, and then calculate the sum.

$$\sum_{k=0}^{4} 3k-1$$

Solution:

$$\begin{aligned}\sum_{k=0}^{4} 3k-1 &= (3\cdot 0-1)+(3\cdot 1-1)+(3\cdot 2-1)+(3\cdot 3-1)+(3\cdot 4-1)\\ &= -1+2+5+8+11 = 25\end{aligned}$$

Example 6

Represent the following infinite series in summation notation:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

Solution:

There are an infinite number of terms in the series, so using an infinity symbol in the upper limit of the sigma is appropriate.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots=\sum_{i=1}^{\infty}\frac{1}{2^i}$$

Example 7

Write the sum in sigma notation.

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}$$

Solution:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}=\sum_{i=1}^{7}\frac{1}{i^2}$$

Summary

- **Sigma notation** $\sum$ is also known as **summation notation,** and is a way to represent a sum of numbers. It is especially useful when the numbers have a specific pattern or would take too long to write out without abbreviation.
- Sigma notation can be defined as follows: $a_1+a_2+a_3+a_4+\ldots+a_n=\sum_{i=1}^{n}a_i$.

Review

For 1-5, write out all the terms of the sigma notation and then calculate the sum.

1. $\sum_{k=1}^{5}2k-3$
2. $\sum_{k=0}^{8}2^k$
3. $\sum_{i=1}^{4}2\cdot 3^i$
4. $\sum_{i=1}^{10}4i-1$

5. $\sum_{n=1}^{11} 9(4)^{n-1}$

Represent the following series in summation notation with a lower index of 0:

6. $1+4+7+10+13+16+19+22$

7. $3+5+7+9+11$

8. $8+7+6+5+4+3+2+1$

9. $5+6+7+8$

10. $3+6+12+24+48+\cdots$

11. $10+5+\frac{5}{2}+\frac{5}{4}$

12. $4-8+16-32+64\cdots$

13. $2\cdot4\cdot6\cdot8\cdot\cdots$

14. $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$

15. $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\frac{2}{81}+\cdots$

s)

Please see the Appendix.

13.5 Arithmetic Series

Learning Objectives

Learn to compute finite arithmetic series more efficiently than just adding the terms together one at a time.

Introduction

Imagine you visit the Grand Canyon and drop a rock off the edge of a cliff. The distance the rock falls is 16 feet the 1st second, 48 feet the next second, 80 feet the 3rd second, and so on. The distance the rock falls can be modeled with an arithmetic series. While it is possible to add arithmetic series one term at a time, it is not feasible or efficient when there are a large number of terms.

How can we use arithmetic series to devise a clever way to add up all the whole numbers between 1 and 100?

Arithmetic Series

The key to adding up a finite arithmetic series is to pair up the 1st term with the last term, the 2nd term with the 2nd-to-last term, and so on. The sum of each pair will be equal. Consider a arithmetic series:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots a_n.$$

When we pair the 1st and the last terms and note that $a_n = a_1 + (n-1)k$, the sum is

$$a_1 + a_n = a_1 + a_1 + (n-1)k = 2a_1 + (n-1)k.$$

When we pair up the 2nd and the 2nd-to-last terms, we obtain the same sum:

$$a_2 + a_{n-1} = (a_1 + k) + (a_1 + (n-2)k) = 2a_1 + (n-1)k.$$

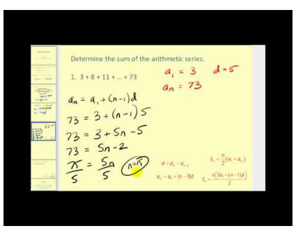
The next logical question to ask is: How many pairs are there? If there are n terms total, then there are exactly $\frac{n}{2}$ pairs. If n happens to be even, then every term will have a partner, and $\frac{n}{2}$ will be a whole number. If n happens to be odd, then every term but the middle one will have a partner, and $\frac{n}{2}$ will include a $\frac{1}{2}$ pair that represents the middle term with no partner.

General Formula for Arithmetic Series

$$\sum_{i=1}^{n} a_i = \frac{n}{2}\left(2a_1 + (n-1)k\right),$$

where k is the common difference for the terms in the series.

The following video introduces arithmetic series:



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Play, Learn, and Explore with Arithmetic Series: Arithmetic Series

Examples

Example 1

Add up the numbers between 1 and 10 (inclusive) in two ways.

Solution:

One way to add up lists of numbers is to pair them up for easier mental arithmetic.

$$\begin{aligned}1+2+3+4+5+6+7+8+9+10 &= 3+7+11+15+19\\ &= 10+26+19\\ &= 36+19\\ &= 55\end{aligned}$$

Another way is to note that $1+10=2+9=3+8=4+7=5+6=11$. There are 5 pairs of 11, which total 55.

Example 2

Evaluate the following sum:

$$\sum_{k=0}^{5} 5k-2.$$

Solution:

The 1st term is -2, the last term is 23, and there are 6 terms making 3 pairs. A common mistake is to forget to count the 0 index.

$$\sum_{k=0}^{5} 5k-2=\frac{6}{2}\cdot(-2+23)=3\cdot 21=63$$

Example 3

Does the technique for finding the sum of an arithmetic series work for finding the sum of a geometric series?

$$\frac{1}{8}+\frac{1}{2}+2+8+32$$

Solution:

The real sum is $\frac{341}{8}$.

When you try to use the technique used for arithmetic sequences, you get $3\left(\frac{1}{8}+32\right)=\frac{771}{8}$.

It is important to know that geometric series have their own method for summing. The method learned in this concept works only for arithmetic series.

Example 4

Recall the problem from the Introduction: How can we use arithmetic series to devise a clever way to add up all the whole numbers between 1 and 100?

Solution:

Carl Friedrich Gauss was a mathematician who lived hundreds of years ago. According to an anecdote about him, as a young boy in school he misbehaved and his teacher asked him to add up all the numbers between 1 and 100. Within a few seconds, Gauss stated 5,050.

You should notice that $1+100=2+99=\cdots=101$, and that there are exactly 50 pairs that sum to be 101. $50\cdot 101=$ 5,050.

Example 5

Sum the 1st 15 terms of the following arithmetic sequence:

$$-1,\frac{2}{3},\frac{7}{3},4,\frac{17}{3}\ldots$$

Solution:

The initial term is -1, and the common difference is $\frac{5}{3}$.

$$\begin{aligned}\sum_{i=1}^{n} a_i &= \frac{n}{2}(2a_1+(n-1)k)\\ &= \frac{15}{2}\left(2(-1)+(15-1)\frac{5}{3}\right)\\ &= \frac{15}{2}\left(-2+14\cdot\frac{5}{3}\right)\\ &= 160\end{aligned}$$

Example 6

Sum the 1st 100 terms of the following arithmetic sequence:

$$-7,-4,-1,2,5,8,\dots$$

Solution:

The initial term is -7, and the common difference is 3.

$$\begin{aligned}\sum_{i=1}^{n} a_i &= \frac{n}{2}(2a_1+(n-1)k)\\ &= \frac{100}{2}(2(-7)+(100-1)3)\\ &= 14150\end{aligned}$$

Example 7

Evaluate the following sum:

$$\sum_{i=0}^{500} 2i-312.$$

Solution:

The initial term is -312, and the common difference is 2.

$$\begin{aligned}\sum_{i=0}^{500} 2i-312 &= \frac{501}{2}(2(-312)+(501-1)2)\\ &= 94188\end{aligned}$$

Summary

- An **arithmetic series** is a sum of numbers whose consecutive terms form an arithmetic sequence.
- The general form for an arithmetic series can be represented by $\sum_{i=1}^{n} a_i = \frac{n}{2}(2a_1+(n-1)k)$, where k is the common difference for the terms in the series.

Review

1. Sum the 1st 24 terms of the sequence $1, 5, 9, 13, \ldots$
2. Sum the 1st 102 terms of the sequence $7, 9, 11, 13, \ldots$
3. Sum the 1st 85 terms of the sequence $-3, -1, 1, 3, \ldots$
4. Sum the 1st 97 terms of the sequence $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \ldots$
5. Sum the 1st 56 terms of the sequence $-\frac{2}{3}, \frac{1}{3}, \frac{4}{3}, \ldots$
6. Sum the 1st 91 terms of the sequence $-8, -4, 0, 4, \ldots$

Evaluate the following sums:

7. $\sum_{i=0}^{300} 3i + 18$
8. $\sum_{i=0}^{215} 5i + 1$
9. $\sum_{i=0}^{100} i - 15$
10. $\sum_{i=0}^{85} -13i + 1$
11. $\sum_{i=0}^{212} -2i + 6$
12. $\sum_{i=0}^{54} 6i - 9$
13. $\sum_{i=0}^{167} -5i + 3$
14. $\sum_{i=0}^{341} 6i + 102$
15. $\sum_{i=0}^{452} -7i - \frac{5}{2}$

Review (Answers)

Please see the Appendix.

13.6 Geometric Series

Learning Objectives

Learn to sum infinite and finite geometric series and categorize geometric series as convergent or divergent.

Introduction

A deposit of $200 is made on the 1st day of January, April, July, and October of every year in an account that pays 4.5% interest, compounded quarterly. The future balance can be modeled using a geometric series. An advanced factoring technique allows you to rewrite the sum of a finite geometric series in a compact formula. An infinite geometric series has two possibilities: It can sum to be a number, or the sum can continue to grow to infinity. When does an infinite geometric series sum to be just a number, and when does it sum to be infinity?

Geometric Series

Recall the advanced factoring technique for the difference of two squares and, more generally, two terms of any power (5 in this case).

$$a^2 - b^2 = (a-b)(a+b)$$
$$a^5 - b^5 = (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$$
$$a^n - b^n = (a-b)(a^{n-1} + \cdots + b^{n-1})$$

If the 1st term is 1, then $a = 1$. If you replace b with the letter r, you end up with

$$1 - r^n = (1-r)(1 + r + r^2 + \cdots r^{n-1}).$$

You can divide both sides by $(1-r)$ because $r \neq 1$.

$$1 + r + r^2 + \cdots r^{n-1} = \frac{1-r^n}{1-r}$$

The left side of this equation is a geometric series with starting term 1 and common ratio of r. Note that even though the ending exponent of r is $n-1$, there are a total of n terms on the left. To make the starting term not 1, just scale both sides of the equation by the 1st term you want, a_1.

Sum of a Finite Geometric Series

$$a_1 + a_1r + a_1r^2 + \cdots a_1r^{n-1} = a_1\left(\frac{1-r^n}{1-r}\right)$$

To sum an infinite geometric series, start by looking carefully at the previous formula for a finite geometric series. As the number of terms gets infinitely large, $(n \to \infty)$, one of two things will happen:

$$a_1\left(\frac{1-r^n}{1-r}\right)$$

Option 1: The term r^n will go to infinity or negative infinity. This will happen when $|r| \geq 1$. When this happens, the sum of the infinite geometric series does not go to a specific number, and the series is said to be **divergent.**

Option 2: The term r^n will go to zero. This will happen when $|r| < 1$. When this happens, the sum of the infinite geometric series goes to a certain number, and the series is said to be **convergent**.

One way to think about these options is think about what happens when you take 0.9^{100} and 1.1^{100}.

$$0.9^{100} \approx 0.00002656$$
$$1.1^{100} \approx 13780$$

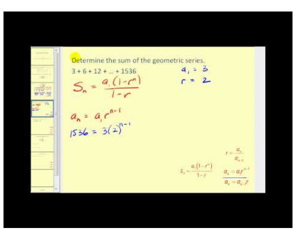
As you can see, even numbers close to 1 either get very small quickly, or very large quickly.

Sum of a Infinite Geometric Series

$$\sum_{i=1}^{\infty} a_1 \cdot r^{i-1} = a_1\left(\frac{1}{1-r}\right)$$

Notice how this formula is the same as the finite version but with $r^n = 0$, just as you reasoned.

The following video introduces geometric series:



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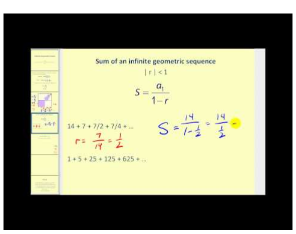
The following video explains how to determine the sum of an infinite geometric series, if the sum exists:



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Examples

Example 1

Compute the sum of the following infinite geometric series:

$$0.2+0.02+0.002+0.0002+\dots$$

Solution:

You can tell just by looking at the sum that the infinite sum will be the repeating decimal $0.\overline{2}$. You may recognize this as the fraction $\frac{2}{9}$, but if you don't, this is how you turn a repeating decimal into a fraction.

Let $x=0.\overline{2}$.

Then, $10x=2.\overline{2}$.

Subtract the two equations and solve for x.

$$\begin{aligned}10x-x&=2.\overline{2}-0.\overline{2}\\9x&=2\\x&=\frac{2}{9}\end{aligned}$$

Example 2

Why does an infinite series with $r=1$ diverge?

Solution:

If $r=1$, this means that the common ratio between the terms in the sequence is 1. This means that each number in the sequence is the same. When you add up an infinite number of any finite numbers (even fractions close to 0), you will always get infinity or negative infinity. The only exception is 0. This case is trivial because a geometric series with an initial value of 0 is simply the following series, which clearly sums to 0:

$$0+0+0+0+\dots.$$

Example 3

What is the sum of the 1st eight terms in the following geometric series?

$$4+2+1+\frac{1}{2}+\dots.$$

Solution:

The 1st term is 4, and the common ratio is $\frac{1}{2}$.

$$a_1\left(\frac{1-r^n}{1-r}\right)=4\left(\frac{1-\left(\frac{1}{2}\right)^8}{1-\frac{1}{2}}\right)=4\left(\frac{\frac{255}{256}}{\frac{1}{2}}\right)=\frac{255}{32}$$

Example 4

Recall the question from the Introduction: When does an infinite geometric series sum to be just a number, and when does it sum to be infinity?

Solution:

An infinite geometric series converges if and only if $|r|<1$. Infinite arithmetic series never converge.

Example 5

Compute the sum from Example 1 using the formula for the sum of an infinite geometric series, and confirm that the sum truly does converge.

Solution:

The 1st term of the sequence is $a_1 = 0.2$. The common ratio is 0.1. Since $|0.1| < 1$, the series does converge.

$$0.2\left(\frac{1}{1-0.1}\right) = \frac{0.2}{0.9} = \frac{2}{9}$$

Example 6

Does the following geometric series converge or diverge? Does the sum go to positive or negative infinity?

$$-2+2-2+2-2+\dots$$

Solution:

The initial term is -2, and the common ratio is -1. Since the $|-1| \geq 1$, the series is said to diverge. Even though the series diverges, it does not approach negative or positive infinity. When you look at the partial sums (the sums up to certain points), they alternate between two values:

$$-2, 0, -2, 0, \dots.$$

This pattern does not go to a specific number. Just like a sine or cosine wave, it does not have a limit as it approaches infinity.

Example 7

You put \$100 in a bank account at the end of every year for 10 years. The account earns 6% interest. How much do you have in total at the end of 10 years?

Solution:

The 1st deposit gains 9 years of interest: $100 \cdot 1.06^9$

The 2nd deposit gains 8 years of interest: $100 \cdot 1.06^8$. This pattern continues, creating a geometric series. The last term receives no interest at all.

$$100 \cdot 1.06^9 + 100 \cdot 1.06^8 + \cdots 100 \cdot 1.06 + 100$$

Note that normally, geometric series are written in the opposite order so you can identify the starting term and the common ratio more easily.

$$a_1 = 100,\ r = 1.06$$

The sum of the 10 years of deposits is:

$$a_1\left(\frac{1-r^n}{1-r}\right) = 100\left(\frac{1-1.06^{10}}{1-1.06}\right) \approx \$1,318.08$$

Summary

- To **converge** means the sum approaches a specific number.
- To **diverge** means the sum does not converge, and so usually goes to positive or negative infinity. It could also mean that the series oscillates infinitely.
- A **partial sum** of an infinite sum is the sum of all the terms up to a certain point. Considering partial sums can be useful when analyzing infinite sums.
- Notation for the sum of a geometric series: $a_1 + a_1 r + a_1 r^2 + \cdots a_1 r^{n-1} = a_1\left(\frac{1-r^n}{1-r}\right)$.

Review

Find the sum of the 1st 15 terms for each geometric sequence below.

1. $5, 10, 20, \ldots$
2. $2, 8, 32, \ldots$
3. $5, \frac{5}{2}, \frac{5}{4}, \ldots$
4. $12, 4, \frac{4}{3}, \ldots$
5. $\frac{1}{3}, 1, 3, \ldots$

For each infinite geometric series, identify whether the series is convergent or divergent. If convergent, find the number where the sum converges.

6. $5+10+20+\cdots$
7. $2+8+32+\cdots$
8. $5+\frac{5}{2}+\frac{5}{4}+\cdots$
9. $12+4+\frac{4}{3}+\cdots$
10. $\frac{1}{3}+1+3+\cdots$
11. $6+2+\frac{2}{3}+\cdots$
12. You put \$5,000 in a bank account at the end of every year for 30 years. The account earns 2% interest. How much do you have in total at the end of 30 years?
13. You put \$300 in a bank account at the end of every year for 15 years. The account earns 4% interest. How much do you have in total at the end of 10 years?
14. You put \$10,000 in a bank account at the end of every year for 12 years. The account earns 3.5% interest. How much do you have in total at the end of 12 years?
15. Why don't infinite arithmetic series converge?

Review (Answers)

Please see the Appendix.

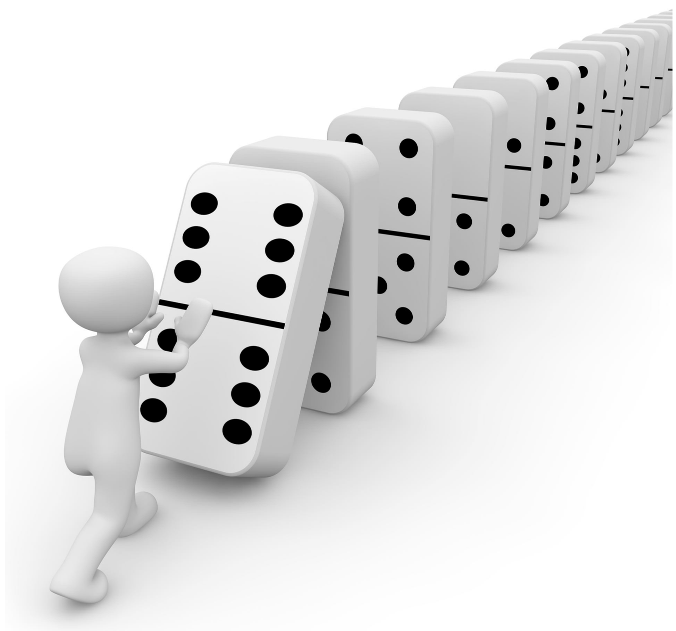
13.7 Induction Proofs

Learning Objectives

Learn how to prove statements about numbers using induction.

Introduction

Induction is one of many methods for proving mathematical statements about numbers. The basic idea is that you prove a statement is true for a small number like 1. This is called the base case. You show that if the statement is true for some random number k, then it must also be true for $k+1$.



An induction proof is like dominoes set up in a line, where the base case starts the falling cascade of truth. Once you have shown that, in general, if the statement is true for k, then it must also be true for $k+1$. It means that once you show the statement is true for 1, then it must also be true for 2. Then it must also be true for 3, and it must also be true for 4, and so on.

What happens when you forget the base case?

Mathematical Induction

Simple mathematical induction has three steps:

1. The base case is where the statement is shown to be true for n equal to some small number like 1. Note that sometimes more than one base case is needed.
2. The inductive hypothesis is where the statement is assumed to be true for $n = k$.

3. The inductive step is where you show that the statement must be true for $n = k+1$.

These three logical pieces will show that the statement is true for every number greater than the base case. In strong mathematical induction, the inductive hypothesis assumes all the statements from $n = 1$ to k for some integer k.

Suppose you want to use induction to prove for $n \geq 1,\ 2+2^2+2^3+\ldots+2^n = 2^{n-1}-2$.

Start with the **base case**. Show the statement works when $n = 1$.

$$2^1 = 2 \quad \text{and} \quad 2^{1+1}-2 = 4-2 = 2.$$

Therefore, $2^1 = 2^{1+1}-2$. (Both sides are equal to 2.)

Next, state your **inductive hypothesis**. Assume the statement works for n equal to some number k:

$$2+2^2+\cdots+2^k = 2^{k+1}-2.$$

(You are assuming this is a true statement.)

Finally, use algebra to manipulate the $n = k$ statement to **prove** the statement is also true for $n = k+1$. So, you will be trying to show that $2+2^2+\cdots+2^{k+1} = 2^{(k+1)+1}-2$. Start with the inductive hypothesis, and multiply both sides of the equation by 2. Then, do some algebra to get the equation looking like you want.

Inductive Hypothesis (starting equation): $2+2^2+\cdots 2^k = 2^{k+1}-2$

Multiply by 2: $2(2+2^2+\cdots+2^k) = 2(2^{k+1}-2)$

Rewrite: $2^2+2^3+\cdots+2^{k+1} = 2^{k+1+1}-4$

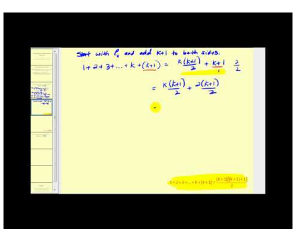
Add 2 to both sides: $2+2^2+2^3+\cdots+2^{k+1} = 2^{k+1+1}-4+2$

Simplify: $2+2^2+\cdots+2^{k+1} = 2^{k+1+1}-2$

This is exactly what you were trying to prove! So, first you showed that the statement worked for $n = 1$. Then you showed that if the statement works for one number, it must work for the next number. This means the statement must be true for all natural numbers greater than or equal to 1.

The idea of induction can be hard to understand at first, and it definitely takes practice. One thing that makes induction tricky is that there is not a clear procedure for the "proof" part. With practice, you will start to see some common algebra techniques for manipulating equations to prove what you are trying to prove.

The following video explains how to prove a mathematical statement using proof by induction, and shows two specific examples:



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Examples

Example 1

There is something wrong with this proof. Can you explain what the mistake is?

For $n \geq 1 : 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Base Case: When $n=1$, $1=1^2=\frac{1(1+1)(2\cdot 1+1)}{6}=\frac{1\cdot 2\cdot 3}{6}=\frac{6}{6}=1$.

Inductive Hypothesis: Assume the following statement is true for $n=k$:

$$1^2+2^2+3^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}.$$

Proof: You want to show the statement is true for $n=k+1$.

"Since the statement is assumed true for $n=k$*, which is any number, then it must be true for* $n=k+1$*. You can just substitute* $k+1$ *in."*

$1^2+2^2+3^2+\cdots+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

Solution:

This is the most common fallacy when doing induction proofs. The fact that the statement is assumed to be true for $n=k$ does not immediately imply that it is true for $n=k+1$, and you cannot just substitute in $k+1$ to produce what you are trying to show. This is equivalent to assuming true for all numbers and then concluding true for all numbers, which is circular and illogical.

Example 2

Write the base case, inductive hypothesis, and what you are trying to show for the statement below. Do not actually prove it.

$$1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$$

Solution:

Base Case: When $n=1$, $1^3=\frac{1^2(1+1)^2}{4}$. (Both sides are equal to 1.)

Inductive Hypothesis: Assume the following statement is true for $n=k$:

$$1^3+2^3+3^3+\cdots+k^3=\frac{k^2(k+1)^2}{4}.$$

Proof: Prove that the following is true for $n=k+1$:

$$1^3+2^3+3^3+\cdots+k^3+(k+1)^3=\frac{(k+1)^2((k+1)+1)^2}{4}.$$

Example 3

Prove the following statement: For $n \geq 1, 1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2$.

Solution:

Base Case(s): Two base cases are shown; however, only one is actually necessary.

$$1^3=1^2$$
$$1^3+2^3=1+8=9=3^2=(1+2)^2$$

Inductive Hypothesis: Assume the statement is true for some number $n = k$. In other words, assume the following is true:

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = (1 + 2 + 3 + \ldots + k)^2.$$

Proof: You want to show the statement is true for $n = k + 1$. It is a good idea to restate what your goal is at this point. Your goal is to show that

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = (1 + 2 + 3 + \ldots + k + (k+1))^2.$$

You need to start with the assumed case and do algebraic manipulations until you have created what you are trying to show (the equation above):

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = (1 + 2 + 3 + \ldots + k)^2.$$

From the work you have done with arithmetic series, you should notice

$$1 + 2 + 3 + 4 + \cdots + k = \frac{k}{2}(2 + (k-1)) = \frac{k(k+1)}{2}.$$

Substitute into the right side of the equation, and add $(k+1)^3$ to both sides:

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3.$$

When you combine the righthand side algebraically, you get the result of another arithmetic series.

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2 = (1 + 2 + 3 + \cdots k + (k+1))^2$$

Example 4

Recall the problem from the Introduction: What happens when you forget the base case?

Solution:

If you forget the base case in an induction proof, then you haven't really proved anything. You can get silly results like this "proof" of the statement: "$1 = 3$."

Base Case: Missing.

Inductive Hypothesis: $k = k + 1$, where k is a counting number.

Proof: Start with the assumption step and add 1 to both sides.

$$\begin{aligned} k &= k + 1 \\ k + 1 &= k + 2 \end{aligned}$$

Thus, by transitivity of equality:

$$\begin{aligned}k &= k+1 = k+2 \\ k &= k+2\end{aligned}$$

Since k is a counting number, k could equal 1. Therefore, $1 = 3$.

Example 5

Write the base case, inductive hypothesis, and what you are trying to show for the statement below. Do not actually prove it. n^3+2n is divisible by 3 for any positive integer n.

Solution:

Base Case: When $n = 1$, $1^3 + 2 \cdot 1 = 3$, which is divisible by 3.

Inductive Step: Assume the following is true for $n = k$: $k^3 + 2k$ divisible by 3.

Proof: Show the following is true for $n = k+1$: $(k+1)^3 + 2(k+1)$ is divisible by 3.

Example 6

Complete the proof for the previous problem.

Solution:

The goal is to show that $(k+1)^3+2(k+1)$ is divisible by 3 if you already know k^3+2k is divisible by 3. Expand $(k+1)^3+2(k+1)$ to see what you get:

$$\begin{aligned}(k+1)^3+2(k+1) &= k^3+3k^2+3k+1+2k+2 \\ &= (k^3+2k)+3(k^2+k+1)\end{aligned}$$

k^3+2k is divisible by 3 by assumption (the inductive step), and $3(k^2+k+1)$ is clearly a multiple of 3, so it is divisible by 3. The sum of two numbers that are divisible by 3 is also divisible by 3.

Example 7

Prove the following statement using induction:

For $n \geq 1,\ 1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$.

Solution:

Base Case: When $n = 1$, $1 = \frac{1(1+1)}{2} = 1 \cdot \frac{2}{2} = 1$.

Inductive Hypothesis: Assume the following is true for $n = k$:

$$1+2+3+4+\cdots+k = \frac{k(k+1)}{2}.$$

Proof: Start with what you know, and work to show it true for $n = k+1$.

Inductive Hypothesis: $1+2+3+4+\cdots+k = \frac{k(k+1)}{2}$.

Add $k+1$ to both sides: $1+2+3+4+\cdots+k+(k+1) = \frac{k(k+1)}{2}+(k+1)$.

Find a common denominator for the right side:

$$1+2+3+4+\cdots+k+(k+1) = \frac{k^2+k}{2}+\frac{2k+2}{2}.$$

Simplify the right side: $1+2+3+4+\cdots+k+(k+1)=\frac{k^2+3k+2}{2}$.

Factor the numerator of the right side: $1+2+3+4+\cdots+k+(k+1)=\frac{(k+1)(k+2)}{2}$.

Rewrite the right side: $1+2+3+4+\cdots+k+(k+1)=\frac{(k+1)((k+1)+1)}{2}$.

Summary

- The **base case** is the anchor step. It is the 1st domino to fall, creating a cascade, and thus proving the statement true for every number greater than the base case.
- The **inductive hypothesis** is the step where you assume the statement is true for k.
- The **inductive step** is the **proof**. It is when you show the statement is true for $k+1$, using only the inductive hypothesis and algebra.

Review

For each of the following statements: a) show the base case is true; b) state the inductive hypothesis; and c) state what you are trying to prove in the inductive step/proof. *Do not prove yet.*

1. For $n \geq 5,\ 4n < 2^n$.

2. For $n \geq 1,\ 8^n - 3^n$ is divisible by 5.

3. For $n \geq 1,\ 7^n - 1$ is divisible by 6.

4. For $n \geq 2,\ n^2 \geq 2n$.

5. For $n \geq 1,\ 4^n + 5$ is divisible by 3.

6. For $n \geq 1,\ 0^2+1^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$.

Prove each of the statements below. Use your answers to problems 1-6 to help you get started.

7. For $n \geq 5,\ 4n < 2^n$.

8. For $n \geq 1,\ 8^n - 3^n$ is divisible by 5.

9. For $n \geq 1,\ 7^n - 1$ is divisible by 6.

10. For $n \geq 2,\ n^2 \geq 2n$.

11. For $n \geq 1,\ 4^n + 5$ is divisible by 3.

12. For $n \geq 1,\ 0^2+1^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$.

13. You should believe the statement below is clearly false. What happens when you try to prove it true by induction?

For $n \geq 2,\ n^2 < n$.

14. Explain why the base case is necessary for proving by induction.

15. The principles of inductive proof can be used for other proofs besides proofs about numbers. Can you prove the following statement from geometry using induction?

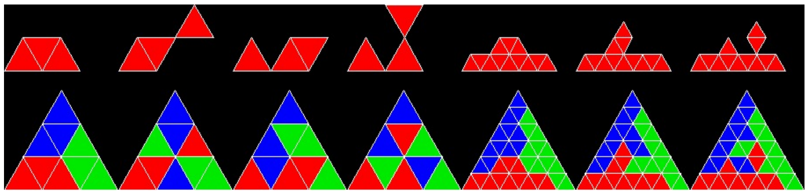
The sum of the interior angles of any $n-gon$ is $180(n-2)$ for $n \geq 3$.

Review (Answers)

Please see the Appendix.

13.8 Project: Sequences and Series

Equilateral triangles can be inscribed within an outer equilateral triangle by constructing midpoints on each side and connecting them.



1. Sketch an equilateral triangle as the 1st of four figures. Label as figure 1.
2. Sketch a 2nd equilateral triangle with an inscribed equilateral triangle constructed using the midpoints. Label as figure 2.
3. If the perimeter of the 1st triangle is 4, what is the perimeter of the 6th inscribed triangle?
4. If the perimeter of the 1st triangle is 2, what is the perimeter of the 10th inscribed triangle?
5. What would be the sum of all the perimeters if the triangles in Step 3 were inscribed to infinity?
6. If the triangles in Step 4 were inscribed to infinity, what would be the sum of all of the perimeters?
7. If the triangles in Step 3 were inscribed to infinity, what would be the sum of all of the areas?
8. If the triangles in Step 4 were inscribed to infinity, what would be the sum of all of the areas?

13.9 Summary: Sequences and Series

In this chapter you learned that recursion, how people see patterns, is where each term in a sequence is defined by the term that came before. You saw, too, that terms in a pattern can also be represented as a function of their term number. Moreover, you learned about two special types of patterns called arithmetic sequences and geometric sequences. Sequences have a wide variety of applications in the real world. Additionally, you saw that series are when terms in a sequence are added together.

Chapter Summary

- A **recursively defined pattern or sequence** is a sequence with terms that are defined based on the prior term(s) in the sequence.
- An **explicit pattern or sequence** is a sequence with terms that are defined based on the term number.
- **Sigma notation** $\sum$ is also known as **summation notation** and is a way to represent a sum of numbers. It is especially useful when the numbers have a specific pattern or would take too long to write out without abbreviation.
- Sigma notation can be defined as follows: $a_1 + a_2 + a_3 + a_4 + \ldots + a_n = \sum_{i=1}^{n} a_i$.
- A **sequence** is a list of numbers separated by commas.
- The common pattern in an **arithmetic sequence** is that the same number is added or subtracted to each number to produce the next number. This is called the **common difference.**
- The common pattern in a **geometric sequence** is that the same number is multiplied or divided to each number to produce the next number. This is called the **common ratio.**
- An **arithmetic series** is a sum of numbers whose consecutive terms form an arithmetic sequence.
- The general form for an arithmetic series can be represented by $\sum_{i=1}^{n} a_i = \frac{n}{2}(2a_1 + (n-1)k)$, where k is the common difference for the terms in the series.
- To **converge** means the sum approaches a specific number.
- To **diverge** means the sum does not converge, and so usually goes to positive or negative infinity. It could also mean that the series oscillates infinitely.
- A **partial sum** of an infinite sum is the sum of all the terms up to a certain point. Considering partial sums can be useful when analyzing infinite sums.
- Notation for the sum of a geometric series: $a_1 + a_1 r + a_1 r^2 + \cdots a_1 r^{n-1} = a_1 \left(\frac{1-r^n}{1-r}\right)$.

Review

Try the following cumulative review problems to practice the concepts we studied in this chapter.

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13.10 References

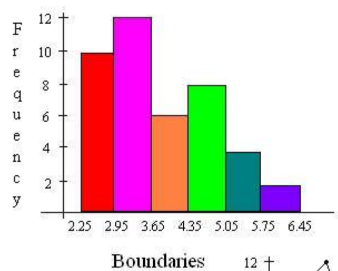
1. CK-12 Foundation. CK-12 Foundation .
2. Tom Bech. https://www.flickr.com/photos/viatorius/5650826199 .
3. JohnnyMrNinja. https://commons.wikimedia.org/wiki/File:Hibiscus_Popsicle.jpg .
4. Tony Hisgett. https://www.flickr.com/photos/hisgett/15542816655/in/photolist-pFsZTv-4DZxKd-f8bLeG-pp3YqE-ppbTbq-pDvg7q-ppctuU-piEdt6-rfrQ7n-5CJB2D-pDkmos-pFwXES-nurpX3-pp41ry-pp1dAY-pFwNsy-oJB6kq-muwG16-pDvdkd-p9Urp1-faZQX-nGaHnw-f8DTU5-oJN1fT-pp2uCK-poWjKv-Tb4mV6-Tb4m8p-oxydG8-nEQpU6-e3FcaD-QXErjw-kivnbu-qwc9E3-7kjhVV-ogkh3s-CQvWw-7kjhYx-trDUU-5CJB7v-oGM82v-7kTNRB-Un91FN-RZv98s-nXJQ2W-bS4kyn-RmhXiG-T6YfAD-SEci2b-SPxkrL .
5. Unknown. http://maxpixel.freegreatpicture.com/Steinchen-Domino-Fun-Play-Stones-Pay-Mikado-1013878 .
6. Unknown. https://overlordoftheuberferal.wordpress.com/tag/right-triangles/ .

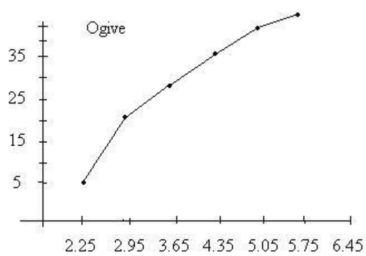
CHAPTER 14 Probability and Statistics

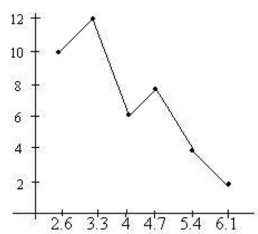
Chapter Outline

14.1 Introduction: Probability and Statistics

Statistics is important for understanding, describing, and predicting the world around us. **Descriptive statistics** is using summaries to present information that you have found to a reader. Summaries can be graphs or small groups of numbers that are easier to understand than long lists of numbers. **Inferential statistics** is using data to make predictions. Both inferential statistics and descriptive statistics help you understand the world around you and communicate information about it effectively.







14.2 Counting with Permutations and Combinations

Learning Objectives

Learn about counting using decision charts, permutations, and combinations.

Introduction

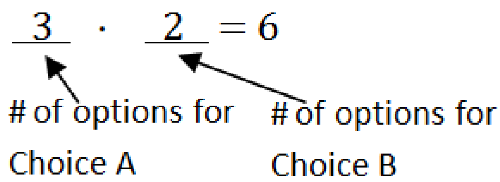
Combinatorics is a branch of mathematics that focuses on the study of finite or countable discrete structures. Combinatorics has applications in optimization, computer science, and statistical physics. There are times that it makes sense to count the number of ways an event could occur by looking at each possible outcome. However, when a large number of outcomes exists, this method becomes inefficient. If someone asked you how many possible regular license plates there are for the state of Oregon, it would not be feasible to count each and every one.

Instead, you could use the fact that on the typical Oregon license plate, there are 4 numbers and 3 letters. Using this information, about how many license plates could there be?

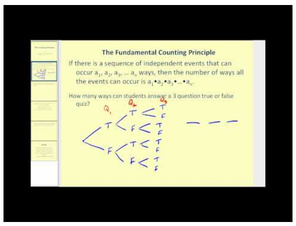
The Counting Principle

Consider choice A with 3 options (A_1, A_2, A_3), and choice B with 2 options (B_1, B_2). If you had to choose an option from A and then an option from B, the overall total number of options would be $3 \cdot 2 = 6$. The options are $A_1B_1, A_1B_2, A_2B_1, A_2B_2, A_3B_1, A_3B_2$.

You can see where the 6 comes from by making a decision tree and using the Fundamental Counting Principle. A **decision tree** is a graph that models the options possible at each stage of an experiment. To make a decision tree, you 1st need to determine how many decisions you are making. Here, there are only two decisions to make: 1) choose an option from A, and 2) choose an option from B, so you will have two "slots" in your decision tree. Next, think about how many possibilities there are for the 1st choice (in this case there are 3), and how many possibilities there are for the 2nd choice (in this case there are 2). The **Fundamental Counting Principle** says that you can multiply those numbers together to get the total number of outcomes.



The following video explains how to find the number of ways an event can occur:



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Combinations and Permutations

Another type of counting question is when you have a given number of objects, you want to choose some (or all) of them, and you want to know how many ways there are to do this. For example, a teacher with a class of 30 students wants 5 of them to do a presentation, and she wants to know how many ways this could happen. These types of questions have to do with **combinations** and **permutations.** The difference between combinations and permutations is whether or not the order you are choosing the objects matters.

- A teacher choosing a group to make a presentation is a **combination** problem, because **order does not matter.**
- A teacher choosing 1st-, 2nd-, and 3rd-place winners in a science fair is a **permutation** problem, because the **order does matter.** (1st place and 2nd place are different outcomes.)

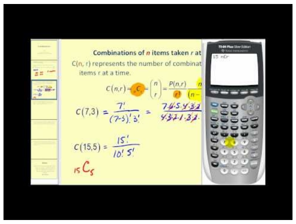
The factorial symbol, !, means to multiply every natural number up to and including that whole number together. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. The factorial symbol is used in the formulas for permutations and combinations.

Combination Formula

The number of ways to choose k objects from a group of n objects where order does not matter is

$$_nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)\,!}.$$

The following video introduces combinations and explains how to evaluate combinations and solve counting problems using combinations:



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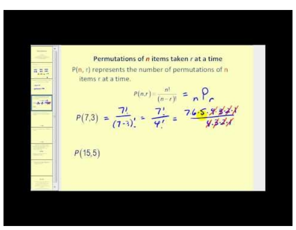
Play, Learn, and Explore with Combinations: Friends at a Party

Permutation Formula

The number of ways to choose and arrange k objects from a group of n objects is

$$_nP_k = k!\binom{n}{k} = k! \cdot \frac{n!}{k!\,(n-k)\,!} = \frac{n!}{(n-k)\,!}.$$

The following video explains how to evaluate factorials, use permutations to solve problems, and determine the number of permutations with indistinguishable items:



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Play, Learn, and Explore with Permutations: Podiums

Notice that in both combination and permutation problems, you are not allowed to repeat your choices. Any time you are allowed to repeat and order does not matter, you can use the Fundamental Counting Principle. (Problems with repetition where order does not matter are more complex and not discussed in this section.)

Whenever you do a counting problem, the 1st thing you should decide is whether the problem is a Fundamental Counting Principle problem, a permutation problem, or a combination problem. You'll find that permutation problems can also be solved with the Fundamental Counting Principle, but the opposite is not true. There are many Fundamental Counting Principle problems (ones where you are allowed to repeat choices) that cannot be solved with the permutation formula.

Examples

Example 1

You are going on a road trip with 4 friends in a car that fits 5 people. How many different ways can everyone sit if you have to drive the whole way?

Solution:

The Fundamental Counting Principle is a great way of thinking about this problem. You have to sit in the driver's seat. There are 4 options for the 1st passenger seat. Once that person is seated, there are 3 options for the next passenger seat. This goes on until there is one person left with 1 seat.

$$1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Example 2

How many different ways can the gold, silver, and bronze medals be awarded in an Olympic event with 12 athletes competing?

Solution:

Since the order does matter with the 3 medals, this is a permutation problem. You will start with 12 athletes and then choose and arrange 3 different winners.

$$_{12}P_3 = \frac{12!}{(12-3)!} = \frac{12!}{9!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot \ldots}{9 \cdot \ldots} = 12 \cdot 11 \cdot 10 = 1{,}320$$

Note that you can also use the Fundamental Counting Principle to decide how many possibilities are there for gold (12), how many possibilities are there for silver (11, since one already has gold), and how many possibilities are there for bronze (10). You can use the Fundamental Counting Principle for any permutation problem.

$$12 \cdot 11 \cdot 10 = 1{,}320$$

Example 3

You are deciding which awards you are going to display in your room. You have 8 awards, but you only have room to display 4 awards. Right now you are not worrying about how to arrange the awards, so the order does not matter. How many ways could you choose the 4 awards to display?

Solution:

Since order does not matter, this is a combination problem. You start with 8 awards and then choose 4.

$$_8C_4 = \binom{8}{4} = \frac{8!}{4!\,(8-4)\,!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 2 \cdot 5 = 70$$

Note that if you try to use the Fundamental Counting Principle with this question, you will need to do an extra step of reasoning. There are 8 options you could choose 1st, then 7 left, then 6, and lastly 5.

$$8 \cdot 7 \cdot 6 \cdot 5 = 1{,}680$$

This number is so big because it takes order into account, which you don't care about. It is the same result you would get if you used the permutation formula instead of the combination formula. To get the right answer, you need to divide this number by the number of ways 4 objects can be arranged, which is $4! = 24$. This has to do with the connection between the combination formula and the permutation formula.

Example 4

Recall the problem from the Introduction: How many Oregon license plates could be created with 3 letters and 4 numbers?

Solution:

A license plate that has 3 letters and 4 numbers can be represented by the Fundamental Counting Principle with 7 spaces. You can use the Fundamental Counting Principle because order definitely does matter with license plates. The 1st spot is a number, the next three spots are letters, and the last three spots are numbers. Note that when choosing a license plate, repetition is allowed.

$$10 \cdot 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 26^3 \cdot 10^4 = 175{,}760{,}000$$

This number is only approximate because, in reality, certain letter and number combinations are not allowed, some license plates have extra symbols, and some commercial and government license plates have more numbers, fewer letters, or blank spaces.

Example 5

There are 20 hockey players on a pro NHL team, 2 of whom are goalies. How many sets of 5 skaters and 1 goalie can be on the ice at the same time?

Solution:

The question asks for how many on the ice, implying that order does not matter. This is combination problem with 2 combinations. You need to choose 1 goalie out of a possible of 2, and choose 5 skaters out of a possible 18.

$$\binom{2}{1}\binom{18}{5} = 2 \cdot \frac{18!}{5! \cdot 13!} = 17{,}136$$

Example 6

How many different ways could you score a 70% on a 10-question test, where each question is weighted equally and is either right or wrong?

Solution:

The order of the questions you got right does not matter, so this is a combination problem.

$$\binom{10}{7} = \frac{10!}{7!3!} = 120$$

Example 7

How many different 4-digit ATM passwords are there? Assume you can repeat digits.

Solution:

Order does matter. There are 10 digits, and repetition is allowed. You can use the Fundamental Counting Principle for each of the 4 options.

$$10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$$

Summary

- The **Fundamental Counting Principle** states that if one event has m possible outcomes and a 2nd event has n possible outcomes, then there are $m \cdot n$ total possible outcomes for the two events together.
- A **combination** is the number of ways of choosing k objects from a total of n objects (order does not matter).

$$_nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

- A **permutation** is the number of ways of choosing and arranging k objects from a total of n objects (order does matter).

$$_nP_k = k!\binom{n}{k} = k! \cdot \frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!}$$

Review

Simplify each of the following expressions so that they do not have a factorial symbol:

1. $\frac{7!}{3!}$
2. $\frac{110!}{105!5!}$
3. $\frac{52!}{49!}$
4. In how many ways can you choose 3 objects from a set of 9 objects?
5. In how many ways can you choose and arrange 4 objects from a set of 15 objects?

First state whether each problem is a permutation/decision tree problem or a combination problem. Then solve.

6. Suppose you need to choose a new combination for your combination lock. You have to choose 3 numbers, each different and between 0 and 40. How many permutations are there?

7. You just won a contest where you can choose 2 friends to go with you to a concert. You have 5 friends who are available and want to go. In how many ways can you choose the friends?

8. You want to construct a 3-digit number from the digits 4, 6, 8, 9. How many possible numbers are there?

9. There are 12 workshops at a conference, and Sam has to choose 3 to attend. In how many ways can he choose the 3 to attend?

10. A contest has 9 girls and 5 boys as finalists. In how many ways can 1st-, 2nd-, and 3rd-place winners be chosen?

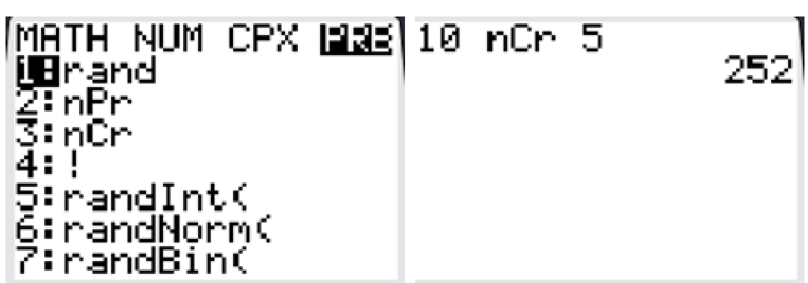
11. For the special at a restaurant, you can choose 3 different items from the 10-item menu. How many different combinations of meals could you get?

12. You visit 12 colleges and want to apply to 4 of them. In how many ways could you choose the 4 to apply to?

13. For the 12 colleges you visited, you want to rank your top 5. In how many ways could you rank your top 5?

14. Explain why the following problem is not strictly a permutation or combination problem: The local ice cream shop has 12 flavors. You decide to buy 2 scoops in a dish. In how many ways could you do this if you are allowed to get 2 of the same scoop?

15. Your graphing calculator has the combination and permutation formulas built in. Push the MATH button and scroll to the right to the PRB list. You should see ${}_nP_r$ and ${}_nC_r$ as options. To use these: 1) On your home screen, type the value for n; 2) Select ${}_nP_r$ or ${}_nC_r$; 3) Type the value for k (r on the calculator). Use your calculator to verify that ${}_{10}C_5 = 252$.



Review (Answers)

Please see the Appendix.

14.3 Binomial Theorem

Learning Objectives

Learn to apply the Binomial Theorem to expand binomials that are raised to a power. In order to do this you will use your knowledge of sigma notation and combinations.

Introduction

The Binomial Theorem is used to generate IP addresses that are assigned to different computers, make economic predictions, and determine the cost and time needed to complete projects. The Binomial Theorem is a formula you can use to expand a binomial such as $(2x-3)^5$ without having to compute the repeated distribution. What is the expanded version of $(2x-3)^5$?

Binomial Theorem

The Binomial Theorem states

$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$.

Writing out a few terms of the summation symbol helps you to understand how this theorem works:

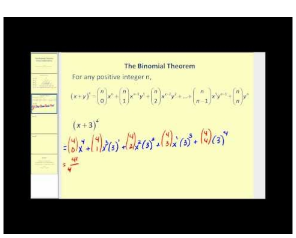
$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

Going from one term to the next in the expansion, you should notice that the exponents of $a$decrease while the exponents of b increase. You should also notice that the coefficients of each term are combinations. Recall that $\binom{n}{0}$ is the number of ways to choose 0 objects from a set of $n$objects.

Binomial Theorem

The **Binomial Theorem** is a theorem that states how to expand binomials that are raised to a power using combinations. The **Binomial Theorem** is $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$.

The following video demonstrates how to apply the Binomial Theorem:



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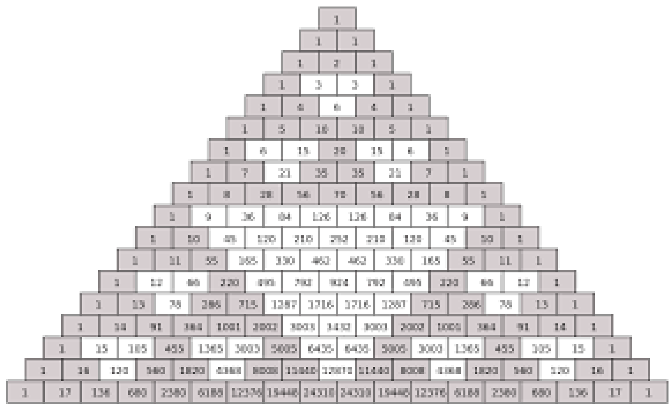
Pascal's Triangle

Another way to think about the coefficients in the Binomial Theorem is that they are the numbers from Pascal's Triangle:



This triangle was named for mathematician Blaise Pascal (shown below).

Look at the expansions of $(a+b)^n$ below, and notice how the coefficients of the terms are the numbers in Pascal's Triangle:

$$(a+b)^0 = 1$$

$$(a+b)^1 = 1a + 1b$$

$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$

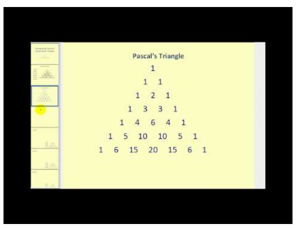
$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

$$\vdots$$

Be extremely careful when working with binomials of the form $(a-b)^n$. You need to remember to capture the negative with the 2nd term as you write out the expansion: $(a-b)^n = (a+(-b))^n$.

The following video explains binomial expansion using Pascal's Triangle.



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Play, Learn, and Explore with Pascal's Triangle: Pascal's Triangle

Examples

Example 1

Expand the binomial below using the Binomial Theorem.

$$(m-n)^6$$

Solution:

$$\begin{aligned}(m-n)^6 &= \binom{6}{0}m^6 + \binom{6}{1}m^5(-n)^1 + \binom{6}{2}m^4(-n)^2 + \binom{6}{3}m^3(-n)^3 \\ &+ \binom{6}{4}m^2(-n)^4 + \binom{6}{5}m^1(-n)^5 + \binom{6}{6}(-n)^6 \\ &= 1m^6 - 6m^5n + 15m^4n^2 - 20m^3n^3 + 15m^2n^4 - 6m^1n^5 + 1n^6\end{aligned}$$

Example 2

What is the coefficient of the term x^7y^9 in the expansion of the binomial $(x+y)^{16}$?

Solution:

The Binomial Theorem allows you to calculate just the coefficient you need.

$$\binom{16}{9} = \frac{16!}{9!7!} = \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 11{,}440$$

Note that symmetry occurs in combinations. That is, 16 choose 9 and 16 choose 7 will give the same result.

Example 3

What is the coefficient of x^6 in the expansion of $(4-3x)^7$?

Solution:

For this problem, you should calculate the whole term, since the 3 and the 4 in $(3-4x)$ will impact the coefficient of x^6 as well. $\binom{7}{6}4^1(-3x)^6 = 7 \cdot 4 \cdot 729x^6 = 20,412x^6$. The coefficient is 20,412.

Example 4

Recall the problem from the Introduction: What is the expanded version of $(2x-3)^5$?

Solution:

The expanded version of $(2x-3)^5$ is

$$\begin{aligned}(2x-3)^5 &= \binom{5}{0}(2x)^5 + \binom{5}{1}(2x)^4(-3)^1 + \binom{5}{2}(2x)^3(-3)^2 \\ &\quad + \binom{5}{3}(2x)^2(-3)^3 + \binom{5}{4}(2x)^1(-3)^4 + \binom{5}{5}(-3)^6 \\ &= (2x)^5 + 5(2x)^4(-3)^1 + 10(2x)^3(-3)^2 \\ &\quad + 10(2x)^2(-3)^3 + 5(2x)^1(-3)^4 + (-3)^5 \\ &= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243.\end{aligned}$$

Example 5

What is the coefficient of x^3 in the expansion of $(x-4)^5$?

Solution:

$$\binom{5}{2} \cdot 1^3(-4)^2 = 160$$

Example 6

Compute the following summation:

$$\sum_{i=0}^{4} \binom{4}{i}$$

Solution:

This problem is asking for $\binom{4}{0} + \binom{4}{1} + \cdots + \binom{4}{4}$, which are the sum of all the coefficients of $(a+b)^4$.

$$1+4+6+4+1 = 16$$

Note that the sum of the coefficients of the expansion of $(a+b)^4$ is the same as what happens when $a = b = 1$. So, computing directly,

$$(1+1)^4 = 2^4 = 16.$$

Example 7

Collapse the following binomial expansion using the Binomial Theorem:

$$32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$$

Solution:

Since the last term is -1 and the power on the 1st term is a 5, you can conclude that the 2nd half of the binomial is $(?-1)^5$. The 1st term is positive and $(2x)^5 = 32x^5$, so the 1st term in the binomial must be $2x$. The binomial is $(2x-1)^5$.

Summary

- The **Binomial Theorem** (or **binomial expansion**) describes the algebraic expansion of powers of a binomial.
- The coefficients in the binomial expansion appear as the entries of Pascal's Triangle. In Pascal's Triangle, each entry is the sum of the two above it.
- The coefficients can also be generated using combinations. The numbers in the combination are associated with a particular row and column in Pascal's Triangle. For example, 5 combinations of 3 would be associated with the 5th row and 3rd column of the Triangle.
- The **Binomial Theorem** is $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$.

Review

Expand each of the following binomials using the Binomial Theorem:

1. $(x-y)^4$
2. $(x-3y)^5$
3. $(2x+4y)^7$
4. What is the coefficient of x^4 in $(x-2)^7$?
5. What is the coefficient of x^3y^5 in $(x+y)^8$?
6. What is the coefficient of x^5 in $(2x-5)^6$?
7. What is the coefficient of y^2 in $(4y-5)^4$?
8. What is the coefficient of x^2y^6 in $(2x+y)^8$?
9. What is the coefficient of x^3y^4 in $(5x+2y)^7$?

Compute the following summations:

10. $\sum_{i=0}^{9} \binom{9}{i}$
11. $\sum_{i=0}^{12} \binom{12}{i}$
12. $\sum_{i=0}^{8} \binom{8}{i}$

Collapse the following polynomials using the Binomial Theorem:

13. $243x^5 - 405x^4 + 270x^3 - 90x^2 = 15x - 1$
14. $x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$
15. $128x^7 - 448x^6y + 672x^5y^2 - 560x^4y^3 + 280x^3y^4 - 84x^2y^5 + 14xy^6 - y^7$

Review (Answers)

Please see the Appendix.

14.4 Basic Probability

Learning Objectives

Learn to calculate the probability of simple and compound events.

Introduction

Assume someone in your neighborhood wins a lottery game on average once every 100 days. What is the probability that someone will win the lottery in the next 100 days?

Probability

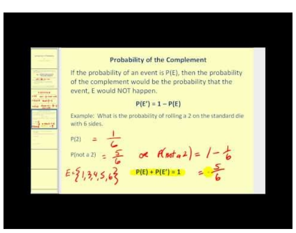
Probability is the chance of an event occurring. Simple probability is defined as the number of successful outcomes divided by the total number of outcomes, assuming all outcomes are equally likely. The notation $P(E)$ is read "the probability of event E."

$$P(E) = \frac{\text{\# successful outcomes}}{\text{\# possible outcomes}}$$

Probabilities can be represented with fractions, decimals, or percents. Since the number of possible outcomes is in the denominator, the probability is always between 0 and 1. A probability of 0 means the event will definitely not happen, while a probability of 1 means the event will definitely happen.

$$0 \leq P(E) \leq 1$$

The following video introduces probability and determining the probability of basic events:



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The probability of something not happening is called the **complement**, P^C or P', and is found by subtracting the probability from 1.

$$P(E^C) = 1 - P(E)$$

You will often be looking at probabilities of two or more independent experiments. Experiments are **independent** when the outcome of one experiment has no effect on the outcome of the other experiment: for instance, drawing a card from a deck and then replacing that card back in the deck prior to drawing a 2nd card. Since the initial card was replaced back in the deck, the 2nd draw is not affected by the 1st draw. Experiments are **dependent** when the outcome of one experiment has an effect on the outcome of the other experiment. Now, suppose after one card is drawn from the deck, it is not replaced. As a result, the deck will have one fewer card, the 1st card drawn, when the 2nd card is drawn, which will have an effect on the probability of this 2nd experiment.

If there are two independent experiments, one with outcome A and the other with outcome B, then the probability of A and B is

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

The probability of A or B is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

Play, Learn, and Explore with Probability: Marbles in the Bag

Examples

Example 1

If you are dealt one card from a 52-card deck, what is the probability that you are dealt a heart? What is the probability that you are dealt a 3? What is the probability that you are dealt the 3 of hearts?

Solution:

There are 13 hearts in a deck of 52 cards.

$$P(heart) = \frac{13}{52} = \frac{1}{4}$$

There are 4 threes in the deck of 52.

$$P(three) = \frac{4}{52} = \frac{1}{13}$$

There is only 1 three of hearts.

$$P(\text{three and heart}) = \frac{1}{52}$$

Example 2

Dean and his friend Randy like to play a special poker game with their friends. Dean goes home a winner 60% of the time, and Randy goes home a winner 75% of the time.

1) What is the probability that they both win on the same night?

2) What is the probability that Randy wins and Dean loses?

3) What is the probability that they both lose?

Solutions:

First represent the information with probability symbols.

Let D be the event that Dean wins. Let R be the event that Randy wins. The complement of each probability is when Dean or Randy loses instead.

$$P(D) = 0.60, \quad P(D^C) = 0.40$$

$$P(R) = 0.75, \quad P(R^C) = 0.25$$

1) $P(D \text{ and } R) = P(D) \cdot P(R) = 0.60 \cdot 0.75 = 0.45$

2) $P(R \text{ and } D^C) = P(R) \cdot P(D^C) = 0.75 \cdot 0.40 = 0.30$

3)$P(D^C \text{ and } R^C) = P(D^C) \cdot P(R^C) = 0.40 \cdot 0.25 = 0.10$

Example 3

Recall the problem from the Introduction: If someone wins the lottery on average once every 100 days, what is the probability that someone will win the lottery in the next 100 days?

Solution:

Since there are 100 days and each day has a probability of 0.01 for a lottery winner, then by this logic, there is a 100% chance that a lottery winner in the next 100 days. However, this isn't true because if on average there is a lottery winner once every 100 days, some stretches of 100 days there will be more winners, and some stretches there will be no winners. The 100% solution does not hold.

To solve this problem, you need to rephrase the question and ask a slightly different one that will help as an intermediate step. What is the probability that there is *not* a lottery winner in the next 100 days?

For this to happen, no one can win on day 1, and not win on day 2, and not win on day 3, etc.

The probability of not winning the lottery on any day is $P(\text{no win}) = 1 - P(\text{win}) = 1 - 0.01 = 0.99$.

The product of each of these probabilities for the 100 days is

$$0.99^{100} \approx 0.366.$$

Therefore, the probability that there is no lottery winner in the next 100 days is about 36.6%. To answer the original question, the probability that no one wins the lottery in the next 100 days is $1 - 0.366 = 0.634$, or about 63.4%.

Example 4

Jack is a basketball player with a free-throw average of 0.77. If the outcome of a particular free throw is independent of the outcomes of others, what is the probability that in a game where he has 8 shots, he makes all 8? What is the probability that he makes only 1?

Solution:

Let J represent the event that Jack makes the free-throw shot, and J^C represent the event that Jack misses the shot.

$$P(J) = 0.77,\ P(J^C) = 0.23$$

The probability that Jack makes all 8 shots is the same as Jack making 1 shot, and making the 2nd shot, and making the 3rd shot, etc.

$$P(J)^8 = 0.77^8 \approx 12.36\%$$

There are 8 ways that Jack could make 1 shot and miss the rest. The probability of each of these cases occurring is

$$P(J^C)^7 \cdot P(J) = 0.23^7 \cdot 0.77.$$

Therefore, the overall probability of Jack making 1 shot and missing the rest is

$$0.23^7 \cdot 0.77 \cdot 8 = 0.0002097 = 0.02097\%.$$

Example 5

If it has a 20% chance of raining on Tuesday, your phone has 30% chance of running out of batteries, and there is a 10% chance that you forget your wallet. If the outcome of these particular events are independent, what is the probability that you are in the rain without money or a phone?

Solution:

While a pessimist may believe that all the improbable negative events will occur at the same time, the actual probability of this happening is less than one percent:

$$0.20 \cdot 0.30 \cdot 0.1 = 0.006 = 0.6\%.$$

Example 6

Consider the previous question with the rain, wallet, and phone. What is the probability that at least one of the three events does occur?

Solution:

We can best find this solution by exploring: What is the probability that none of the events occur?

$$0.8 \cdot 0.7 \cdot 0.9 = 0.504$$

The probability that at least one occurs is the complement of none occurring.

$$1 - 0.504 = 0.496 = 49.6\%$$

Summary

- The **probability** of an event is the number of outcomes you are looking for (called successes) divided by the total number of outcomes.
- The notation $P(E)$ is read "the probability of event E."

$$P(E) = \frac{\text{\# successful outcomes}}{\text{\# possible outcomes}}$$

- The **complement of an event** is the event not happening.
- **Independent events** are events where the occurrence of the 1st event does not impact the probability of the 2nd event.
- **Dependent events** are events where the occurrence of the 1st event does impact the probability of the 2nd event.

Review

A card is chosen from a standard deck.

1. What's the probability that the card is a queen?

2. What's the probability that the card is a queen or a spade?

You toss a nickel, a penny, and a dime.

3. List all the possible outcomes.

4. What is the probability that the nickel comes up heads?

5. What is the probability that none of the coins comes up heads?

6. What is the probability that at least one of the coins comes up heads?

A bag contains 7 red marbles, 9 blue marbles, and 10 green marbles. You reach into the bag and choose 4 marbles, one after another, without replacement.

7. What is the probability that all 4 marbles are red?

8. What is the probability that you get a red marble, then a blue marble, then 2 green marbles?

You take a 40-question multiple choice test, and believe that for each question, you have a 55% chance of getting it right.

9. What is the probability that you get all the questions right?

10. What is the probability that you get all the questions wrong?

A player rolls a pair of standard dice. Find each probability.

11. $P(sum\ is\ even)$

12. $P(sum\ is\ 7)$

13. $P(sum\ is\ at\ least\ 3)$

14. You want to construct a 3-digit number at random from the digits 4, 6, 8, 9 without repeating digits. What is the probability that you construct the number 684?

15. In poker, a straight is 5 cards in a row (for example, 3, 4, 5, 6, 7), NOT all the same suit. (If they are all the same suit, it is considered a straight flush or a royal flush.) A straight can start or end with an ace. What's the probability of a straight? For an even bigger challenge, see if you can calculate the probabilities for all the poker hands.

Review (Answers)

Please see the Appendix.

14.5 Expected Value and Payoffs

Learning Objectives

Learn to apply what you know about mean and averages to weighted averages and expected value.

Introduction

Suppose you record approximately how long it takes you to drive to work each day.

TABLE 14.1:

Time	Number of Days
5-7 minutes	1
7-8 minutes	4
8-9 minutes	7
9-10 minutes	9
10-12 minutes	2

How long does it take you to drive to work on average? To answer this question, you could find the weighted average of the expected value. If you choose expected value, consider that the situation gives you frequency rather than probability. You can calculate the probability of each of the categories by dividing each frequency by the total number of days (23). Since the time occurs in intervals, it is reasonable to use the average time in each interval as representative of the category when calculating the expected value.

TABLE 14.2:

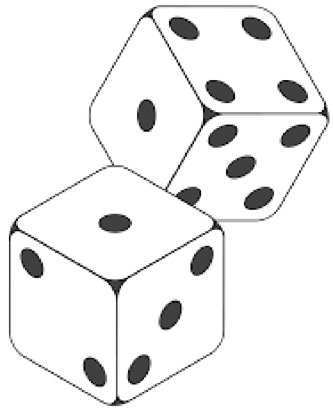
Time	Number of Days	Probability
5-7 minutes	1	$\frac{1}{23}$
7-8 minutes	4	$\frac{4}{23}$
8-9 minutes	7	$\frac{7}{23}$
9-10 minutes	9	$\frac{9}{23}$
10-12 minutes	2	$\frac{2}{23}$

$6 \cdot \frac{1}{23} + 7.5 \cdot \frac{4}{23} + 8.5 \cdot \frac{7}{23} + 9.5 \cdot \frac{9}{23} + 11 \cdot \frac{2}{23} = \frac{203}{23} \approx 8.8$

On average, it takes you 8-9 minutes to get to school.

When playing a game of chance, there are three basic elements. There is the cost to play the game (usually), the probability of winning the game, and the amount you receive if you win. If games of chance with these three elements are played repeatedly, you can use probability and averages to calculate how much you can expect to win or lose in the long run.

Consider a dice game that pays you triple your bet if you roll a 6, and double your bet if you roll a 5. If you roll anything else, you lose your bet. What is your expected return on a $1 wager?



Expected Value

A weighted average is like a regular average, except the data are often given to you in summary form.

Data in Raw Form:

1, 3, 5, 3, 2, 1, 2, 5, 6, 4, 5, 2, 6, 1, 4, 3, 6, 1, 2, 4, 6, 1, 3, 1, 3, 5, 6

Data in Summary Form:

TABLE 14.3:

Number	***Occurrence Count***
1	6
2	4
3	5
4	3
5	4
6	5
Total Occurrences:	27

Notice that the summary data indicates, for example, how many times a 1 was rolled (6 times). To calculate the total number of occurrences of data:

- In raw form: Count how many data points you have.
- In summary form: Find the sum of the occurrence column.

To calculate the average:

- In raw form: Find the sum of the data points, and divide by the total number of occurrences.
- In summary form: Find the sum of the data points by finding the sum of the product of each number and its occurrence:

$$1 \cdot 6 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 3 + 5 \cdot 4 + 6 \cdot 5 = 91$$

Then, divide that sum by the total number of occurrences. *In a sense, you are assigning a weight to each of the 6 numbers based on their frequency in your 27 trials.*

The same logic of finding the average of data given in summary form applies when doing theoretical expected value for a game or a weighted average. Consider a game of chance with four prizes ($1, $2, $3, and $4), where each outcome has a specific probability of happening, shown in the table below:

TABLE 14.4:

Number	***Probability***
$1	50%
$2	20%
$3	20%
$4	10%

Note that the probabilities must add up to 100%! To calculate the expected value of this game, weight the outcomes by their assigned probabilities.

$$\$1 \cdot 0.50 + \$2 \cdot 0.20 + \$3 \cdot 0.20 + \$4 \cdot 0.10 = \$1.90$$

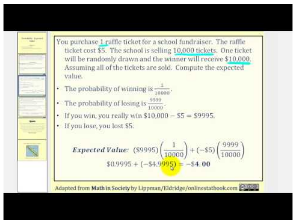
This means that if you were to play this game many times, your average amount of winnings should be $1.90. Note there will be no game that you actually get $1.90, because that was none of the options. Expected value is a measure of what you should expect to get per game in the long run.

Play, Learn, and Explore with Expected Value: Playing Darts

The **payoff** of a game is the expected value of the game minus the cost. If you expect to win about $1.90 on average when you play a carnival game repeatedly, and it costs only $1.50 to play, then the expected payoff is 40 cents per game.

In general, to find the **expected value** for a game or other scenario, find the sum of all possible outcomes, each multiplied by the probability of its occurrence.

The following video defines expected value, and provides two examples of how to find the expected value of an event:



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Play, Learn, and Explore with Expected Value and Payoff: Game of Chance

Examples

Example 1

A teacher has five categories of grades that each make up a specific percentage of the final grade. Calculate Owen's grade.

TABLE 14.5:

Category	Weight	Owen's grade
Quizzes and Tests	30%	78%
Homework	25%	100%
Final	20%	74%
Projects	20%	90%
Participation	5%	100%

Solution:

Using the concept of weighted average, weight each of Owen's grades by the weight of the category.

$$0.78 \cdot 0.3 + 1 \cdot 0.25 + 0.74 \cdot 0.20 + 0.90 \cdot 0.20 + 1 \cdot 0.05 = 0.862$$

Owen gets an 86.2%.

Example 2

Courtney plays a game where she flips a coin. If the coin comes up heads, she wins \$2. If the coin comes up tails, she loses \$3. What is Courtney's expected payoff each game?

Solution:

The probability of getting heads is 50%, and the probability of getting tails is 50%. Using the concept of weighted averages, you should weight winning \$2 and losing \$3 dollars by 50% each. In this case, there is no initial cost to the game.

$$2 \cdot 0.50 - 3 \cdot 0.50 = -0.50$$

This means that while sometimes she might win and sometimes she might lose, on average she is expected to lose about 50 cents per game.

Example 3

Paul is deciding whether or not to pay the parking meter when he is going to the movies. He knows that a parking ticket costs \$30, and he estimates there is a 40% chance that the traffic police spot his car and write him a ticket. If he chooses to pay the meter, it will cost \$4, and he'll have a 0% chance of getting a ticket.

Is it cheaper to pay the meter or risk the fine?

Solution:

Since there are two possible scenarios, calculate the expected cost in each case.

$$\textit{Paying the meter}: \$4 \cdot 100\% = \$4$$

$$\textit{Risking the fine}: \$0 \cdot 60\% + \$30 \cdot 40\% = \$12$$

Risking the fine has an expected cost three times that of paying the meter.

Example 4

Recall the problem from the Introduction: Suppose you are playing a dice game that pays you triple your bet if you roll a 6, and double your bet if you roll a 5. If you roll anything else, you lose your bet. What is your expected return on a $1 wager?

Solution:

In a game that pays you triple your bet if you roll a 6, and double your bet if you roll a 5, the expected return on a $1 wager is

$$\$0 \cdot \frac{2}{3} + \$2 \cdot \frac{1}{6} + \$3 \cdot \frac{1}{6} = \frac{5}{6}.$$

If you spend $1 to play the game, and you play the game multiple times, you can expect a return of $\frac{5}{6}$ of $1, or about 83 cents on average.

Example 5

What is the payoff of a slot machine that costs $1 to play and pays out $5 with probability 4%, $10 with probability of 2%, and $30 with probability 0.5%?

Solution:

$$0 \cdot 0.935 + 5 \cdot 0.04 + 10 \cdot 0.02 + 30 \cdot 0.005 - 1 = -0.45$$

You will lose 45 cents on average if you play the slot machine many times.

Example 6

What is the expected value of an experiment with the following outcomes and corresponding probabilities?

TABLE 14.6:

Outcome	31	35	37	39	43	47	49
Probability	0.1	0.1	0.1	0.2	0.2	0.2	0.1

Solution:

$$31 \cdot 0.1 + 35 \cdot 0.1 + 37 \cdot 0.1 + 39 \cdot 0.2 + 43 \cdot 0.2 + 47 \cdot 0.2 + 49 \cdot 0.1 = 41$$

Summary

- A **weighted average** is an average that multiplies each component by a factor representing its frequency or probability.
- The **expected value** is the return or cost you can expect on average, given many trials.
- The **payoff** of a game is the expected value of the game minus the cost.

Review

1. Explain how to calculate expected value.

2. True or false: If the expected value of a game is 50 cents, then you can expect to win 50 cents each time you play.

3. True or false: The greater the number of games played, the closer the average winnings will be to the theoretical expected value.

4. A player rolls a standard pair of dice. If the sum of the numbers is a 6, the player wins \$6. If the sum of the numbers is anything else, the player has to pay \$1. What is the expected value for this game?

5. What is the payoff of a slot machine that costs 25 cents to play and pays out \$1 with probability 10%, \$50 with probability of 1%, and \$100 with probability 0.01%?

6. A slot machine pays out \$1 with probability 5%, \$100 with probability of 0.5%, and \$1,000 with probability 0.01%. If the casino wants to guarantee that it won't lose money on this machine, how much should it charge people to play?

7. What is the expected value of an experiment with the following outcomes and corresponding probabilities?

TABLE 14.7:

Outcome	12	14	18	20	21	22	23
Probability	0.05	0.1	0.6	0.1	0.1	0.03	0.02

Calculate the final grades for each of the students given the information in the table.

TABLE 14.8:

Category	Weight	Sarah	Jason	Kimmy	Maria	Kayla
Quizzes and Tests	30%	74%	85%	90%	80%	75%
Homework	25%	95%	40%	100%	90%	95%
Final	20%	68%	80%	85%	70%	50%
Projects	20%	85%	70%	95%	75%	85%
Participation	5%	95%	100%	100%	80%	60%

8. What is Sarah's final grade?

9. What is Jason's final grade?

10. What is Kimmy's final grade?

11. What is Maria's final grade?

12. What is Kayla's final grade?

13. Look back at the grades and final grades for the five students. Do the grades seem fair to you given how each student performed in each area? Do you think the category weights should be changed?

14. You are in charge of a game booth at the fair. In the game, players pick a card at random from the deck. If the card is a jack, queen, or king, the player wins $5. What is the minimum amount you should charge to feel confident you will make a profit by the end of the fair?

15. Make up your own game that has at least two possible outcomes with an expected payoff of 50 cents.

16. Explain why it makes sense for a casino to consider the concept of expected value when designing its games.

Review (Answers)

Please see the Appendix.

14.6 Graphic Displays of Data

Learning Objectives

Learn to display data using bar charts, histograms, pie charts, and boxplots.

Introduction

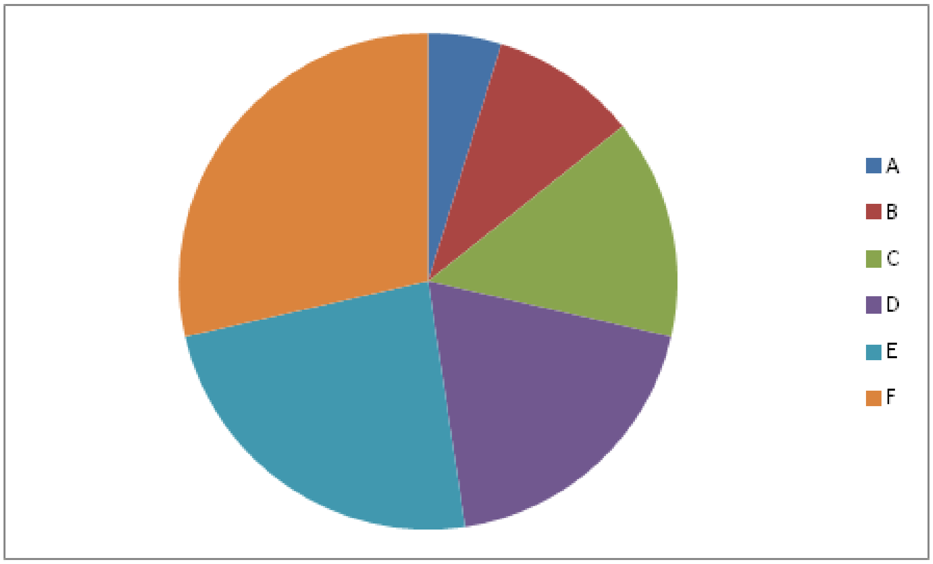
Data related to sports events, health crises, political campaigns, and many other situations can be collected and analyzed in a number of different ways. While two common types of graphic displays are bar charts and histograms, several other options may be useful, depending on one's goal in analyzing and presenting data. These different types of displays can help us summarize data and tell a story, but choosing among them is key. For example, both bar charts and histograms use vertical or horizontal bars to represent the number of data points in each category or interval.

Suppose Dr. Alameda has given all her computer science classes an exam. She has created a frequency table for the scores, but needs a better way to display the data. How can she do this?

Graphic Displays of Data

There are a few common ways of displaying data graphically that you should be familiar with:

1) A **pie chart** shows the relative proportions of data in different categories. Pie charts are excellent ways of displaying categorical data with easily separable groups. The pie chart below shows six categories labeled $A-F$. The size of each pie slice is determined by the central angle.

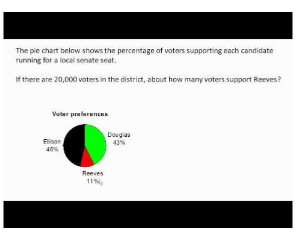


Determining the Central Angle for a Pie Chart

Since there 360 in a circle, the size of the central angle θ_A of category A can be found by

$$\frac{\theta_A}{360} = \frac{\text{\# data points in category A}}{\text{Total number of data points}}.$$

The following video discusses a specific example of a pie chart related to voters in a particular district:



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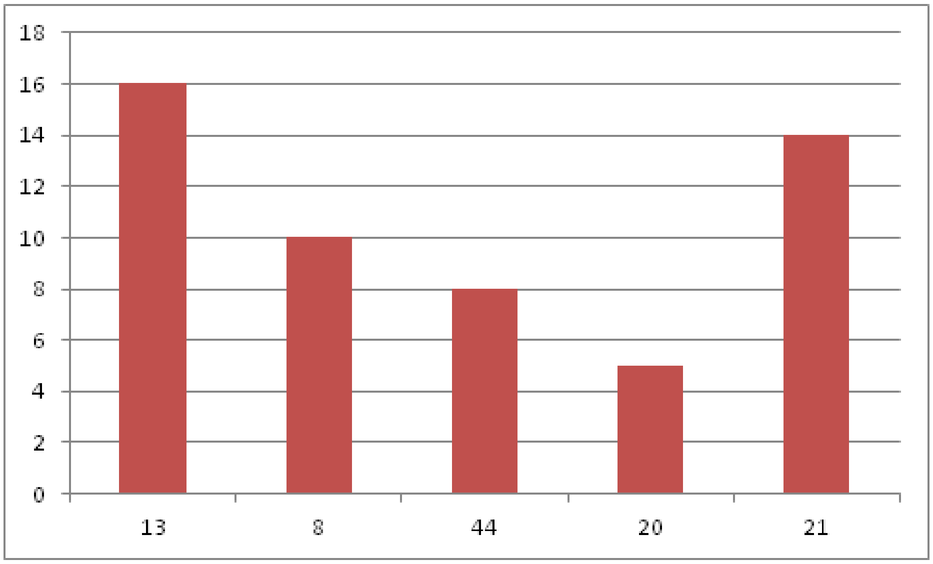
Click image to the left or use the URL below.

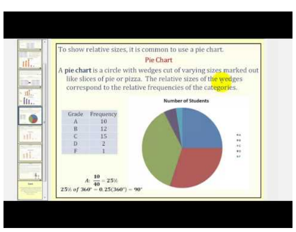
URL: http://www.ck12.org/flx/render/embeddedobject/190313

Play, Learn, and Explore with Pie Charts: Soda Party

2) A **bar chart** displays frequencies of categories of data. The bar chart below has 5 categories, and shows the TV channel preferences for 53 adults. The horizontal axis could also have been labeled *News, Sports, Local News, Comedy, Action Movies.* The bars in a bar graph do not touch, because the bars represent categories and not continuous numbers. For example, just because you split your time between channel 8 and channel 44 does not mean on average you watch channel 26. Categories can be numbers, so you need to be very careful.



The following video gives other examples of graphical displays, including pie charts and bar graphs:

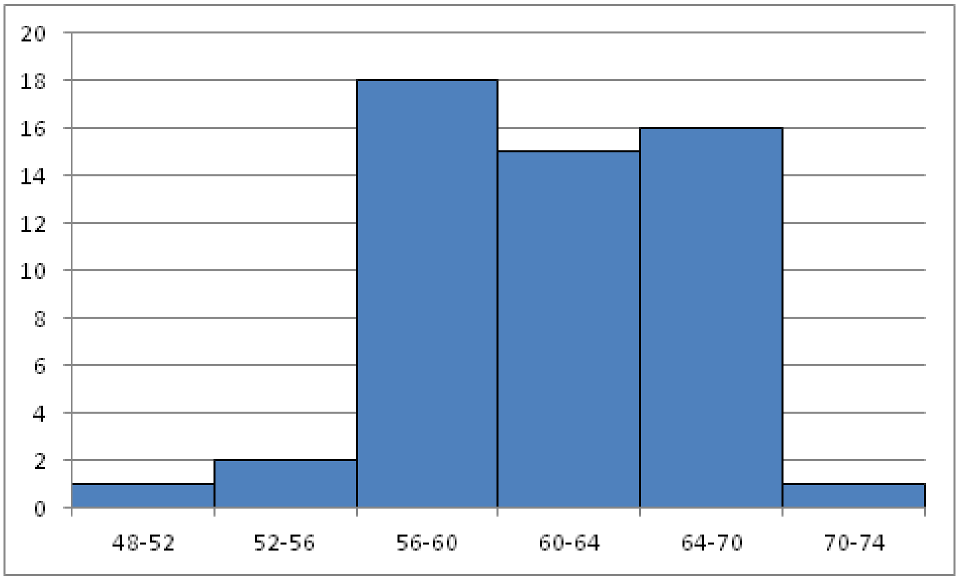


MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/196249

3) A **histogram** displays frequencies of quantitative data that have been sorted into intervals. Unlike the bar graph, the bars in a histogram do touch, because the bars represent categories of continuous numbers. Below is a histogram that shows the heights of the 53 students in a class. Notice that in this histogram, an interval ends at the same number that the next interval begins. For instance, 52 is the end of the 1st interval and the beginning of the 2nd interval. This overlapping number does not mean that the data point is counted in each interval. Instead, the 1st interval would include data points between $48 \leq x < 52$, and the 2nd interval would include data points between $52 \leq x < 56$. In other words, 52 is the upper limit of the 1st interval, but the data point itself is included in the interval in which it is the lower limit. Also, note that in this histogram, the largest category is 56-60 inches, with 18 people.

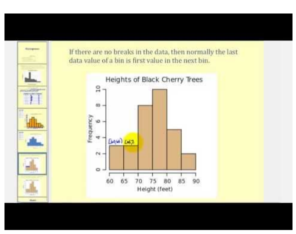


If you wanted to create a frequency table from this histogram, you would need to tally the number of people whose height falls into each interval. For the example above, you would get the following frequency table:

TABLE 14.9:

Interval	**Frequency**
48-52	1
52-56	2
56-60	18
60-64	15
64-70	16
70-74	1

The following video goes into more depth on defining, interpreting, and creating a histogram from data:



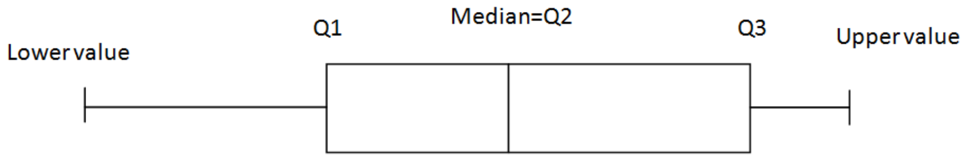
MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/190315

Play, Learn, and Explore with Histograms: Weights of Students

4) A **boxplot** (also known as a **box-and-whiskers plot**) is another way to display quantitative data. It displays the five-number summary (data minimum, $Q1$, median of the data, $Q3$, and data maximum). The box can either be vertically or horizontally displayed, depending on the labeling of the axis. The box does not need to be perfectly symmetrical because it represents data that might not be perfectly symmetrical.



Play, Learn, and Explore with Boxplot: Babies in a Waiting Room

Examples

Example 1

Create a pie chart to represent the preferences of 43 hungry students.

- Other - 5
- Burritos - 7
- Burgers - 9
- Pizza - 22

Solution:

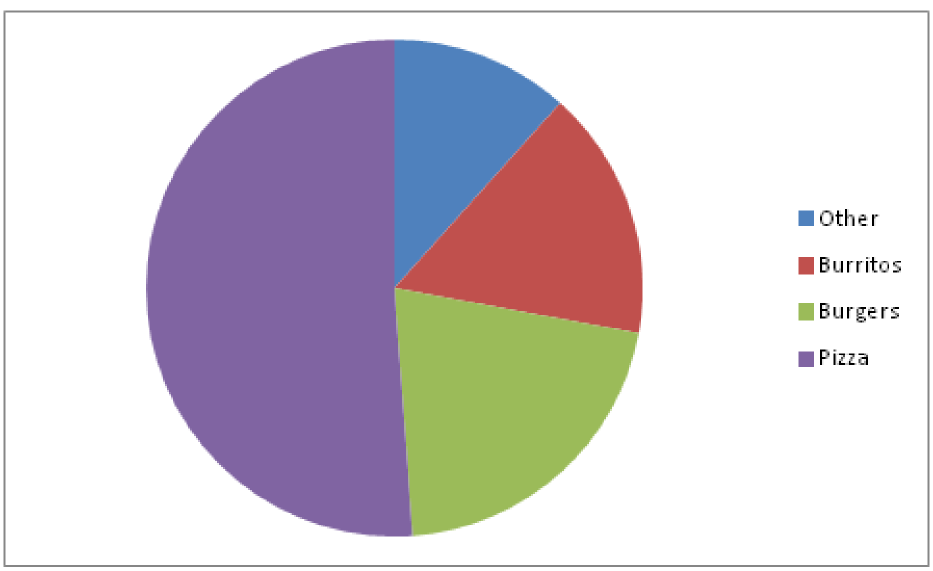
Since there are a total of 43 students, each category is going to be based out of 43. Divide the amount by 43 to find the percent of the circle that each segment represents, or multiply the fraction by 360 to get the central angle.

For Other: $\frac{5}{43} = 0.116279... \approx 11.6\%$ or $\frac{5}{43} \times 360 \approx 41.86^\circ$

For Burritos: $\frac{7}{43} = 0.16279... \approx 16.3\%$ or $\frac{7}{43} \times 360 \approx 58.60^\circ$

For Burgers: $\frac{9}{43} = 0.2093... \approx 20.9\%$ or $\frac{9}{43} \times 360 \approx 75.35^\circ$

For Pizza: $\frac{22}{43} = 0.5116279... \approx 51.2\%$ or $\frac{22}{43} \times 360 \approx 184.19^\circ$



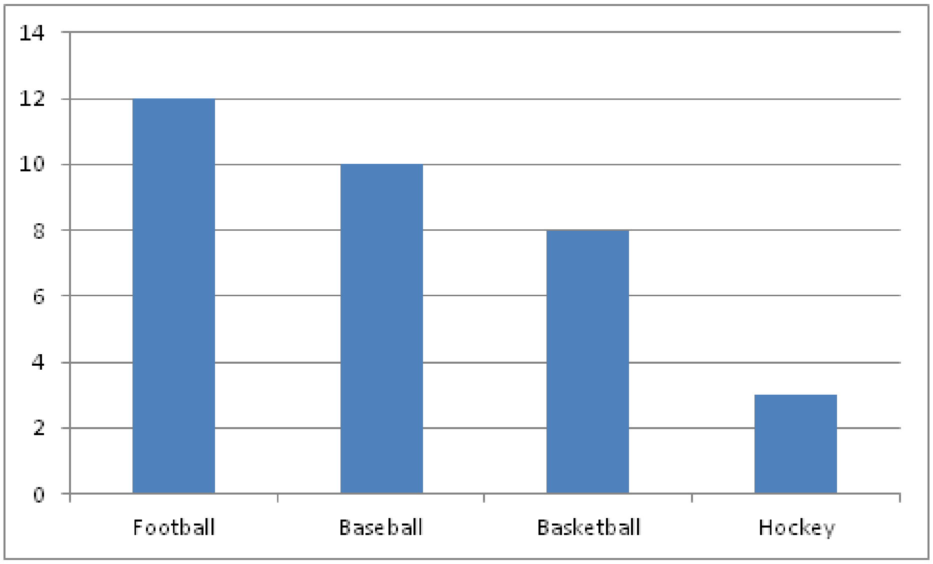
Example 2

Create a bar chart representing the preference for sports of a group of 33 people.

- Football - 12
- Baseball - 10
- Basketball - 8
- Hockey - 3

Solution:

Create a bar for each sport. The height should match the value for that category.



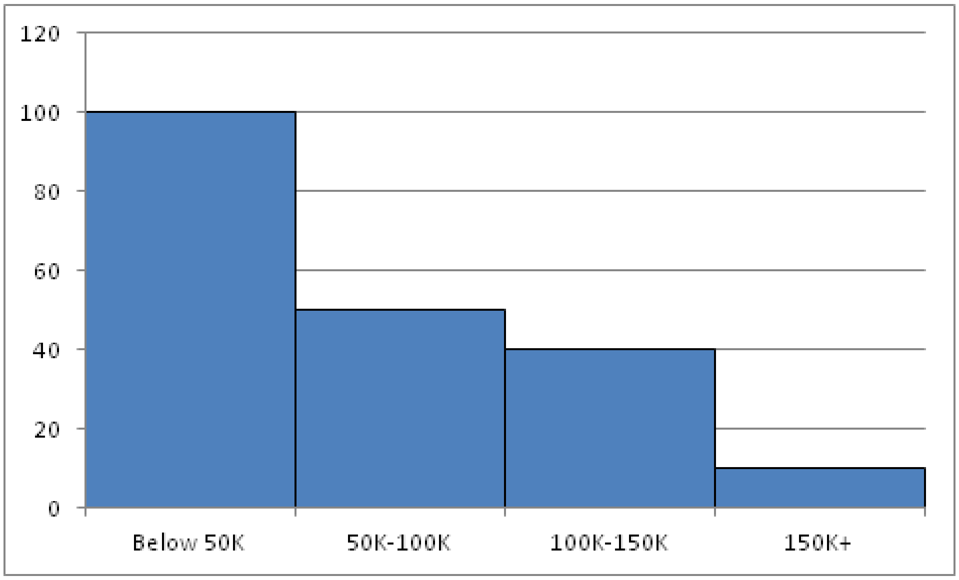
Example 3

Create a histogram for the income distribution of 200 million people.

- Below $50,000 is 100 million people
- Between $50,000 and $100,000 is 50 million people
- Between $100,000 and $150,000 is 40 million people
- Above $150,000 is 10 million people

Solution:

Create a bar for each interval. The height should match the number of people (in millions) for that interval.



Example 4

Recall the problem from the Introduction: Dr. Alameda created a frequency table for the exam scores for her computer science class, but needs a better way to display the data. How can she do this?

Solution:

Dr. Alameda has created a frequency table representing the scores on the computer science exam for all her classes. The frequency table is difficult to read and interpret, so Dr. Alameda wants to create a graphical display.

TABLE 14.10: Computer Science Test Scores

Bins: Test Scores	Frequency
0-50	17
51-60	9
61-70	35
71-80	81
81-90	123
91-100	81

Dr. Alameda can make a histogram to represent the data. This is the best method for graphically illustrating frequency data.

To make a histogram, first draw the horizontal (x) and vertical (y) axes.

Next, label the horizontal axis. The horizontal axis lists the different categories of data. In this case, the category will be "Scores."

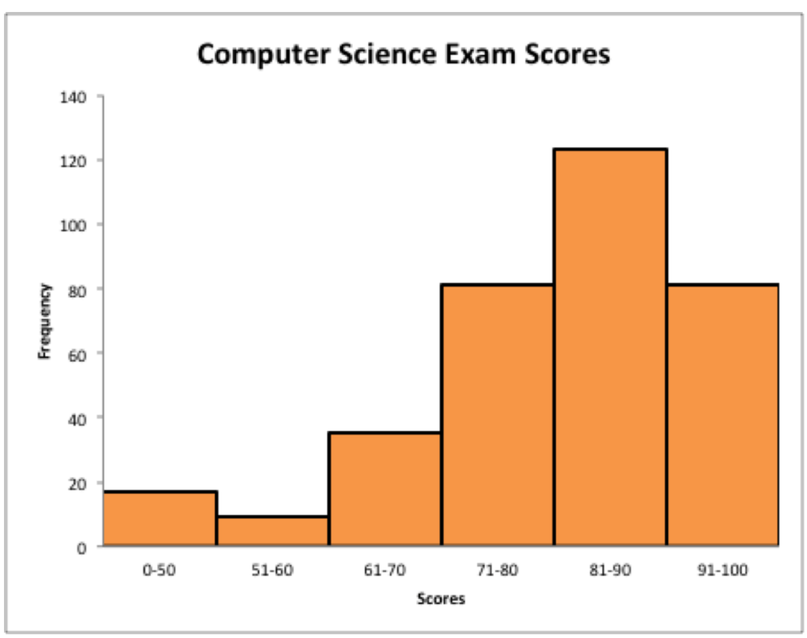
Next, label the vertical axis. The vertical axis lists the quantity or amount of the data. In this case, the category will be "Frequency."

Next, title the graph. The title of the graph should be short and clear. It should explain what data are presented in the graph. In this case, the title will be "Computer Science Exam Scores."

Then, determine the units on the vertical axis. To do this, start by reviewing the smallest and largest frequencies in the table. The smallest value is 9, and the largest is 123. Based on these values, label the vertical axis from 0 to 140. Since the range of the frequency values is large, the units should also be large. The vertical axis should use a unit of 20.

Next, draw the vertical columns. To do this, write each bin along the horizontal axis. Then draw each column vertically until it reaches the frequency for that score. For example, draw a vertical column to the number 17 for the bin 0-50. Do not leave spaces between the columns.

The answer is the graph should look like the one below.



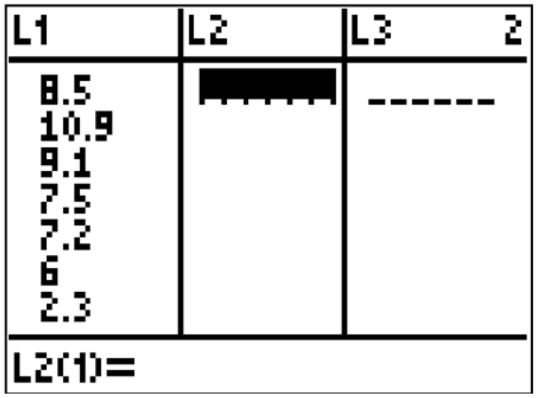
Example 5

Create a boxplot of the following numbers in your calculator:

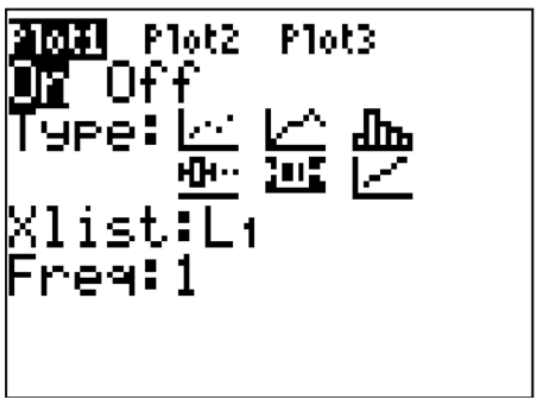
8.5, 10.9, 9.1, 7.5, 7.2, 6, 2.3, 5.5

Solution:

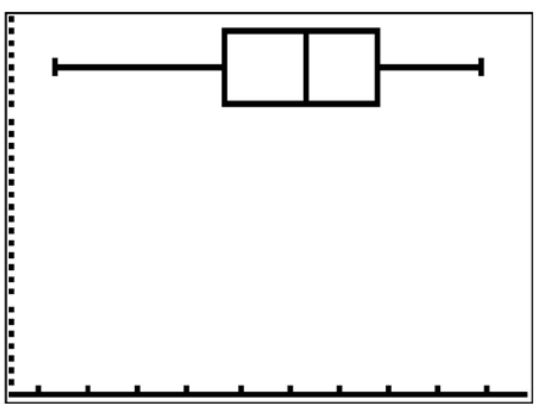
Enter the data into L_1 by going into the Stat menu.



Then, turn on the statplot and choose boxplot.

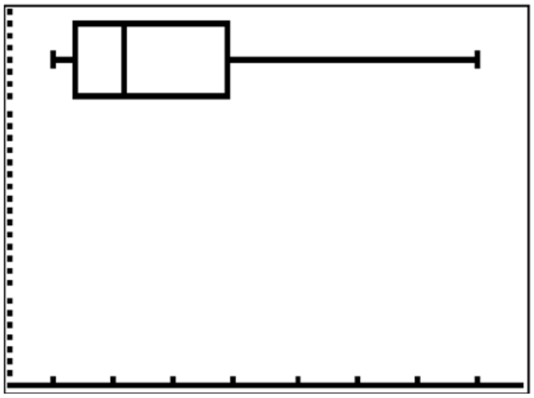


Use Zoomstat to automatically center the window on the boxplot.



Example 6

Identify the interesting characteristics of the following boxplot:



Solution:

The lower bound, $Q1$ and $Q2$, all seem to be relatively close together. $Q3$ seems to be stretched a little to the right, and the upper bound is significantly stretched to the right.

Summary

- A ***pie chart*** is a graphic display of categorical data where the relative size of each pie slice corresponds to the frequency of each category.
- A ***bar chart*** is a graphic display of categorical variables that uses bars to represent the frequency of the count in each category.
- A ***histogram*** is a graphic display of quantitative variables that uses bars to represent the frequency of the count of the data in each interval.
- A ***boxplot*** is a graphic display of quantitative data that demonstrates the five-number summary.

Review

A math class of 30 students had the following grades:

TABLE 14.11:

Grade	Number of Students with Grade
A	10
B	10
C	5
D	3
F	2

1. Create a bar chart for this data.

2. Create a pie chart for this data.

3. Which graph do you think makes a better visual representation of the data?

4. A simulation of a large number of runs of rolling three dice and adding the numbers results in the following five-number summary: 3, 8, 10.5, 13, 18. Make a box-and-whisker plot for the data.

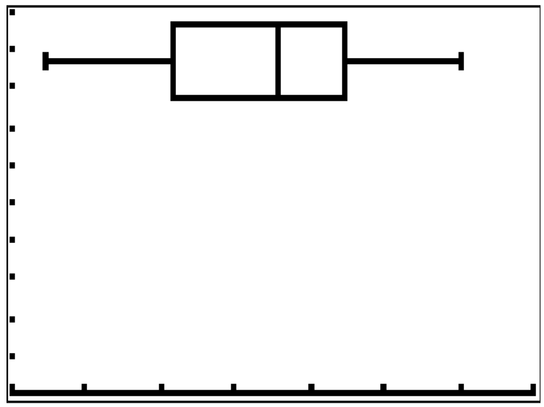
A set of 20 exam scores is 67, 94, 88, 76, 85, 93, 55, 87, 80, 81, 80, 61, 90, 84, 75, 93, 75, 68, 100, 98

5. Create a histogram for this data. Use your best judgment to decide what the intervals should be.

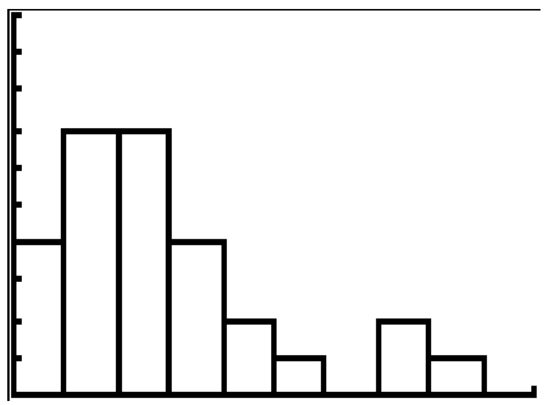
6. Find the five-number summary for this data.

7. Use the five-number summary to create a boxplot for this data.

8. Describe the data shown in the boxplot below.



9. Describe the data shown in the histogram below.



A math class of 30 students has the following eye colors:

TABLE 14.12:

Eye Color	**Number of Students with Eye Color**
Brown	20
Blue	5
Green	3
Other	2

10. Create a bar chart for this data.

11. Create a pie chart for this data.

12. Which graph do you think makes a better visual representation of the data?

13. Suppose you have data that show the breakdown of registered Republicans by state. What types of graphs could you use to display this data?

14. From which types of graphs could you obtain information about the spread of the data? Note that spread is a measure of how spread out all of the data are.

15. The data collected depict the number of hours 12 families traveled this summer to their vacation destination. Create a frequency table to display the data.

7 3 10 5 12 9 8 4 3 11 3 9

16. Create a histogram to display the data.

17. Write a few sentences to explain any conclusions you can draw from the data.

18. Generate a question that you will use to survey 20 people. Which graphic display is the most appropriate for the data?

Review (Answers)

Please see the Appendix.

14.7 Mean, Median, and Mode

Learning Objectives

Learn to calculate three measures of the center of univariate data and decide which measure is best based on context.

Introduction

An amusement park is designing a new section for children over 3 years old and under 8 years old. As part of the park's research, it used a survey of the heights and weights of a thousand children in that age group. Which measure of central tendency should it use to accommodate the greatest number of children on a roller coaster?

Mean, Median, and Mode

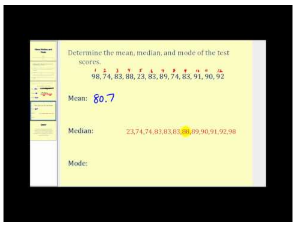
With descriptive statistics, your goal is to describe the data you find in a sample or given in a problem. Because it would not make sense to present your findings as long lists of numbers, you summarize important aspects of the data. One important aspect of the data is the **measure of central tendency**, which is a measure of the "middle" value of a set of data. A measure of central tendency is helpful if we want to summarize a set of data or to compare different sets of data. For example, suppose a restaurant manager wants to know what dish on the menu is the most popular. Alternatively, suppose a coach wants to know how fast a sprinter can run a given distance. Moreover, a real estate agent may want to know the price of houses in a certain area. There are three ways to measure central tendency:

1. Use the **mean**, which is the arithmetic average of the data.

2. Use the **median**, which is the number exactly in the middle of the data. When the data have an odd number of counts, the median is the middle number after the data have been ordered. When the data have an even number of counts, the median is the arithmetic average of the two most central numbers.
3. Use the **mode**, which is the most often occurring number in the data. If two or more numbers occur equally frequently, then the data are said to be bimodal or multimodal.

Calculating the mean, median, and mode is straightforward. What is challenging is determining when to use each measure, and knowing how to interpret the data using the relationships between the three measures.

The following video explains how to determine the mean, median, and mode:



MEDIA
Click image to the left or use the URL below.
URL: http://www.ck12.org/flx/render/embeddedobject/169058

Play, Learn, and Explore with Mean, Median, and Mode: Trees

Examples

Example 1

Given different situations and datasets, how do we decide which measure to use?

Solution:

To decide which measure of central tendency to use, it is a good idea to calculate and interpret all three of the numbers.

For example, if someone asked you how many people can sit in the typical car, it would make more sense to use mode than to use mean. With mode, you can find out that a 5-person car is the most frequent car driven, and determine that the answer to the question is 5. If you calculate the mean for the number of seats in all cars, you will end up with a decimal like 5.4, which makes less sense in this context.

On the other hand, if you were finding the central heights of professional basketball players, using mean might make a lot more sense than mode.

Example 2

Compute the mean, median, and mode for the following numbers:

3, 5, 1, 6, 8, 4, 5, 2, 7, 8, 4, 2, 1, 3, 4, 6, 7, 9, 4, 3, 2

Solution:

Mean: The sum of all these numbers is 94, and there are 21 numbers total, so the mean is $\frac{94}{21} \approx 4.4762$.

Median: When you order the numbers from least to greatest, you get

1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9.

The 11th number has 10 numbers to the right and 10 numbers to the left, so it is the median. The median is the number 4.

Mode: The most frequently occurring number is the number 4.

Example 3

You write a computer code to produce a random number between 0 and 10 with equal probability. Unfortunately, you suspect your code doesn't work perfectly because in your first few attempts at running the code, it produces the following numbers:

1, 9, 1, 1, 9, 2, 9, 1, 9, 9, 9, 2, 2.

How would you argue using mean, median, or mode that this code is probably not producing a random number between 0 and 10 with equal probability?

Solution:

This question is very similar to questions you will see when you study statistical inference.

First, you would note that the mean of the data is 4.9231. If the data were truly random, then the mean would probably be right around the number 5, which it is. This is not strong evidence to suggest that the random number generating code is broken.

Next, you would note that the median of the data is 2. This should make you suspect that something is wrong. You would expect that the median is of random numbers between 0 and 10 to be somewhere around 5.

Lastly, you would note that the mode of the data is 9. By itself, this is not strong data to suggest anything. Every sample will have to have at least one mode. What should make you suspicious, however, is the fact that only two other numbers were produced and were almost as frequent as the number 9. You would expect a greater variety of numbers to be produced.

Example 4

Recall the problem from the Introduction: Which measure of central tendency should the park use to accommodate the greatest number of children on a roller coaster?

Solution:

To attract more customers, the amusement park should accommodate as many children as possible. For this reason, it should use mode to determine the most common height and weight. However, knowing the most common height and weight may still not accommodate the greatest number of children, so it should also consider the mean to determine the average height and weight.

Example 5

Ross and his friends want to play basketball. They decide to choose teams based on the number of cousins everyone has. One will be the team with fewer cousins, and the other will be the team with more cousins. Should they use the mean, median, or mode to compute the cutoff number that will separate the two teams?

Solution:

Ross and his friends should use the median number of cousins as the cutoff number, because this will allow each team to have the same number of players. If there is an odd number of people playing, then the extra person will just join either team or switch in later.

Example 6

Compute the mean, median, and mode for the following numbers:

1, 4, 5, 7, 6, 8, 0, 3, 2, 2, 3, 4, 6, 5, 7, 8, 9, 0, 6, 5, 3, 1, 2, 4, 5, 6, 7, 8, 8, 8, 4, 3, 2.

Solution:

The mean is 4.6061. The median is 5. The mode is 8.

Example 7

The cost of fresh blueberries at different times of the year are

$2.50, $2.99, $3.20, $3.99, $4.99.

If you bought blueberries regularly, what would you typically pay?

Solution:

The word "typically" is used instead of "average" to allow you to choose whether mean, median, or mode would make the most sense. In this case, mean does make the most sense. The average cost is $3.53.

Summary

- The **mean** is the arithmetic average of the data.
- The **median** is the number in the middle of a dataset. When the data have an odd number of counts, the median is the middle number after the data have been ordered. When the data have an even number of counts, the median is the average of the two most central numbers.
- The **mode** is the most often occurring number in the data. If two or more numbers occur equally frequently, then the data are said to be **bimodal** or **multimodal**.

Review

You surveyed the students in your English class to find out how many siblings each student has. Here are your results:

0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 10, 12.

1. Find the mean, median, and mode of this data.

2. Why does it make sense that the mean number of siblings is greater than the median number of siblings?

3. Which measure of central tendency do you think is best for describing the typical number of siblings?

4. So far in math this semester, you have taken 10 quizzes. The mean of the scores is 88.5. What is the sum of the scores?

5. Find x if 5, 9, 11, 12, 13, 14, 16, and x have a mean of 12.

6. Meera drove an average of 22 miles a day last week. How many miles did she drive last week?

7. Find x if 2, 6, 9, 8, 4, 5, 8, 1, 4, and x have a median of 5.

Calculate the mean, median, and mode for each set of numbers:

8. 11, 15, 19, 12, 21, 34, 15, 28, 24, 15, 27, 19, 20, 13, 15

9. 3, 5, 7, 5, 5, 17, 8, 9, 11, 5, 3, 7

10. -3, 0, 5, 8, 12, 4, 2, 1, 6

Calculate the mean and median for each set of numbers:

11. 12, 88, 89, 90

12. 16, 17, 19, 20, 20, 98

13. For which of the previous two questions was the median **less than** the mean? What in the set of numbers caused this?

14. For which of the previous questions (11 or 12) was the median **greater than** the mean? What in the set of numbers caused this?

15. In each of the sets of numbers for problems 11 and 12, one number could be considered an **outlier**. Which numbers do you think are the outliers and why? What would happen to the mean and median if you removed the outliers?

Review (Answers)

Please see the Appendix.

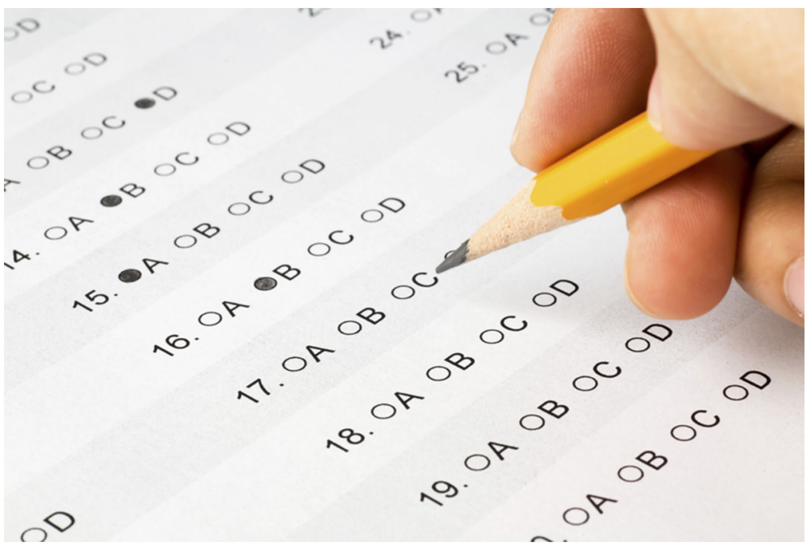
14.8 Five-Number Summary

Learning Objectives

Learn to calculate quartiles and produce five number summaries for data sets.

Introduction

When given a long list of numbers, it is useful to summarize the data. One way to summarize the data is to give the lowest number, the highest number, and the middle number. In addition to these three numbers, it is also useful to give the median of the lower half of the data, and the median of the upper half of the data. These five numbers give a very concise summary of the data.



What is the five-number summary of the following test scores?

0, 0, 1, 2, 63, 61, 27, 13

The Five-Number Summary

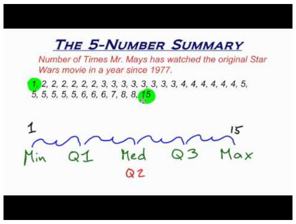
Suppose you have ordered data with m observations. The rank of each observation is shown by its index.

$$y_1 \leq y_2 \leq y_3 \leq \cdots \leq y_m$$

In datasets that are large enough, the data can be divided into four equal groups comprising a quarter of the data by three distinct points called **quartiles**. These three quartiles are the 1st quartile, $Q1$, the 2nd quartile, $Q2$, and the 3rd quartile, $Q3$. The 2nd quartile, $Q2$, is defined to be the median of the data. The 1st quartile, $Q1$, is defined to be the median of the lower half of the data, which are all the values less than the value of $Q2$. The 3rd quartile, $Q3$, is similarly defined to be the median of the upper half of the data, which are all the values greater than the value of $Q2$.

These three numbers in addition to the minimum and maximum values are the five-number summary. Note there are variations of the five-number summary, which you can study in a statistics course.

The following video explains how to determine the five-number summary of a set of data:



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/62539

Examples

Example 1

Calculate the five-number summary for the ages of grandchildren in a family: 2, 7, 17, 19, 25, 26, 26, 32.

Solution:

There are 8 observations total.

- Lowest value (minimum) : 2
- $Q1 : \frac{7+17}{2} = 12$ *(Note this is the median of the first half of the data: 2, 7, 17, 19.)*
- $Q2 : \frac{19+25}{2} = 22$ *(Note this is the median of the full set of data.)*
- $Q3 : 26$ *(Note this is the median of the second half of the data: 25, 26, 26, 32.)*
- Upper value (maximum): 32

Example 2

Calculate the five-number summary for the following numbers called during a bingo game: 4, 8, 11, 11, 12, 14, 16, 20, 21, 25.

Solution:

There are 10 observations total.

- Lowest value (minimum) : 4
- $Q1$: 11 *(Note this is the median of the first half of the data: 4, 8, 11, 11, 12.)*
- $Q2 : \frac{12+14}{2} = 13$ *(Note this is the median of the full set of data.)*
- $Q3$: 20 *(Note this is the median of the second half of the data: 14, 16, 20, 21, 25.)*
- Upper value (maximum): 25

Example 3

Calculate the five-number summary for the following data: 3, 7, 10, 14, 19, 19, 23, 27, 29.

Solution:

There are 9 observations total. To calculate $Q1$ and $Q3$, include the median in both the lower-half and upper-half calculations.

- Lowest value (minimum): 3
- $Q1$: 10 *(This is the median of 3, 7, 10, 14, 19.)*
- $Q2$: 19
- $Q3$: 23 *(This is the median of 19, 19, 23, 27, 29.)*
- Upper value (maximum): 29

Example 4

Recall the problem from the Introduction: What is the five-number summary of the following test scores?

0, 0, 1, 2, 63, 61, 27, 13

Solution:

To calculate the five-number summary, it helps to order the data.

0, 0, 1, 2, 13, 27, 61, 63

- Since there are 8 observations, the median is the average of the 4th and 5th observations: $\frac{2+13}{2} = 7.5$
- The lowest observation is 0.
- The highest observation is 63.

- The middle of the lower half is $\frac{0+1}{2} = 0.5$
- The middle of the upper half is $\frac{27+61}{2} = 44$.

The five-number summary is 0, 0.5, 7.5, 44, 63.

Example 5

Create a set of data that meets the following five-number summary: $\{2, 5, 9, 18, 20\}$.

Solution:

Suppose there are 8 data points. The lowest point must be 2, and the highest point must be 20. The middle two points must average to be 9, so they could be 8 and 10. The 2nd and 3rd points must average to be 5, so they could be 4 and 6. The 6th and 7th points need to average to be 18, so they could be 18 and 18. Here is one possible answer:

2, 4, 6, 8, 10, 18, 18, 20

Example 6

Calculate the five-number summary for the following data: 1, 1, 1, 2, 2, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7, 8, 9, 10, 15.

Solution:

There are 20 observations.

- Lower : 1
- $Q1 : \frac{2+3}{2} = 2.5$
- $Q2 : \frac{4+5}{2} = 4.5$
- $Q3 : \frac{6+7}{2} = 6.5$
- Upper: 15

Example 7

Calculate the five-number summary for the following data:

1, 4, 96, 356, 2557, 9881, 14420, 20100.

Solution:

There are 8 observations.

- Lower : 1
- $Q1 : \frac{4+96}{2} = 50$
- $Q2 : \frac{356+2557}{2} = 1456.5$
- $Q3 : \frac{9881+14420}{2} = 12150.5$
- Upper: 20100

Summary

- The **rank** of an observation is the number of observations that are less than or equal to the value of that observation.
- The greatest value in the 1**st quartile** is called $Q1$, in the **2nd quartile,** $Q2$**,** and the **3rd quartile,** $(Q3)$**.** The **2nd quartile** is also known as the median.

Review

Calculate the five-number summary for each of the following sets of data:

1. 0.16, 0.08, 0.27, 0.20, 0.22, 0.32, 0.25, 0.18, 0.28, 0.27
2. 77, 79, 80, 86, 87, 87, 94, 99
3. 79, 53, 82, 91, 87, 98, 80, 93
4. 91, 85, 76, 86, 96, 51, 68, 92, 85, 72, 66, 88, 93, 82, 84
5. 335, 233, 185, 392, 235, 518, 281, 208, 318
6. 38, 33, 41, 37, 54, 39, 38, 71, 49, 48, 42, 38
7. 3, 7, 8, 5, 12, 14, 21, 13, 18
8. 6, 22, 11, 25, 16, 26, 28, 37, 37, 38, 33, 40, 34, 39, 23, 11, 48, 49, 8, 26, 18, 17, 27, 14
9. 9, 10, 12, 13, 10, 14, 8, 10, 12, 6, 8, 11, 12, 12, 9, 11, 10, 15, 10, 8, 8, 12, 10, 14, 10, 9, 7, 5, 11, 15, 8, 9, 17, 12, 12, 13, 7, 14, 6, 17, 11, 15, 10, 13, 9, 7, 12, 13, 10, 12
10. 49, 57, 53, 54, 49, 67, 51, 57, 56, 59, 57, 50, 49, 52, 53, 50, 58
11. 18, 20, 24, 21, 5, 23, 19, 22
12. 900, 840, 880, 880, 800, 860, 720, 720, 620, 860, 970, 950, 890, 810, 810, 820, 800, 770, 850, 740, 900, 1070, 930, 850, 950, 980, 980, 880, 960, 940, 960, 940, 880, 800, 850, 880, 760, 740, 750, 760, 890, 840, 780, 810, 760, 810, 790, 810, 820, 850
13. 13, 15, 19, 14, 26, 17, 12, 42, 18
14. 25, 33, 55, 32, 17, 19, 15, 18, 21
15. 149, 123, 126, 122, 129, 120

Review (Answers)

Please see the Appendix.

14.9 Variance

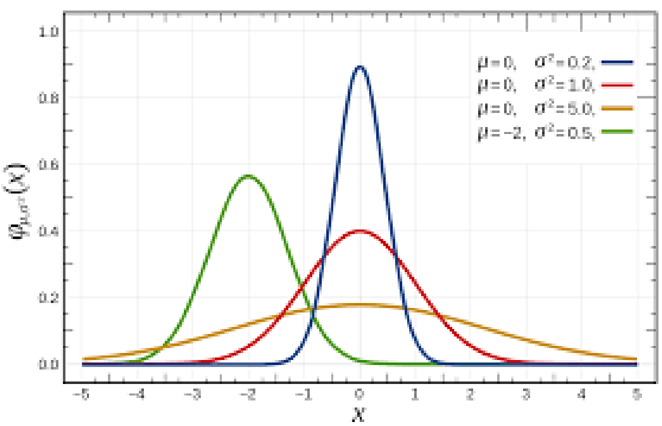
Learning Objectives

Learn to calculate population variance, sample variance, and standard deviation from univariate data.

Introduction

Two groups of students, each with an average test score of 75, might have a score distribution that looks remarkably different. One group might be made up entirely of grades between 72 and 78, while the other group may have half the students around 50, with the other half near 100. Variance is a way of measuring the variation in a set of data. What are the mean and variance for the following sample test scores taken from a larger student population?

75, 73, 78, 90, 60, 51, 87, 79, 80, 77



Variance

The thought process of a person trying to describe the spread of some data for the 1st time must have been something like this:

Well, the average is 75. What if I try to just add up how different each number is from 75?

Calculating the numbers, the person realizes pretty quickly that this sum will be zero, essentially by definition. This is because the numbers that occur below 75 precisely cancel out with the numbers above 75.

Since I cannot add the differences directly, why don't I just sum the absolute value of the differences?

This is a legitimate method for describing the spread of the data. The sum of the absolute differences from the mean is the total distance of the data points from the point representing the mean of the set. It is called absolute deviation, and is simply the sum of the absolute values of each of the differences.

If I take the average absolute difference, I will be able to judge on average how far away each data point is from the mean. A larger difference means more spread out.

If you take the average of the absolute deviation, you get the mean absolute deviation. The mean absolute variation is a legitimate, but limited, way of describing the spread of data. Eventually, a person trying to describe the spread of data for the 1st time might consider a method called population variance.

What if instead of using absolute value to solve the issue, I square each difference and then add them together? Of course I'd have to divide by the number of data points to get the average difference squared.

This method turns out to be extraordinarily powerful in statistics. One downside is that most of the time you cannot get data from the entire population; you usually only get it from a sample. Over time, people realized that samples were typically less variable than their populations, and dividing by the number of data points was consistently underestimating the true variance of the population. In other words, if n is the size of the sample, then multiplying the sum of the square differences by $\frac{1}{n}$ makes the variance too small. Research and theory progressed until it was realized that multiplying the sum of the square differences by $\frac{1}{n-1}$ made the fraction slightly larger and properly estimated the variance of the population. Thus, there are two ways to calculate variance, one for populations and one for samples.

Hey, wait, by squaring the differences, doesn't that mean the units are squared? What if I want to describe the spread in the regular units? Should I just take the square root of the variance?

This is why the Greek letter lowercase sigma, σ, is used for standard deviation of a population (which is the square root of the variance), and σ^2 is the symbol for variance of a population. The letters s and s^2 are used for sample standard deviation and sample variance. The Greek letter mu, μ, is the symbol used for mean of a population, while $\bar{x}$ is the symbol used for mean of a sample.

Mean and variance for the population: $x_1, x_2, x_3, \ldots, x_n$

$$\mu = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$$

$$\sigma^2 = \frac{1}{n} \cdot \sum_{i=1}^{n} (\mu - x_i)^2$$

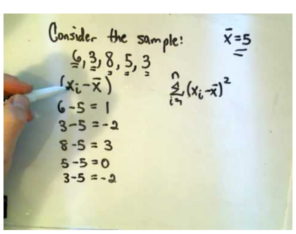
Mean and variance for a sample from a population: $x_1, x_2, x_3, \ldots, x_m$

$$\bar{x} = \frac{1}{m} \cdot \sum_{i=1}^{m} x_i$$

$$s^2 = \frac{1}{m-1} \cdot \sum_{i=1}^{m} (\bar{x} - x_i)^2$$

Variance is a measure of the spread of data. The bigger the variance, the more spread out the data points.

The following video explains how to find the variance of a set of data:



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/190317

Play, Learn, and Explore with Variance: Variance

Examples

Example 1

Calculate the variance and mean for rolling a fair, six-sided die.

Solution:

$$\mu = \frac{1}{6}(1+2+3+4+5+6) = \frac{1}{6} \cdot 21 = 3.5$$

Since the population for a six-sided die is entirely known, you would use the population variance.

$$\begin{aligned}\sigma^2 &= \frac{1}{6}\left[(3.5-1)^2+(3.5-2)^2+(3.5-3)^2+(3.5-4)^2+(3.5-5)^2+(3.5-6)^2\right]\\ &= \frac{1}{6}[6.25+2.25+0.25+0.25+2.25+6.25]\\ &\approx 2.9167\end{aligned}$$

Example 2

Calculate the mean and variance of the following data sample of lap times:

59.8, 57.1, 58.2, 58.6, 57.8, 57.9, 58.0, 57.3.

Solution:

$$\bar{x} = \frac{1}{8}(59.8+57.1+58.2+58.6+57.8+57.9+58.0+57.3) = 58.0875$$

This is a *sample*, so you should use the sample variance formula.

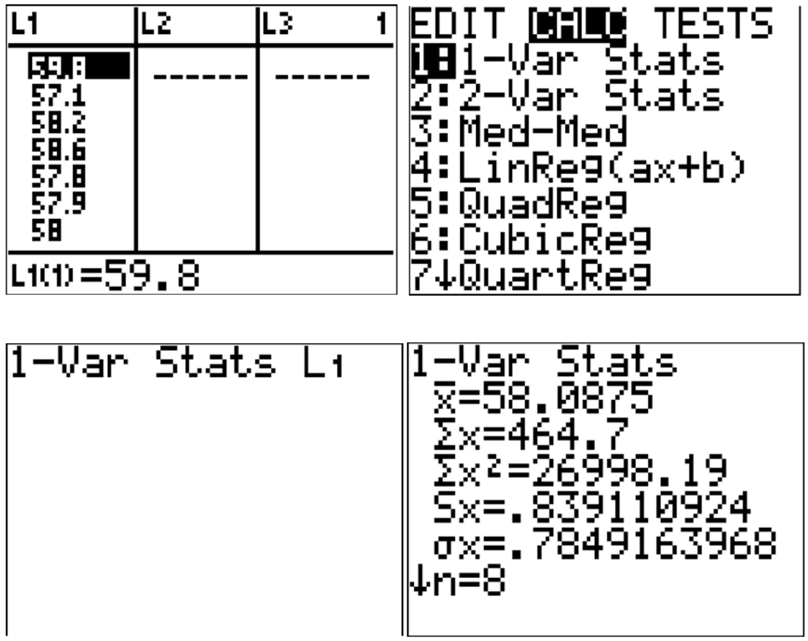
$$\begin{aligned}s^2 &= \frac{1}{8-1} \cdot \left[(\bar{x}-59.8)^2+(\bar{x}-57.1)^2+(\bar{x}-58.2)^2+(\bar{x}-58.6)^2+(\bar{x}-57.8)^2\right.\\ &\quad \left.+(\bar{x}-57.9)^2+(\bar{x}-58.0)^2+(\bar{x}-57.3)^2\right]\\ &= \frac{1}{7}\left[(-1.7125)^2+0.9875^2+(-0.1125)^2+(-0.5125)^2+0.2875^2+0.1875^2+0.0875^2+0.7875^2\right]\\ &\approx \frac{1}{7}[2.9327+0.9751+0.0126+0.2626+0.0826+0.0351+0.0076+0.6201]\\ &\approx \frac{1}{7}[4.9288]\\ &\approx 0.7041\end{aligned}$$

Example 3

Use a calculator to calculate the variance from Example 2.

Solution:

To calculate variance on a TI-83/84 calculator, enter the data in a list, choose 1-Var Stats, and run the 1-Var Stats on the list you entered the data.



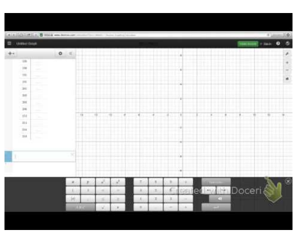
The two outputs that are important for you to interpret are:

$$Sx = 0.839110924 \quad \text{and} \quad \sigma x = 0.7848163968.$$

The calculator does not know whether the data is a population or a sample, so it produces both. Since this problem is about a sample, the number of interest is Sx. The calculator produces standard deviation. You need to square that number to produce the appropriate variance.

$$0.8391^2 \approx 0.7041$$

The following video explains how to calculate statistics in the free online calculator Desmos:



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/195515

Example 4

Recall the problem from the Introduction: What are the mean and variance for the following sample test scores taken from a larger student population?

75, 73, 78, 90, 60, 51, 87, 79, 80, 77

Solution:

The mean of the test scores is 75. The variance is calculated by taking the difference of each number from the mean, squaring, and summing these differences.

$$0^2+2^2+3^2+15^2+15^2+24^2+12^2+4^2+5^2+2^2=1{,}228$$

Since the data is a sample, you divide the sum by one fewer than the number of terms.

$$\frac{1228}{10-1}\approx 136.4444$$

If you knew the variances for two samples, each from a different group, you could quickly determine which group had test scores that were more spread out.

Example 5

Calculate the standard deviation for the following six numbers by hand. Assume the numbers are a population.

2, 4, 6, 8, 12, 19

Solution:

$$\begin{aligned}
\mu &= \frac{1}{6}(2+4+6+8+12+17)=8\\
\sigma^2 &= \frac{1}{6}((8-2)^2+(8-4)^2+(8-6)^2+0+(8-12)^2+(8-17)^2)\\
&= \frac{1}{6}(6^2+4^2+2^2+4^2+9^2)\\
&= \frac{1}{6}(36+16+4+16+81)\\
&= \frac{1}{6}(153)\\
&= 25.5\\
\sigma &\approx 5.0498
\end{aligned}$$

Example 6

Use an Excel spreadsheet to organize your calculations for computing the variance of the numbers below. Assume these numbers are a true population.

14, 15, 7, 15, 2, 0, 6, 5, 12, 3

Solution:

After entering the data in a column, you can use the power of the embedded programming of the spreadsheet to make a 2nd column of just the average.

- The average command is: "= average(A2:A11)"

You can subtract one cell from another cell to find the difference. You can then square the difference to find the difference squared. You can then sum these values using the sum command.

- The sum command is: "= sum(D2:D11)"

Finally, just divide the sum by the number of observations (which is 10) to get the variance.

	A	B	C	D	E	F	G	H
1	Data	Average	Difference	Difference squared		Sum of difference squared		
2	14	7.9	-6.1	37.21		288.9		
3	15	7.9	-7.1	50.41				
4	7	7.9	0.9	0.81				
5	15	7.9	-7.1	50.41		Variance		
6	2	7.9	5.9	34.81		28.89		
7	0	7.9	7.9	62.41				
8	6	7.9	1.9	3.61				
9	5	7.9	2.9	8.41				
10	12	7.9	-4.1	16.81				
11	3	7.9	4.9	24.01				

Summary

- **Variance** is a measure of how spread out the data are.
- The square root of the variance is the **standard deviation**.
- Both the variance and the standard deviation can be calculated from a **sample** or from the whole **population**. The formulas are slightly different in each case, so it is important to know whether your data is just a sample or is from the whole population.
- The **absolute deviation** is the sum total of how different each number is from the mean.
- The **mean absolute deviation** is an alternate measure of how spread out the data are. While this method might seem more intuitive, in statistics it has been found to be too limited and is not commonly used.
- **Mean and variance for the population:** $x_1, x_2, x_3, \dots, x_n$
-

$$\mu = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$$

$$\sigma^2 = \frac{1}{n} \cdot \sum_{i=1}^{n} (\mu - x_i)^2$$

Mean and variance for a sample from a population: $x_1, x_2, x_3, \dots, x_m$

$$\overline{x} = \frac{1}{m} \cdot \sum_{i=1}^{m} x_i$$

$$s^2 = \frac{1}{m-1} \cdot \sum_{i=1}^{m} (\overline{x} - x_i)^2$$

Review

1. What are the similarities and differences between standard deviation and variance?

2. Dataset A has a mean of 30 and a standard deviation of 10. Dataset B also has a mean of 30, but a standard deviation of 2. What does this mean about Dataset A compared to Dataset B?

Calculate the variance of each set of data by hand:

3. Sample: 1, 4, 7, 10, 3, 6, 12, 5, 8, 16, 21, 3, 1, 5

4. Population: 23, 27, 19, 24, 20, 22, 31, 30, 28

5. Sample: 64, 62, 60, 58, 54, 60, 61, 63, 47, 100, 29, 59

Calculate the variance of each set of data using your calculator. Compare your answers to your answers to 3-5.

6. Sample: 1, 4, 7, 10, 3, 6, 12, 5, 8, 16, 21, 3, 1, 5

7. Population: 23, 27, 19, 24, 20, 22, 31, 30, 28

8. Sample: 64, 62, 60, 58, 54, 60, 61, 63, 47, 100, 29, 59

9. If $\sigma^2 = 16$, what is the population standard deviation?

10. Which dataset has the largest standard deviation?

1. 10 10 10 10 10
2. 0 0 10 10 10
3. 0 9 10 11 20
4. 20 20 20 20 20

11. What will a large variance look like on a histogram? What will a small variance look like on a histogram?

12. You find some data organized in a bar graph. Could you calculate the variance of this data? Explain.

13. A sample set of 20 exam scores is 67, 94, 88, 76, 85, 93, 55, 87, 80, 81, 80, 61, 90, 84, 75, 93, 75, 68, 100, 98. Calculate the mean, variance, and standard deviation for this data.

14. All of Mike's game scores are 1, 1, 2, 10, 12, 1, 9, 6, 7, 8, 4, 3, 4, 1, 4, 1, 6, 7, 11, 5. Calculate the mean, variance, and standard deviation for this data.

15. Why can't you always calculate the population variance and standard deviation? Why do you sometimes have to calculate the sample variance and standard deviation?

Review (Answers)

Please see the Appendix.

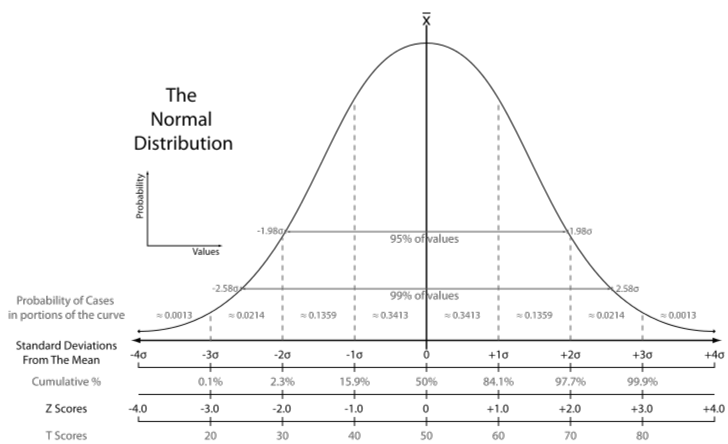
14.10 The Normal Curve

Learning Objectives

Learn the definition of the standard normal distribution and learn how standard deviation and the area under the curve are connected.

Introduction

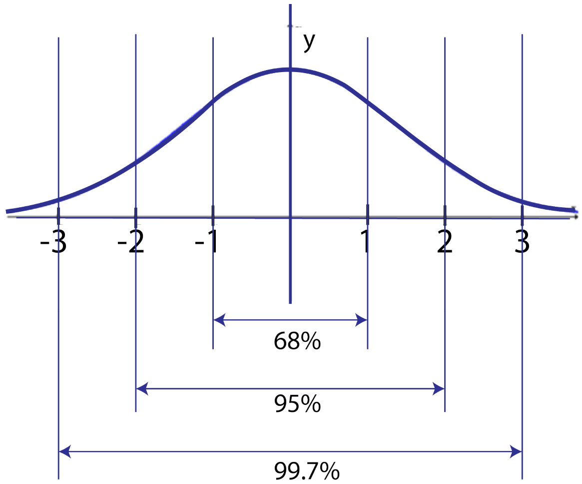
When students ask their teachers to curve exams, what they often mean is they want everyone to simply get a higher grade. Curving a grade can also mean fitting to a bell curve, where lots of people get Cs, some people get Ds and Bs, and very few people get As and Fs. Even though this 2nd interpretation is not what most students mean, the normal curve is one of the most widely used and applied probability distributions. What other examples follow a normal distribution?



Standard Normal Distribution

The **Standard Normal Distribution** is graphed from the following function and is represented by the Greek letter phi, ϕ.

$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$



This distribution represents a population with a mean of 0 and a standard deviation of 1. The numbers along the x-axis represent standard deviations. For data that are normally distributed, the **empirical rule** states that:

- Approximately 68% of the data will be within 1 standard deviation of the mean.
- Approximately 95% of the data will be within 2 standard deviations of the mean.
- Approximately 99.7% of the data will be within 3 standard deviations of the mean.

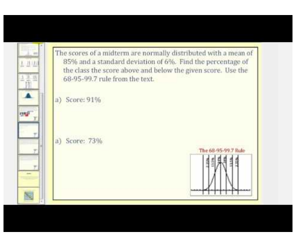
Some other important points about the normal distribution:

- The total area between the normal curve and the x-axis is 1, and this area represents all possible probabilities.
- If data are distributed normally, you can use the normal distribution to determine the percentage of the data between any two values by calculating the area under the curve between those two values.
- Many histograms approximate a normal curve, but a true normal curve is infinitely smooth.

On a TI calculator, the **normalcdf** command calculates the normal cumulative distribution function, which is the area between any two values for data that is normally distributed, as long as you know the mean and standard deviation for the data. Your calculator has this function built in, and it produces an exact answer as opposed to the empirical rule. When using normalcdf, four numbers are needed to be inputted: **normalcdf**(x, y, μ, σ). The 1st number is the lower boundary of the interval, the 2nd number is the upper boundary of the interval, the 3rd number is the mean, and the 4th number is the standard deviation.

There is a 2nd programmed feature, **invNorm**, in the distribution menu that performs the inverse of normalcdf. Instead of being given the standard deviation and asked to find the probability, you are given the probability and asked to find the standard deviation.

The following video introduces the standard normal distribution:

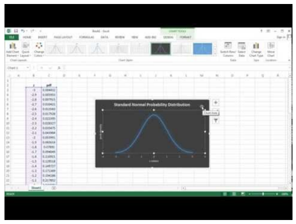


MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/190318

The following video explains how to draw a normal distribution using Microsoft Excel:



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/195534

Play, Learn, and Explore with The Normal Curve using the topic of food safety: www.ck12.org/a/2516900 .

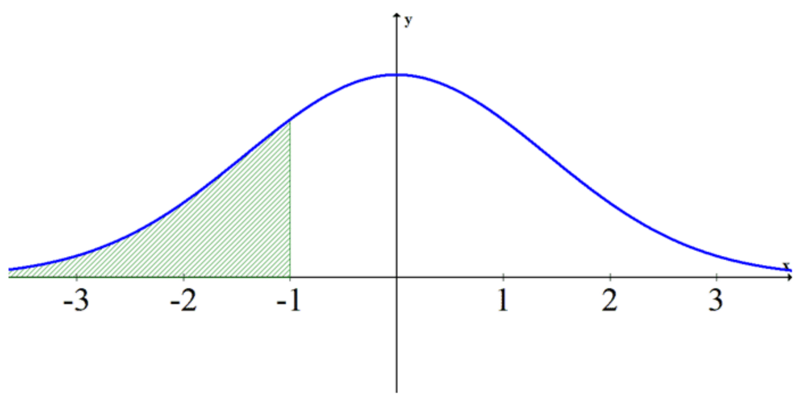
Examples

Example 1

The amount of rain each year in Connecticut follows a normal distribution. What is the probability of getting more than 1 standard deviation below the normal amount of rain?

Solution:

You are looking for the area of the shaded portion of the normal distribution shown below. By the empirical rule, you know that approximately 34% of the data is in between -1 and 0 standard deviations from the mean. Also, 50% of the data is above 0 standard deviations from the mean. Therefore, approximately 84% of the data is unshaded. Therefore, $100\% - 84\% = 16\%$ of the data is shaded. The approximate probability is 16%.

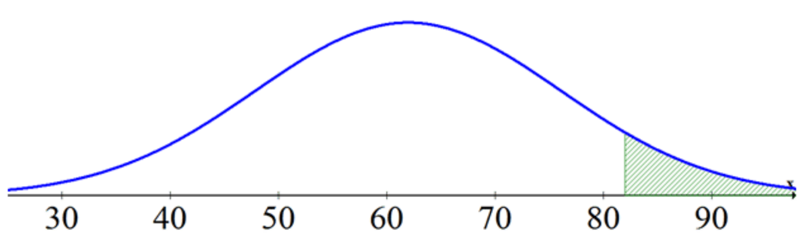


Example 2

On your 1st college exam, you score an 82. After the exam, the professor tells the class that the mean was a 62, and the standard deviation was 10. Assuming the data are normally distributed, what percentage of the class did better than you?

Solution:

An 82 is 20 away from the mean, so is 2 standard deviations from the mean. Therefore, this question is asking for the percentage of students that are above +2 standard deviations above the mean.



Either you can use the fact that your score was exactly 2 standard deviations above the mean, or you can calculate the probability using the actual numbers.

- normalcdf(2, 1E99, 0, 1) = 0.022750 or 2.275%
- normalcdf(82, 1E99, 62, 10) = 0.022750 or 2.275%

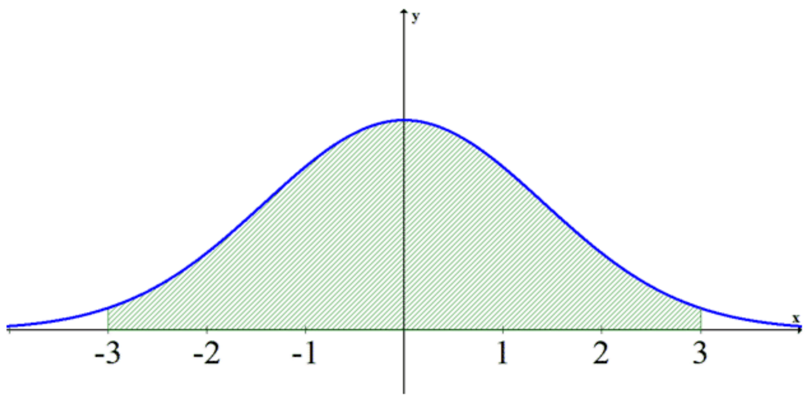
2.275% of the class did better than you on the exam.

Example 3

The quality control technician of a widget-making factory observes that widgets that are 3 standard deviations too large or 3 standard deviations too small from the precise widget size are unusable. Assuming the data are normally distributed, what is the probability of producing a usable widget?

Solution:

This question is essentially asking for the area between -3 standard deviations and +3 standard deviations. The empirical rule says this should be 99.7%. Use the normalcdf function to find the exact value.



normalcdf(-3, 3, 0, 1) = 0.997300 or 99.73%

The quality control technician would decide if this is a high enough success rate for producing a usable widget.

Example 4

Recall the question from the Introduction: What other examples follow a normal distribution?

Solution:

Height, weight, and other measures of people, animals, or plants are normally distributed.

Example 5

What is the probability that a person in Texas is exactly 6 feet tall?

Solution:

Since height is a continuous variable, meaning any number within a reasonable domain interval is possible, the probability of choosing any single number is 0. Many people may be close to 6 feet tall, but in reality they are 5.99 or 6.0001 feet tall. There must be someone in Texas who is the closest to being exactly 6 feet tall, but even that person when measured accurately enough will still be slightly off from 6 feet. This is why, instead of calculating the probability for a single outcome, you calculate the probability between a certain interval, like between 5.9 feet and 6.1 feet. For continuous variables, the probability of any specific outcome, like 6 feet, will always be 0.

Example 6

Two percent of high-school football players are invited to play at a competitive college level. How many standard deviations above the average player would someone need to be to have this opportunity?

Solution:

This situation is the inverse of the previous questions. Thus, you use the invNorm command on your calculator. You are looking for how many standard deviations above the mean include 98% of the data.

invNorm(0.98) = 2.0537

A person would have to be greater than about 2 standard deviations above the mean to be in the top 2 percent.

Example 7

On average, a pumpkin at your local farm weighs 10 pounds with a standard deviation of 6 pounds. You go and find a pumpkin weighing 26 pounds. Of all the pumpkins at the farm, what percent weigh less than this enormous pumpkin?

Solution:

Normalcdf(-1E99, 26, 10, 6) = 0.9961 or 99.61%

The vast majority of the pumpkins weigh less than the 26-pound pumpkin you found.

Summary

- A **standard normal distribution** is a normal distribution with mean of 0 and a standard deviation of 1.
- The **empirical rule** states that for data that are normally distributed, approximately 68% of the data will fall within 1 standard deviation of the mean, approximately 95% of the data will fall within 2 standard deviations of the mean, and approximately 99.7% of the data will fall within 3 standard deviations of the mean. It is a good way to quickly approximate probabilities.

Review

Consider the standard normal distribution for the following questions:

1. What is the mean?

2. What is the standard deviation?

3. What is the percentage of the data below 1?

4. What is the percentage of the data below -1?

5. What is the percentage of the data above 2?

6. What is the percentage of the data between -2 and 2?

7. What is the percentage of the data between -0.5 and 1.7?

8. What is the probability of a value of 2?

Assume that the mean weight of one-year-old girls in the United States is normally distributed, with a mean of about 9.5 kilograms and a standard deviation of approximately 1.1 kilograms.

9. What percent of one-year-old girls weigh between 8 and 12 kilograms?

10. What percent of girls weigh above 12 kilograms?

11. Girls in the bottom 5% by weight need their weight monitored every two months. How many standard deviations below the mean would a girl need to be to have her weight monitored?

Suppose that adult women's heights are normally distributed with a mean of 65 inches and a standard deviation of 2 inches.

12. What percent of adult women have heights between 60 inches and 65 inches?
13. Use the empirical rule to describe the range of heights for women within 1 standard deviation of the mean.
14. What is the probability that a randomly selected adult woman is more than 64 inches tall?
15. What percent of adult women are either less than 60 inches or greater than 72 inches tall?

Review (Answers)

Please see the Appendix.

14.11 Linear Correlation

Learning Objectives

Learn to work with data that has two variables as you learn about linear correlation, correlation coefficients, and regression.

Introduction

Statistics is largely concerned with the relationship between the values of one variable and the values of an associated variable. **Bivariate data** are data that have two variables that are often paired up like coordinate points. Is there any relationship between the following data? If there is, does it mean that doctors cause cancer?

TABLE 14.13:

Number of Doctors	27	30	36	60	81	90	156	221	347
Cancer Rate	0.02	0.07	0.16	0.20	0.43	0.87	1.21	2.80	3.91

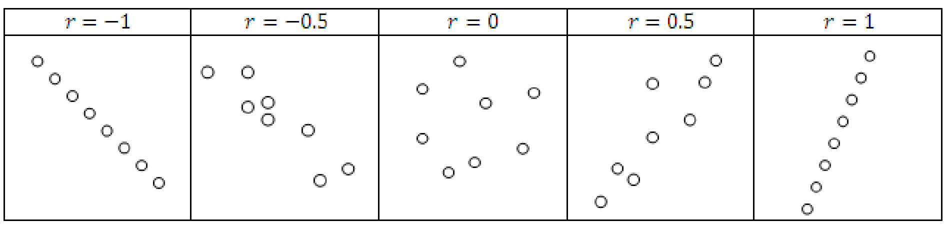
Linear Correlation

A **scatterplot** creates an (x,y) point from each data pair. When making a scatterplot, you can try to assign the independent variable to x, and the dependent variable to y; however, it often will not be obvious which variable is the dependent variable, so you will just have to pick one.

Once you plot the data and zoom appropriately, you will see the points scattered about. Sometimes there will be a clear linear relationship, and sometimes it will appear random. The **correlation coefficient**, r, is a number that quantifies two aspects of the relationship between the data:

- The correlation coefficient is either negative, zero, or positive.
  - If the correlation coefficient is negative, then the data are negatively correlated. This means that as one variable tends to increase, the other variable tends to decrease.
  - If the correlation coefficient is zero, then the data have no linear correlation. This means that the two variables tend to not be related in a linear pattern.
  - If the correlation coefficient is positive, then the data are positively correlated. This means that the two variables tend to increase together.
- The correlation coefficient is a number between $-1 \leq r \leq 1$, indicating the strength of correlation. If $r = 1$ or $r = -1$, then the data are perfectly linear. Note that a perfectly linear relationship includes lines with slopes other than 1.

Consider the examples below to see what different correlation coefficients will look like in data:

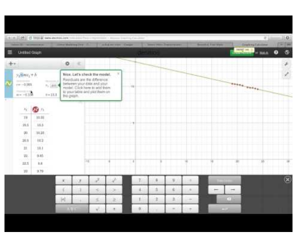


Once the data are determined to be sufficiently linear, the **regression line** or **line of best fit** can be used to approximate data within the set of beyond the set. In other words, this line represents the linear relationship of the data, and is the line closest to all the data points. Your calculator can perform a regression to produce the equation of a line that attempts to model the trend of the data. The regression line may actually pass through all, some, or none of the data points. This regression line is represented in statistics by

$$\hat{y} = a + bx.$$

The symbol $\hat{y}$ is pronounced "y-hat" and is the predicted y-value based on a given x-value. Occasionally, you may also calculate the predicted x-value given a y-value, however, this is less mathematically sound. Also notice that the linear regression model is simply a rearrangement of the standard equation of a line, $y = mx + b$.

The following video explains how to create a scatterplot using Desmos, a free online calculator:



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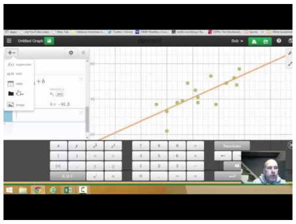
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/195606

The following video explains how to find the equation for the line of best fit, using Desmos:



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Click image to the left or use the URL below.

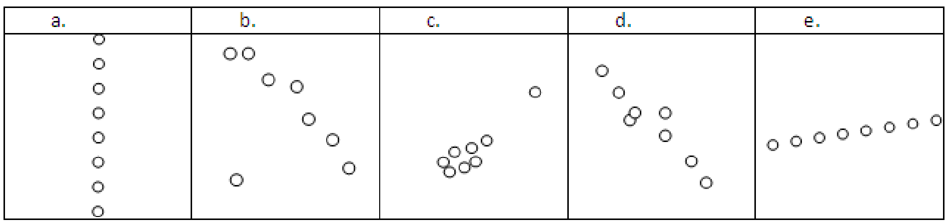
URL: http://www.ck12.org/flx/render/embeddedobject/195607

Play, Learn, and Explore with Regression and Linear Correlation: www.ck12.org/a/2112809/ .

Examples

Example 1

Estimate a correlation coefficient for the following scatterplots.

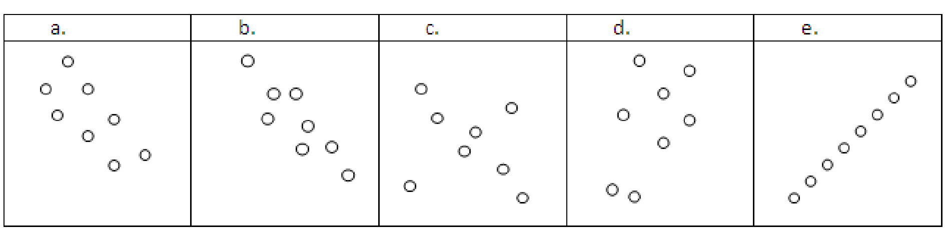


Solution:

1. $r \approx 0$. Because the height (y) does not seem to be dependent on the x, the data are uncorrelated. Another way to see this is that the slope appears to be undefined.
2. $r \approx -0.7$. If the solo point in the bottom left is an outlier, you could choose to not include it in the data. Then the r value would be closer to -1.
3. $r \approx +0.8$. The clump of data seems to be slightly positive correlated, and the single point in the upper left has a strong effect indicating positive slope.
4. $r \approx -0.8$. The data seem to be fairly strongly negatively correlated.
5. $r \approx 1$. The data seem to be perfectly linearly correlated.

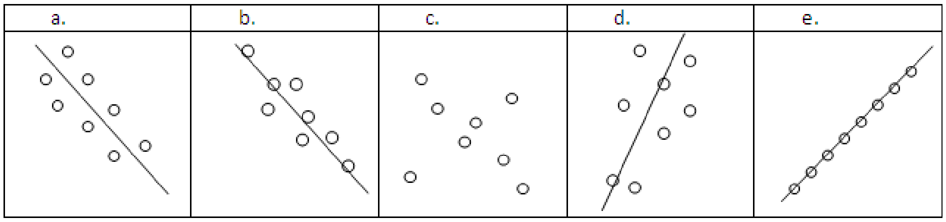
Example 2

Draw a regression line through the following scatterplots.



Solution:

Visualize and sketch the "line of best fit" for each set of points.



Note that in part a, the regression line does not touch any point. Instead, it captures the general trend of the data. In part c, the correlation is not high enough in any direction to produce a regression line. The calculator may give a regression line for scatterplots that look like part c, but you need to be very skeptical that there is actually a relationship between the two variables.

Example 3

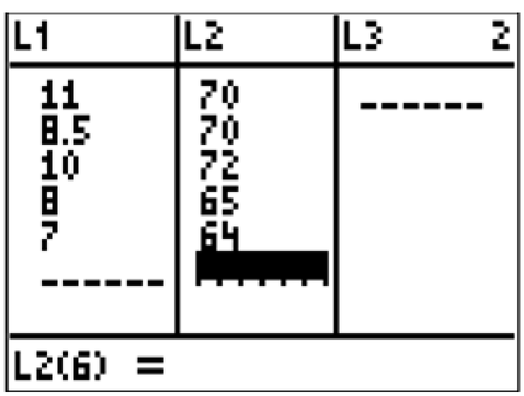
Use your calculator to perform a linear regression on the data below. Then, predict the height of someone who has shoe size 9.

TABLE 14.14:

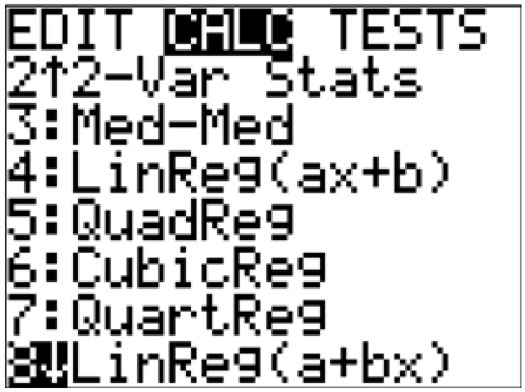
Shoe Size	Height (in)
11	70
8.5	70
10	72
8	65
7	64

Solution:

First, enter the data.



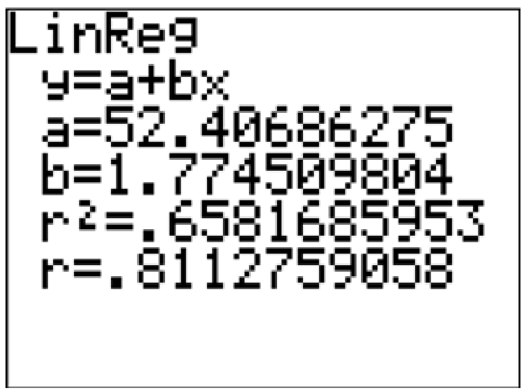
Next, perform the regression. Notice that the calculator can perform linear regression in two ways that are essentially the same. To keep consistent with $\hat{y} = a + bx$, use linear regression. This is option 8 in the [STATS], [CALC] menu.



Now you need to tell the calculator to perform the regression on the two lists you want, and where to copy the equation. The syntax is

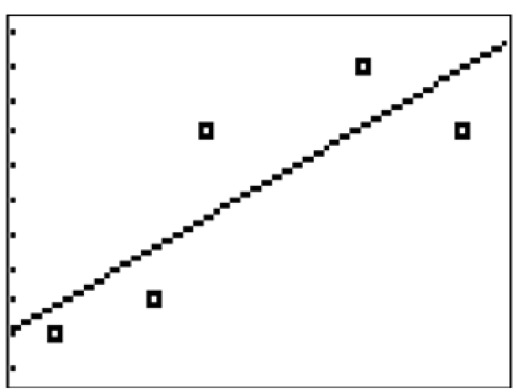
$$\text{LinReg}(a+bx)L_1, L_2, Y_1.$$

Note: To find Y1, go to [VARS], [Y-VARS], [FUNCTION], [Y1].



Notice that the r value is about 0.8. This indicates that there is a fairly strong positive correlation between shoe size and height. *If your calculator does not display the r and r^2 lines, then you need to go into the catalog and run the program "DiagnosticOn." This will enable the display of the correlation coefficient.*

You can then graph the scatterplot and the regression line:



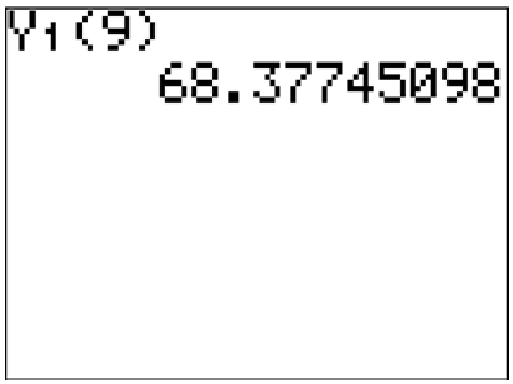
The regression equation is

$$\hat{y} = 52.4069 + 1.7745x,$$

where x represents shoe size and $\hat{y}$ represents predicted height. The predicted height for someone with size 9 shoe is 68.3774:

$$\hat{y} = 52.4069 + 1.7745 \cdot 9 = 68.3774$$

An easy way to use the power of the calculator is to use function notation from the home screen:



Example 4

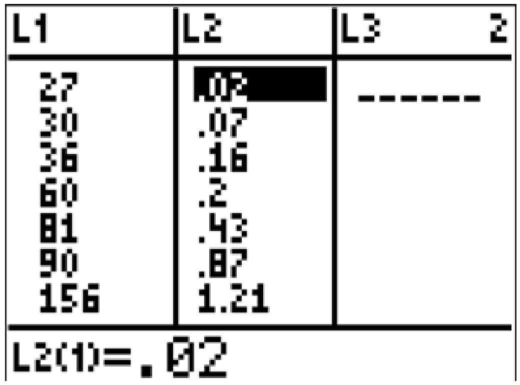
Recall the problem from the Introduction: Is there any relationship between the data below? If there is, does it mean that doctors cause cancer?

TABLE 14.15:

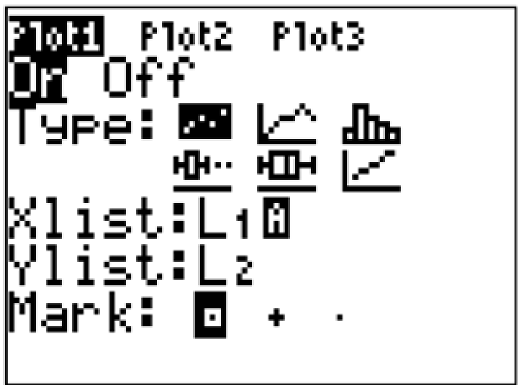
Number of Doctors	27	30	36	60	81	90	156	221	347
Cancer Rate	0.02	0.07	0.16	0.20	0.43	0.87	1.21	2.80	3.91

Solution:

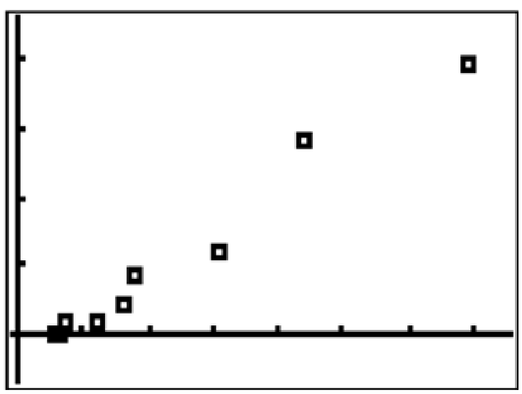
Enter the data onto lists in your calculator:



Turn on the [STAT PLOT] that compares the two lists of data:



Note that the data are extremely linear with a positive correlation coefficient:



One of the most misunderstood concepts in statistics is that correlation does not imply causation. Just because there is a correlation between the number of doctors and the cancer rate doesn't mean that the number of doctors *cause* cancer. There are dozens of reasons why more doctors might correlate with higher cancer rates. In general, remember that correlation is not the same as causation. Be careful before making any conclusions about change in one variable *causing* change in another variable.

Example 5

The data below represent the average number of working words in an elementary-school student's vocabulary as it relates to shoe size. Perform a linear regression that models the data.

TABLE 14.16:

Shoe Size	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
Vocabulary	1135	1983	2501	4113	5431	7891	9320	11041

Solution:

Let x represent shoe size and y represent vocabulary.

$$\hat{y} = -2{,}660.4167 + 2{,}940.9333x$$

$$r = 0.9865$$

The correlation coefficient is very close to positive one. This is a strong indication that the data can be modeled by a linear relationship.

Example 6

Use the equation from Example 1 to predict the vocabulary for someone who has a 1.0 shoe size (see table in Example 5). Does this prediction seem reasonable given the data? Why or why not?

Solution:

$$\hat{y} = -2{,}660.4167 + 2{,}940.9333 \cdot 1$$

$$\hat{y} = 280.4167$$

This number seems remarkably low considering the data. This point is very close to the x-intercept, which can be found using algebra:

$$0 = -2{,}660.4167 + 2{,}940.9333x$$

$$0.9046 = x$$

The interpretation of the point (0.9046, 0) from the model is that when someone has a shoe size of just under 1.0, then that person's predicted vocabulary is 0. Shoe sizes below 0.9046 will have a negative vocabulary. Is this reasonable? It certainly does not make sense that someone could have a negative number of words in their vocabulary. Newborn babies are born without knowing any words, and this number stays flat at 0 for some length of time. Therefore, this model is not accurate for very low shoe sizes.

Example 7

Shaquille O'Neal has size 23 shoes. Based on this sentence, and using the data from Example 5, can you make any inferences about his vocabulary? Does a larger shoe size result in a larger vocabulary?

Solution:

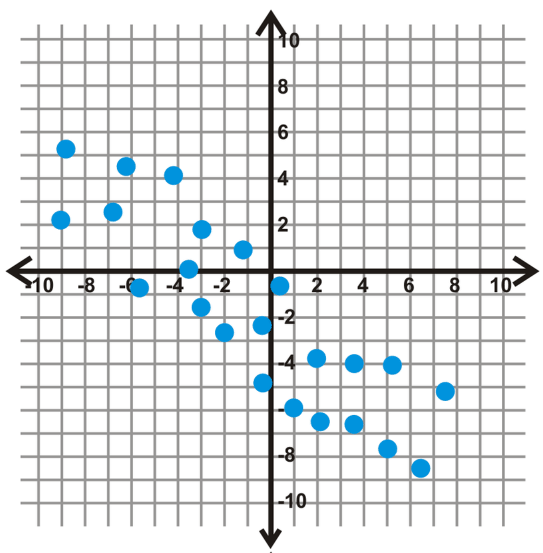
Shaquille's shoe size is significantly beyond the scope of the data that the model is based on. The data relate to elementary-school students, and a size 23 shoe is beyond the relevant domain. This means it wouldn't make sense to use this model to predict the size of Shaquille's vocabulary. Shoe size does not cause vocabulary, but the two variables are strongly correlated because over time both tend to grow.

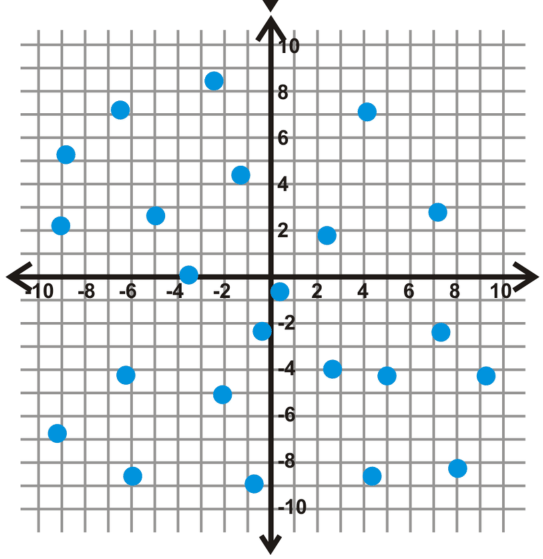
Summary

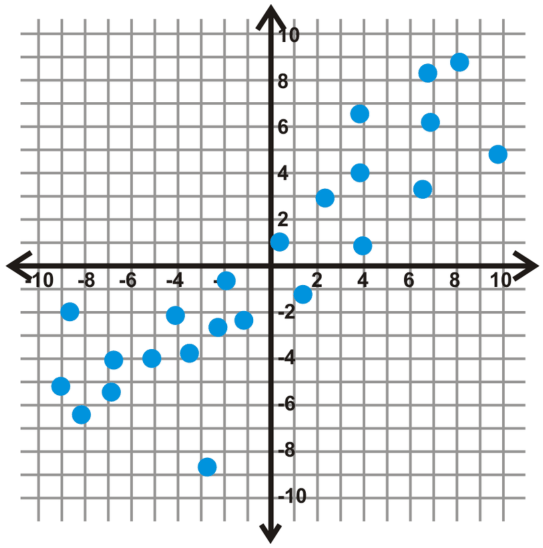
- A **scatterplot** creates an (x, y) point from each data pair.
- **Bivariate data** are data that have two variables that are often paired up like coordinate points.
- The **correlation coefficient**, r, is a number in the interval [-1, 1]. It indicates the strength of the correlation between two variables.

Review

Determine if the scatterplots below have positive, negative, or no correlation.

1. 

2. 



3.

For each correlation coefficient, describe what it means for data to have that correlation coefficient, and sketch a scatterplot with that correlation coefficient.

4. $r = 1$

5. $r = -0.5$

6. $r = -1$

7. $r = 0$

8. $r = 0.8$

The data below show the SAT math score and GPA for seven different students.

TABLE 14.17:

SAT math score	595	520	715	405	680	490	565
GPA	3.4	3.2	3.9	2.3	3.9	2.5	3.5

9. Use your calculator to perform a linear regression that models the data. What is the regression equation? What is the correlation coefficient?

10. Use the equation from the previous problem to predict the GPA for a student with an SAT score of 500. Does this prediction seem reasonable given the data? Why or why not?

11. Does a high SAT math score cause a high GPA?

The Price of Apple Stock from Oct. 2009 to Sept. 2011 (Source: Yahoo! Finance)

TABLE 14.18:

10/09	**11/09**	**12/09**	**1/10**	**2/10**	**3/10**	**4/10**	**5/10**	**6/10**	**7/10**	**8/10**	**9/10**
$181	$189	$198	$214	$195	$208	$236	$249	$266	$248	$261	$258
10/10	**11/10**	**12/10**	**1/11**	**2/11**	**3/11**	**4/11**	**5/11**	**6/11**	**7/11**	**8/11**	**9/11**
$282	$309	$316	$331	$345	$352	$344	$349	$346	$349	$389	$379

12. Use your calculator to perform a linear regression that models the data. What is the regression equation? What is the correlation coefficient?

13. Use the equation from Number 12 to predict: What would be the price of the stock in January 2012?

14. What conclusions can you make about these data?

Total Number of Home Runs Hit in Major League Baseball, 2000-2010 (Source: Baseball Almanac)

TABLE 14.19:

2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
5,693	5,458	5,059	5,207	5,451	5,017	5,386	4,957	4,878	4,655	4,613

15. Use your calculator to perform a linear regression that models the data. What is the regression equation? What is the correlation coefficient?

16. Use the equation from Number 15 to predict, how many total home runs should be hit in 2011?

17. What conclusions can you make about this data?

18. Explain in your own words what the correlation coefficient measures.

19. Explain why a larger sample size will cause a more accurate correlation coefficient.

Review (Answers)

Please see the Appendix.

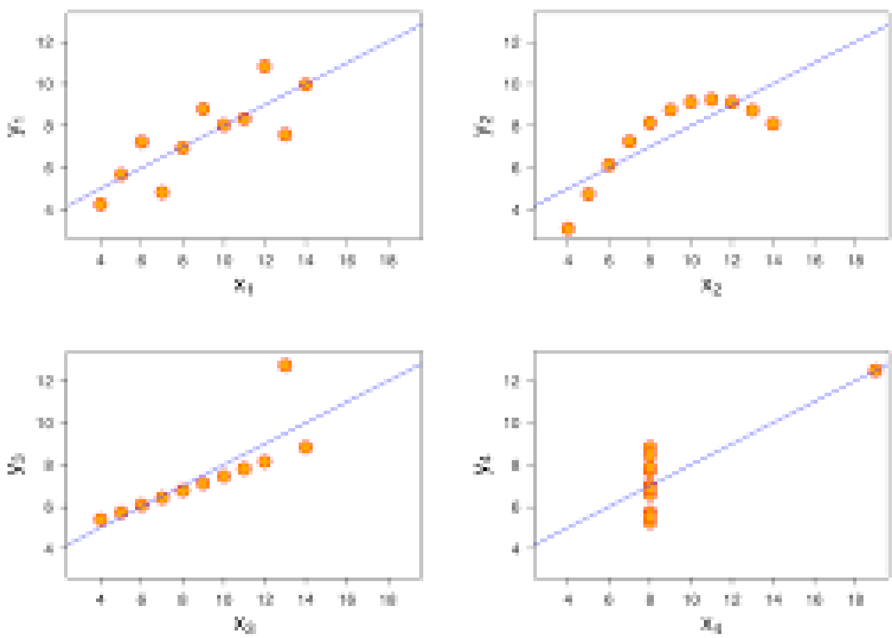
14.12 Modeling with Regression

Learning Objectives

Learn to use regression on a variety of different types of data to make reasonable predictions.

Introduction

Linear correlation is just one type of relationship between two variables. Online graphing software and graphing calculators have the power to use a variety of different function families to find other relationships and create many different types of models. How do you choose which function family is best for a given situation?



Regression

Once you understand how to do linear regression with your calculator, you already know the technical mechanics to perform other regressions in the [STAT] [CALC] menu. The most common regressions correspond to the function families.

- QuadReg: Quadratic function family
- CubicReg: Cubic function family
- QuarticReg: Quartic function family or 4th-degree polynomial
- LnReg: Natural Log function family
- ExpReg: Exponential function family
- PwrReg: Power function family
- Logistic: Logistic function family

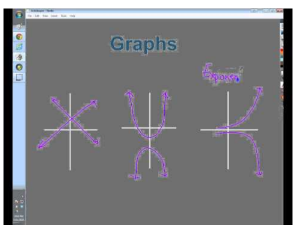
- SinReg: Sinusoidal function family

When you perform these types of regressions, it will be incredibly important for you to interpret and explain parts of the graph. Here are some points to keep in mind:

1. The y-intercept may have a particular meaning that may or may not be reasonable.
2. When you use your model to make predictions, it is important to remember the relevant domain of your model. If your data is about elementary-school students, then it might extend to middle- and high-school students, but it might not.
3. The calculator may produce a correlation coefficient for each of these non-linear regressions, but you should be very careful. Technically, the correlation coefficient is only supposed to be calculated with linear regression, so the calculator is doing some fancy linearization to produce it. You can learn more about this process in future statistics courses.

In general, at this point you should use your best judgment when choosing a function family to model a given set of data and deciding how good a fit the model is based on context.

The following video explains how to determine whether a linear, quadratic, or exponential function is the best fit for a scatter plot:



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/190321

Examples

Example 1

The data below represent the speed at which Ben can kick a soccer ball at different ages. Determine the best regression function to use and determine its equation.

TABLE 14.20:

Age(Years)	Speed (mph)
4	15
10	32
20	65
30	70
50	45
60	35

Solution:

The best regression to use is a quadratic relationship, because when Ben is young or old he cannot kick the ball very fast. Ben can kick the ball the fastest when he is an adult between the ages of 20 and 40.

$$\hat{y} = -0.05981x^2 + 4.0679x + 0.6191$$

Example 2

What are two weaknesses and two strengths of the model used to predict Ben's kicking speed from Example 1?

Solution:

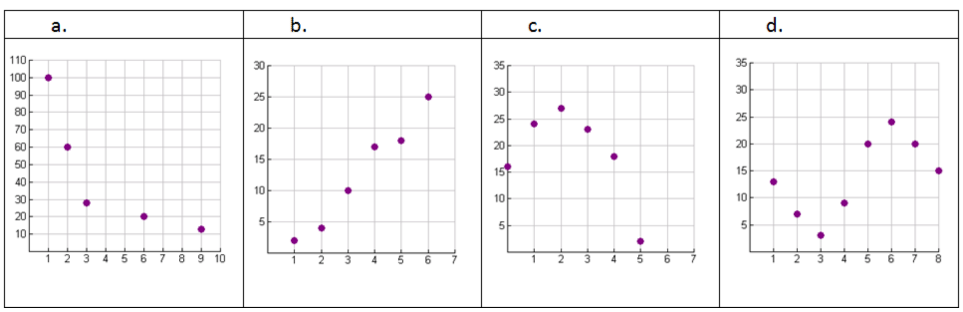
One strength is that a quadratic model correctly describes the peak of kicking speed occurring in the middle of Ben's life. A linear regression might forecast Ben's kicking speed increasing forever, and a logistic regression might forecast Ben's kicking speed always staying fast despite his old age. A 2nd strength of the model could be the y-intercept of 0.6191. Even though this number is not really in the relevant domain, it implies that as a newborn baby, Ben could kick the ball very slowly, which is arguably true.

One weakness of the model is that it predicts that Ben will kick the ball at 0 miles per hour at age 68.1660. This implies that Ben will not be able to kick the ball at all, which isn't necessarily true.

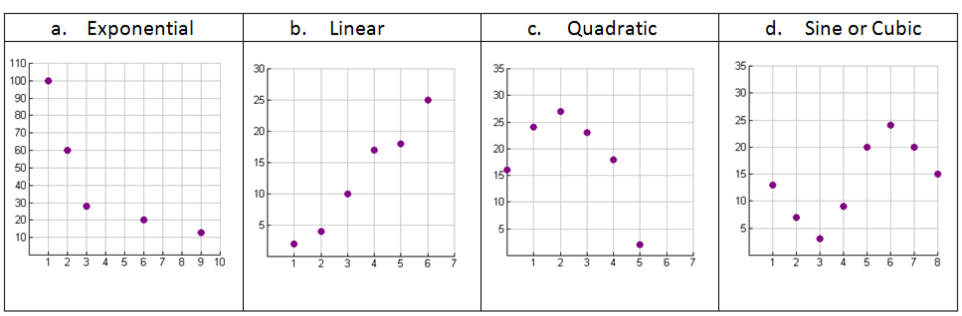
A 2nd weakness of the model is that it predicts negative speed at either age extreme, which doesn't make sense. A better model would be flat at 0, when Ben is born, and also at the end of Ben's life, when he is no longer able to kick the ball.

Example 3

Use your knowledge of function families to predict the best model for each of the following scatterplots:



Solution:

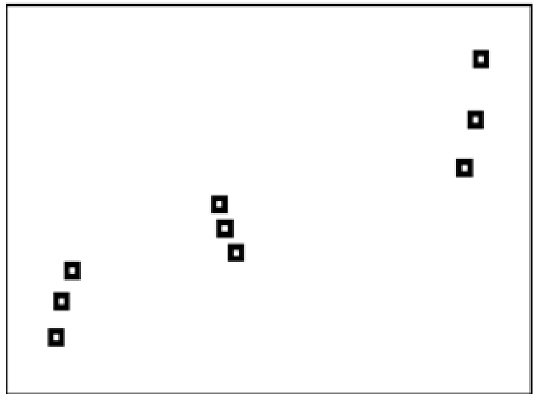


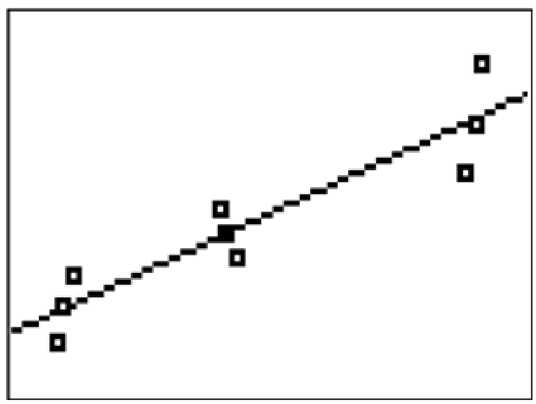
Example 4

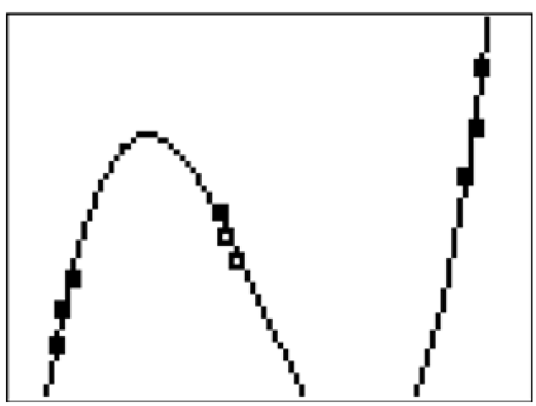
Recall the question from the Introduction: How do you choose which function family is best for a given situation?

Solution:

For some datasets, it is possible to use a polynomial or other complicated shape to exactly intersect every data point. The downside is that the model will miss the overall relationship. Consider the following data and modeling a linear relationship or cubic relationship:







The linear relationship describes the upward positive relationship in the data very well, but some points are slightly off of the line. The cubic relationship is much more accurate at the specific data points; however, there are features of the cubic relationship that differ significantly from reality when interpreted in context. To choose the best regression model, you need to use context clues and the reasonableness of the various features of the model that fit each situation.

Example 5

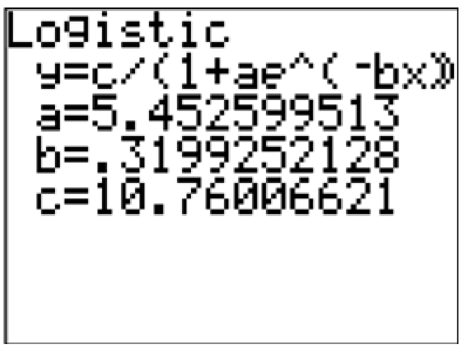
The following data represent the height of an elephant over time. Determine the best regression function to use and determine its equation.

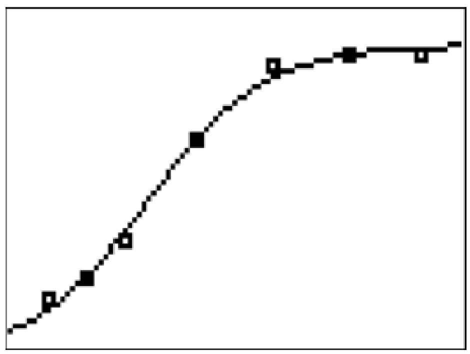
TABLE 14.21:

Age(Years)	Height (ft)
0	2
2	2.8
4	4
8	7.5
12	10
16	10.4
20	10.45

Solution:

Logistic is the best function family because it levels off over time, indicating that the elephant ceases to grow once it matures.





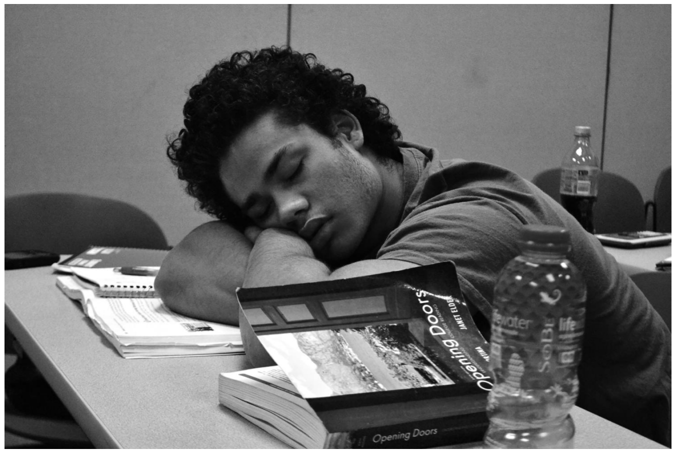
$$\hat{y} = \frac{10.7601}{1 + 5.4526 \cdot e^{-0.3199x}}$$

Example 6

Given the following data about SAT scores and number of hours slept the night before, use an appropriate function family to produce a reasonable model. Defend your choice of function families.

TABLE 14.22:

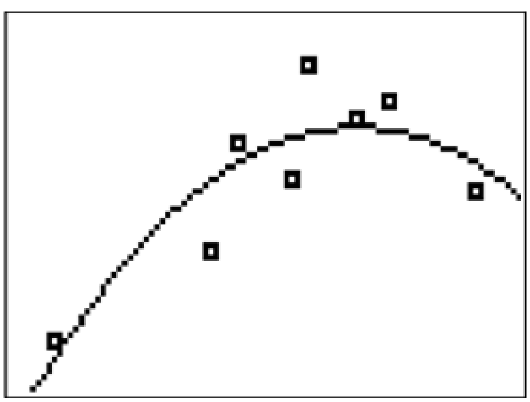
# Hours Slept	SAT Score
8.5	1840
10.9	1510
9.1	1900
7.5	2070
7.2	1550
6.0	1720
2.3	840
5.5	1230



Solution:

Let x be the number of hours slept, and y be the SAT score.

After plotting the points, you should choose a function family to use as a model. In this case, it would be appropriate to try a quadratic relationship.



A quadratic model makes sense because there seems to be a peak in the model and in the data around 8.5 hours of sleep. It makes sense that someone who does not get enough sleep will do worse, and someone who gets too much sleep might also do worse.

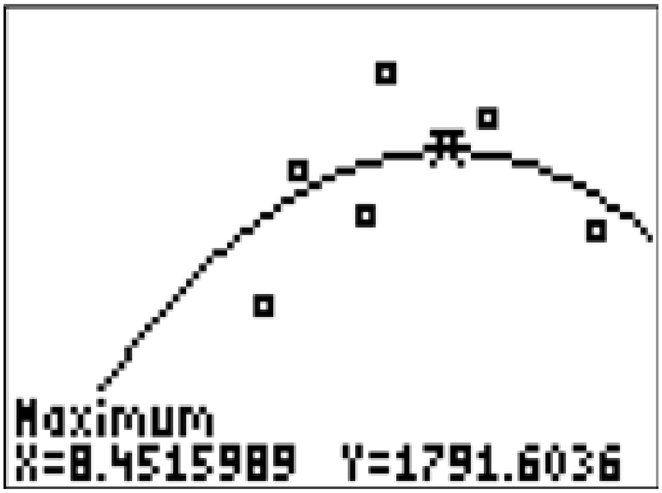
Example 7

Using the model from Example 6, answer the following questions:

1) What is the perfect amount of sleep to get before the SATs?

Solution:

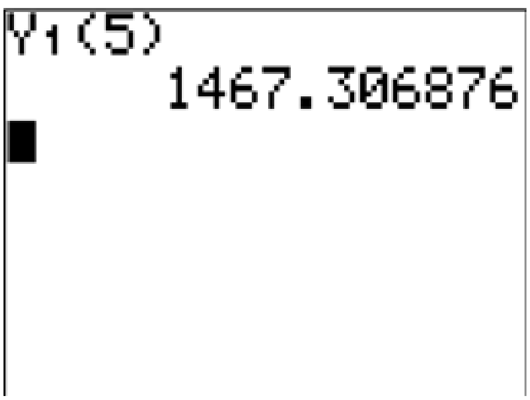
Use the calculator to find the maximum of the parabola. The x-coordinate represents the "perfect" amount of sleep.



2) Calculate the score you are predicted to get if you sleep for 5 hours.

Solution:

You can substitute $x = 5$ into the equation, or you can let the function you created and stored in y_1 simply act on the 5.



3) What is the relevant domain of the model?

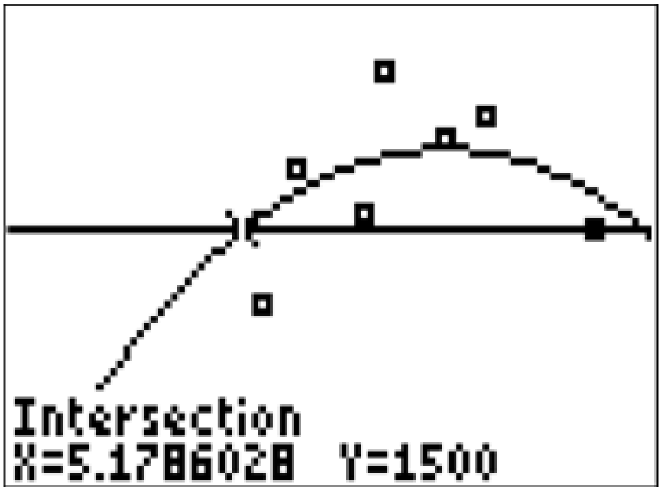
Solution:

The relevant domain is between about 2 hours and 10 hours of sleep. Beyond those numbers of sleep, the model will probably not make a whole lot of sense. How could someone get negative hours of sleep?

4) The average SAT score is about 1500. According to the model, what amount of sleep predicts this score? Does this number represent the average number of hours that people sleep before the SATs?

Solution:

You can substitute $\hat{y} = 1500$ into the equation and solve for x using the quadratic formula, or you can graph the line $y = 1500$ and use the calculator to produce the two intersecting points.



5.1786 hours and 11.7246 hours are the number hours of sleep that predict a score of 1500.

When using the model in this direction, the results do not make as much sense, and you need to be extremely careful about what you say.

5) Compare the actual and predicted score for someone with 6 hours of sleep.

Solution:

The actual score for someone who got 6 hours of sleep can be found in the original data to be 1720. The model predicts 1627.9970. The difference between the model and what actually happened is $1720 - 1627.9970 = 92.0030$.

Summary

- **Regression** is a method of attempting to fit a model to observed data in order to predict new values.
- The best fit for a scatter plot may be a linear, quadratic, or exponential model.

Review

The table below shows the average height of an American female by age.

TABLE 14.23:

Age (Years)	Height (inches)
2	34
8	50
11	57
15	63
23	64
35	64

1. Determine two different equations that model the height over time using two different function families.

2. Which function is a better fit for this data? Why?

3. Use both equations to predict the y-intercepts. What does the y-intercept represent in each case? Are your predictions reasonable for this part of the graph?

4. Use your "better fit" equation to predict the height of a 70-year-old woman. Is your prediction reasonable for this part of the graph? Why or why not? What do you really need your model to do for the domain [16,100]?

Alice is in Wonderland and drinks a potion that approximately halves her height for each sip she takes, as shown in the table below.

TABLE 14.24:

# of sips	Height (inches)
0	60
1	29
2	16
3	8
4	4.1

5. Do an exponential regression to determine an appropriate model. What is the equation?

6. Explain why exponential regression is a good choice in this case.

7. How many sips did she take if she is 2 inches tall?

8. How tall will she be if she has 6 sips?

A rumor is spreading around your 400-person school. The following table shows the number of people who know the rumor each day.

TABLE 14.25:

Day	# of people who know the rumor
1	2
2	8
3	29
5	161
6	372
7	378
8	391

9. Use logistic regression to determine an equation that models the number of people who know the rumor over time.

10. Why is the logistic model appropriate in this case?

11. Use your regression equation to predict the time when only one person knew the rumor. Does this make sense?

The data table below represents how the tide changes the depth of the ocean water at a beach. At a certain place in the water, a scientist measures the depth of the water for 10 consecutive hours.

TABLE 14.26:

Hours	Depth of Ocean Water (ft)
0	9
1	11.2
2	12.4
3	12.9
4	12.5
5	11
6	8.9
7	7
8	5.5

TABLE 14.26: (continued)

9	4.9
10	5.4

12. Choose the function family that is the best model for this situation and determine the regression equation.

13. Use your regression equation to predict the depth of the water at 10 hours. What is the difference between the actual depth from the data and the predicted depth from your equation (residual)?

14. Do a cubic regression on the calculator. What is the cubic regression equation? Is this a better or worse model than the model you originally chose?

15. Why might statisticians do modeling with regression for their data?

Review (Answers)

Please see the Appendix.

14.13 Project: Probability and Statistics

For this project, we will use our knowledge of probability and statistics to analyze the data from the 100-meter freestyle swimming for men and women since 1980.

Here is the dataset for the men's world record progression, as of December 14, 2016:

TABLE 14.27:

Name	Time	Country Represented	Date
Cesar Cielo	46.91	Brazil	07/30/2009
Eamon Sullivan	47.05	Australia	08/13/2008
Alain Bernard	47.20	France	08/13/2008
Eamon Sullivan	47.24	Australia	08/11/2008
Alain Bernard	47.50	France	03/22/2008
Alain Bernard	47.60	France	03/21/2008
Pieter Van Den Hoogen-band	47.84	Netherlands	09/19/2000
Michael Klim	48.18	Australia	09/16/2000
Alexander Popov	48.21	Russia	06/18/1994
Matt Biondi	48.42	United States	08/10/1988
Matt Biondi	48.74	United States	06/24/1986
Matt Biondi	48.95	United States	08/06/1985
Matt Biondi	49.24	United States	08/06/1985
Rowdy Gaines	49.36	United States	04/03/1981

Source: USA Swimming http://www.usaswimming.org/_Rainbow/Documents/3bdb177b-e9aa-4a67-805e-0ab046e5e265/men_lcm.PDF Downloaded January 9, 2017.

Here is the data for the women's world record progression, as of December 14, 2016:

TABLE 14.28:

Name	Time	Country	Date
Cate Campbell	52.06	Australia	07/01/2016
Britta Steffen	52.07	Germany	07/31/2009
Britta Steffen	52.56	Germany	06/27/2009
Britta Steffen	52.85	Germany	06/25/2009
Libby Trickett	52.88	Australia	03/27/2008
Britta Steffen	53.30	Germany	08/02/2006
Libby Trickett	53.42	Australia	01/30/2006
Jodie Henry	53.52	Australia	08/18/2004
Libby Trickett	53.66	Australia	03/31/2004
Inge De Bruijn	53.77	Netherlands	08/01/2000
Inge De Bruijn	53.80	Netherlands	05/29/2000
Jingyi Le	54.01	China	09/05/1994
Jenny Thompson	54.48	United States	03/01/1992
Kristin Otto	54.73	Germany	08/19/1986
Barbara Krause	54.79	Germany	07/21/1980

Source: USA Swimming http://www.usaswimming.org/_Rainbow/Documents/1e379e54-2328-44bc-aafd-21ca80f00698/women_lcm.PDF Downloaded January 9, 2016.

Directions

1. Calculate the five number summary for each of the time data sets.
2. Create a box-and-whisker plot of the times for each sex. What do you notice about the plots?
3. Create two pie graphs depicting the countries represented (one for men and one for women).
4. Create a scatter plot with both sets of data on one set of axes, where time is the dependent variable and year is the independent variable.
5. Calculate regression lines for each dataset.
6. Make a prediction about what year the two sexes will tie.
7. Explain what you think will happen after that year.

14.14 Summary: Probability and Statistics

This chapter addressed concepts in probability as well as statistics.

The probability lessons addressed counting permutations and combinations, as well as the Fundamental Counting Principle. Then the chapter moved on to the Binomial Theorem. Finally, you learned how to calculate the predicted cost of events using the expected value formula.

You worked with univariate data and learned how to display it graphically and summarize it numerically. Additionally, you learned how to calculate mean, median, mode, and variance—and also learned when to use each. Moreover, you explored bivariate data and used the regression capabilities of your calculator to create mathematical models for real-world phenomena, while also exploring the difference between correlation and causation.

Chapter Summary

- A **combination** is the number of ways of choosing k objects from a total of n objects. (Order does not matter.)
- The number of ways to find the combination or choose k objects from a group of n objects is:

$$_nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)\,!}$$

- A **permutation** is the number of ways of choosing and arranging k objects from a total of n objects. (Order does matter.)
- The formula to calculate how to choose and arrange k objects from a group of n objects (or calculate a permutation):

$$_nP_k = k!\binom{n}{k} = k!\cdot\frac{n!}{k!\,(n-k)\,!} = \frac{n!}{(n-k)\,!}$$

.

- The **Fundamental Counting Principle** states that if one event has m possible outcomes and a 2nd independent event has n possible outcomes, then there are $m \times n$ total possible outcomes for the two events together.

- The **probability** of an event is the number of outcomes you are looking for (called successes) divided by the total number of outcomes.
- The notation $P(E)$ is read "the probability of event E."

$$P(E) = \frac{\#\ successes}{\#\ possible\ outcomes}$$

- The **complement of an event** is the event not happening.
- **Independent events** are events where the occurrence of the 1st event does not impact the probability of the 2nd event.

- The **Binomial Theorem** (or **binomial expansion**) describes the algebraic expansion of powers of a binomial.
- The coefficients in the binomial expansion appear as the entries of Pascal's Triangle. In Pascal's Triangle, each entry is the sum of the two above it.

- The coefficients can also be generated using combinations. The numbers in the combination are associated with a particular row and column in Pascal's Triangle. For example, 5 combinations of 3 would be associated with the 5th row and 3rd column of the triangle.
- The **Binomial Theorem** is:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

- A **weighted average** is an average that multiplies each component by a factor representing its frequency or probability.
- The **expected value** is the return or cost you can expect on average, given many trials.
- The **payoff** of a game is the expected value of the game minus the cost.

- The **mean** is the arithmetic average of the data.
- The **median** is the number in the middle of a dataset. When the data has an odd number of counts, the median is the middle number after the data have been ordered. When the data has an even number of counts, the median is the average of the two most central numbers.
- The **mode** is the most often occurring number in the data. If two or more numbers occur equally frequently, then the data is said to be **bimodal** or **multimodal**.

- With **descriptive statistics**, your goal is to describe the data that you find in a sample or is given in a problem.
- With **inference statistics**, your goal is use the data in a sample to draw conclusions about a larger population.
- The **rank** of an observation is the number of observations that are less than or equal to the value of that observation.

- Data are divided into four parts by the 1**st quartile** $(Q1)$**, 2nd quartile** $(Q2)$**,** and 3**rd quartile** $(Q3)$**.** The 2**nd quartile** is also known as the median.
- **Variance** is a measure of how spread out the data are.
- The square root of the variance is the **standard deviation**.
- Both the variance and the standard deviation can be calculated from a **sample** or from the whole **population**. The formulas are slightly different in each case, so it is important to know whether your data is just a sample or is from the whole population.
- The **absolute deviation** is the sum total of how different each number is from the mean.
- The **mean absolute deviation** is an alternate measure of how spread out the data are. While this method might seem more intuitive, in statistics it has been found to be too limited and is not commonly used.
- **Mean and variance for the population:** $x_1, x_2, x_3, \ldots, x_n$

$$\mu = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$$

$$\sigma^2 = \frac{1}{n} \cdot \sum_{i=1}^{n} (\mu - x_i)^2$$

- **Mean and variance for a sample from a population:** $x_1, x_2, x_3, \ldots, x_m$

$$\bar{x} = \frac{1}{m} \cdot \sum_{i=1}^{m} x_i$$
$$s^2 = \frac{1}{m-1} \cdot \sum_{i=1}^{m} (\bar{x} - x_i)^2$$

- A **standard normal distribution** is a normal distribution with mean of 0 and a standard deviation of 1.
- The **empirical rule** states that for data that are normally distributed, approximately 68% of the data will fall within 1 standard deviation of the mean, approximately 95% of the data will fall within 2 standard deviations of the mean, and approximately 99.7% of the data will fall within 3 standard deviations of the mean. It is a good way to quickly approximate probabilities.
- **Normalcdf** is the normal cumulative distribution function and calculates the area between any two values for data that are normally distributed, as long as you know the mean and standard deviation for the data. Your calculator has this function built in, and it produces an exact answer as opposed to the empirical rule.

- A **scatterplot** creates an (x, y) point from each data pair.
- **Bivariate data** are two sets of data that are paired.
- The **correlation coefficient**, r, is a number in the interval [-1, 1]. It indicates the strength of the correlation between two variables.
- **Regression** is a method of attempting to fit a model to observed data in order to predict new values.
- The best fit for a scatter plot may be a linear, quadratic, or exponential model.

Review

Try the following cumulative review problems to practice the concepts studied in this chapter:

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/195610

14.15 References

1. By Kwevans54 (Own work) [Public domain], via Wikimedia Commons. https://commons.wikimedia.org/wiki/File%3ADogs_reaction_time.JPG .
2. Oregon Department of Transportation. https://www.flickr.com/photos/oregondot/10728525246 .
3. Unsplash. https://pixabay.com/en/volkswagen-adventure-travel-vw-569315/ .
4. The U.S. Army. https://commons.wikimedia.org/wiki/File:Steven_Holcomb_and_Steve_Langton(12626788343).jpg .
5. Jamie C2009. https://www.flickr.com/photos/jamiec2009/8636260958 .
6. Nonenmac. https://commons.wikimedia.org/wiki/File:Pascal%27s_Triangle_divisible_by_3.svg .
7. Unknown. https://commons.wikimedia.org/wiki/File:Blaise_Pascal_Versailles.JPG .
8. Guts Gaming. https://www.flickr.com/photos/125363144@N08/14906650705 .
9. Erik Drost. https://www.flickr.com/photos/edrost88/7116805785/in/photolist-bQTuq2-4xaH8V-ka7gjc-7Q36z7-7hjWm-9miPVW-7hjnv-juDHAo-9ReYu9-4kvVM1-dueKNw-9ur17Y-4tb2JN-ka5ysV-qRAso5-qUgYFD-bvewBM-98PVF-6XkdN-6Xka1-bvex6Z-bBJcUC-8v8gpc-9fXe4-8uPC3x-9omXJz-hwrQ52-8vmFHW-9QUfR-8vmEdU-8voJC1-huXb7J-8vkF4z-8uPbCZ-hwBeTD-7hkq8-5CdhN5-8voKVo-aEp3CS-4s3XUg-yTz6E-9fUsP-6Xkhi-7GyZM-bssuv-9QUPV-i1zMhK-hCZ1wM-8voLo5-hRwua9 .
10. Gabriele Diwald. https://www.flickr.com/photos/52214493@N03/8038516482/in/photolist-dfkuTL-dfkvKm-oWT4bF-oWT4DV-jJctyW-aE43CD-da9Q8z-7AeipX-okonkW-TXvqmV-pnZNyn-SfJBk4-7T5vhb-Rb2SY3-gp9ybn-Sjsczd-6TQp3m-SaMMig-d4FqAb-kPZ9R2-b1fj48-nbdD9N-91kWGN-o7znFp-jNFm66-nDRmYi-5a3QvE-dC2ao7-9djJRY-9zDijK-ebsxQo-asjh12-fm19gz-bWvfRG-8XyEeU-RWdHe5-UcYQaK-6gDqrF-qQJWYk-cMCHx7-n819Mn-eXzn45-f2BLW1-TMzvuH-j4XSbq-cBZmHm-e4QAxj-ehGdbC-8XFEjk-rhjEiY .
11. Steaphan Greene. https://commons.wikimedia.org/wiki/File:2-Dice-Icon.svg .
12. Sarah_Ackerman. https://www.flickr.com/photos/sackerman519/4276504331/in/photolist-7vUcKi-cMLoaG-8jkHfK-vidYV8-j3faDk-8oBgLP-f5uPAv-f5uNxt-E5NZTa-f5K5pf-4zc4Ya-f5K71h-7rUmuj-8in4yv-jg1EWN-f5uLoX-j3fa8v-qBGU4F-f5K2nY-bqZ5g1-SBfp93-9fLuzd-8is4Ab-deoUr-abZUXU-f5K5R1-4zgj3h-8io17x-f5uMBx-8irik5-8ioEzM-hjsYqT-f5uP8D-jg1pMb-f5uK8D-8inDCv-f5K52Y-cMLtjm-8inwSH-jfW51z-66RUdV-8io2Dv-9VByo5-4YoB7i-f5uR6X-f5K7AL-f5uLUt-8irM75-f5K6oJ-8inynD .
13. TomK32. https://commons.wikimedia.org/wiki/File:Parking_meter_pd_med.jpg .
14. NikeChillemi. https://pixabay.com/en/slot-machines-gambling-casino-701707/ .
15. Pexels. https://pixabay.com/en/apple-brick-wall-computer-cup-1854101/ .
16. Flickr: slgckgc. https://www.flickr.com/photos/slgc/16173880801 .
17. Mazku. https://commons.wikimedia.org/wiki/File:Cobra_Powerpark.JPG .
18. Tim Patterson. https://www.flickr.com/photos/timpatterson/313655887 .
19. Alberto G.. https://www.flickr.com/photos/albertogp123/5843577306/in/photolist-9UnRWS-4nFTB6-4ocshf-4pZ4GJ-7kuz2S-7kaEY-xLUqv-6ojgRs-9k2saA-6ojh25-is1M9-arbtDz-naBjky-4Yg7oj-73AM5f-HwBe-PyrEu-nsWjTr-4JTs6B-nMErQZ-6kDAjS-7xRMxr-naBoL9-6kDCvb-6kDGpC-ntSsFP-mJmYyn-52u6Xs-6kzw4X-6kzrpM-77BmSA-612yQa-6kztwZ-6kzs8D-6kzyy8-3HFmP8-3HFmyP-4WmYM3-nox8KF-3wVvk-nu8eMq-4aPLHB-nUubX1-2PZ8b-f2fCU-6kDF4j-b9fHwg-6kzxRP-ntSt8R-6kDGWQ .
20. Adam Baker. https://www.flickr.com/photos/atbaker/1315463822/in/photolist-4dGmNc-4rq7VE-dHAnc6-31f6yJ-7sPiNj-c7GdZQ-4AQVuL-6QfnN9-pBuDMM-64vVTA-bAxMF3-baBCTK-oYfaoQ-fpTrKW-bezLLi-87YreB-fpDj4z-GmC4J-g15kh-7ABDCc-4MgDkK-GmFbk-cz7xSm-2egmh-dsWzMD-fpTu5N-991Uux-6ha1kZ-dUZYHR-bzJcAH-3DBciD-7yLRPn-6i4Adh-93bTCE-Lq434-fLJqfc-4xSKEd-oeEZZR-hzoiXs-7yLRG4-gdyjJb-fpTxc7-dY6jN5-3p3nWD-9X3Ljp-7yQCuy-3gJE7Y-dVdkRx-4VrhJF-938M7B .
21. Edwin Torres. https://commons.wikimedia.org/wiki/File:Classic_BINGO_game_(6679739315)_(2).jpg .
22. Inductiveload. https://de.wikipedia.org/wiki/Datei:Normal_Distribution_PDF.svg .
23. By Heds 1 at English Wikipedia [Public domain], via Wikimedia Commons. https://upload.wikimedia.org

/wikipedia/commons/2/25/The_Normal_Distribution.svg .
24. CK-12 Foundation. .
25. Rhoda Baer . https://commons.wikimedia.org/wiki/File:Doctors_examine_teenager.jpg .
26. Schutz. https://commons.wikimedia.org/wiki/File:Anscombe.svg .
27. CJ. https://www.flickr.com/photos/cjdjkobe/5624576451 .
28. skeeze. https://pixabay.com/en/swimming-competition-swimmers-pool-659903/ .

CHAPTER 15 A Preview of Calculus

Chapter Outline

15.1 Introduction: A Preview of Calculus

While calculus has ancient roots, Sir Issac Newton (below left) and Gottfried Wilhelm Leibniz (below right) developed modern calculus a few hundred years ago to account for subtle paradoxes in their representations of the physical world. Calculus is Latin for "small pebble used for counting." It was originally based on the summation of counting infinitesimal differences. In other words, what happens when you add an infinite number of infinitely small numbers together? What happens when you divide an infinitely small number (dy) by another infinitely small number (dx)? Calculus is basically the study of change. It includes a system of tools and a way of thinking about infinity that help make sense of these perceived paradoxes.

15.2 Limit Notation

Learning Objectives

Learn to write and read limit notation and use limit notation to describe the behavior of a function at a point and at infinity.

Introduction

The total worldwide box-office receipts for a blockbuster movie can be approximated by the function $f(x) = \frac{120x^2-10}{4x^2+1}$, where $f(x)$ is measured in millions of dollars, and x is the number of months since the movie's release. Determine the movie's gross earnings for the long run. As x gets extremely large, the function $f(x)$ approaches $\frac{120}{4} = 30$, because the greatest powers are equal and $\frac{120}{4}$ is the ratio of the leading coefficients. However, how is this statement represented using limit notation?

Limit Notation

When learning about the end behavior of a rational function, you describe the function as either having a horizontal asymptote at some number, or going to infinity. Limit notation is a way of describing this end behavior mathematically. It is a way of expressing the fact that the function gets arbitrarily close to a value.

Limit Notation

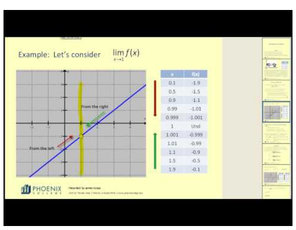
The limit of f of x as x approaches a is b.

$$\lim_{x \to a} f(x) = b$$

Notice that the function $f(x)$ is any function in terms of x. The notion of x approaching a means x is getting sufficiently close to a, but $x \neq a$. Sufficiently close consists of the values that are less than a (to the left of a) and greater than a (to the right of a). The letter a can be any number or infinity.

The letter b can only be a number. If the function goes to infinity, then the function has no limit because infinity is technically not a number. In this case, you should write that the limit does not exist, or DNE.

While a function may never actually reach the value of b, it will get arbitrarily close to b. One way to think about the concept of a limit is to use a physical example. If you stand some distance away from a wall, then take a big step to get halfway to the wall. Take another step to go halfway to the wall again. If you keep taking steps that take you halfway to the wall, then two things will happen. First, you will get extremely close to the wall but never actually reach the wall regardless of how many steps you take. Second, an observer who wishes to describe your situation would notice that the wall acts as a limit to how far you can go.



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/62302

Play, Learn, and Explore Limit Definitions: www.ck12.org/a/1886264

Examples

Example 1

Translate the following statement into limit notation: The limit of $y = 4x^2$ as x approaches 2 is 16.

Solution:

$\lim_{x \to 2} 4x^2 = 16$

Example 2

Translate the mathematical statement below into words.

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i = 1$$

Solution:

The limit of the sum of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ as the number of terms approaches infinity is 1.

Example 3

Use limit notation to represent the following mathematical statement:

$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{1}{2}$.

Solution:

$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{3}\right)^i=\frac{1}{2}$

Example 4

Recall the question from the Introduction: The total worldwide box-office receipts for a blockbuster movie can be approximated by the function $f(x)=\frac{120x^2-10}{4x^2+1}$, where $f(x)$ is measured in millions of dollars, and x is the number of months since the movie's release. As $x$approaches infinity, the function $f(x)$ approaches $\frac{120}{4}=30$. However, how is this statement represented using limit notation?

Solution:

The limit of $f(x)=\frac{120x^2-10}{4x^2+1}$ as x approaches infinity is $\frac{120}{4}=30$ can be written using limit notation as

$$\lim_{x\to\infty}\left(\frac{120x^2-10}{4x^2+1}\right)=\frac{120}{4}=30.$$

Example 5

Describe the end behavior of the following rational function at infinity and negative infinity using limits:

$f(x)=\frac{-5x^3+4x^2-10}{10x^3+3x^2+98}$.

Solution:

Since the function has equal powers of x in the numerator and in the denominator, the end behavior is $-\frac{1}{2}$ as x goes to both positive and negative infinity.

$\lim_{x\to\infty}\left(\frac{-5x^3+4x^2-10}{10x^3+3x^2+98}\right)=\lim_{x\to-\infty}\left(\frac{-5x^3+4x^2-10}{10x^3+3x^2+98}\right)=-\frac{1}{2}$

Example 6

Translate the following limit expression into words:

$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)=x.$

Solution:

The limit of the ratio of the difference between f of quantity x plus h and f of x and h as h approaches 0 is x.

Example 7

What do you notice about the limit expression in Example 6?

Solution:

You should notice that $h\to 0$ does not mean $h=0$, because if it did you could not have a 0 in the denominator. You should also note that in the numerator, $f(x+h)$ and $f(x)$ are going to be super close together as h approaches 0. Calculus will enable you to deal with problems that seem to look like $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Summary

- **Limit notation** is a way of expressing the fact that the function gets arbitrarily close to a value.
- The limit of f of x as x approaches a is b is written as $\lim_{x \to a} f(x) = b$.
- If the function goes to infinity, then the limit does not exist, or DNE.

Review

Describe the end behavior of the following rational functions at infinity and negative infinity using limits:

1. $f(x) = \frac{2x^4+4x^2-1}{5x^4+3x+9}$

2. $g(x) = \frac{8x^3+4x^2-1}{2x^3+4x+7}$

3. $f(x) = \frac{x^2+2x^3-3}{5x^3+x+4}$

4. $f(x) = \frac{4x+4x^2-5}{2x^2+3x+3}$

5. $f(x) = \frac{3x^2+4x^3+4}{6x^3+3x^2+6}$

Translate the following statements into limit notation:

6. The limit of $y = 2x^2 + 1$ as x approaches 3 is 19.

7. The limit of $y = e^x$ as x approaches negative infinity is 0.

8. The limit of $y = \frac{1}{x}$ as x approaches infinity is 0.

Use limit notation to represent the following mathematical statements:

9. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1}{3}$

10. The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

11. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$

12. $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = 1$

Translate the following mathematical statements into words:

13. $\lim_{x \to 0} \frac{5x^2-4}{x+1} = -4$

14. $\lim_{x \to 1} \frac{x^3-1}{x-1} = 3$

15. If $\lim_{x \to a} f(x) = b$, is it possible that $f(a) = b$? Explain.

Review (Answers)

Please see the Appendix.

15.3 Graphs to Find Limits

Learning Objectives

Learn to use graphs to help you evaluate limits and refine your understanding of what a limit represents.

Introduction

To ship a package overnight, a delivery service charges \$18 for the 1st pound, and \$2 for each additional pound or portion of a pound. The total cost can be represented by the function $f(x) = \{\$18 \quad 02,$ where x is the number of pounds of the package. If the package weighs 5 pounds, what is the limit of the cost function?

One-sided Limits

A one-sided limit can be evaluated either from the left or from the right. Since left and right are not absolute directions, a more precise way of thinking about direction is from the negative side or from the positive side. The notation for these one-sided limits is

$$\lim_{x \to a^-} f(x) \text{ and } \lim_{x \to a^+} f(x).$$

The negative superscript on a is not an exponent, but rather it indicates from the negative side. Likewise, the positive superscript is not an exponent, but rather it indicates from the positive side. When evaluating one-sided limits, consider only what value the function is approaching on the one side of the x-value, regardless of what the function is doing at the actual point or on the other side of the number.

One-sided Limits

The limit of $f(x)$ as x approaches a from the left side is L_1:

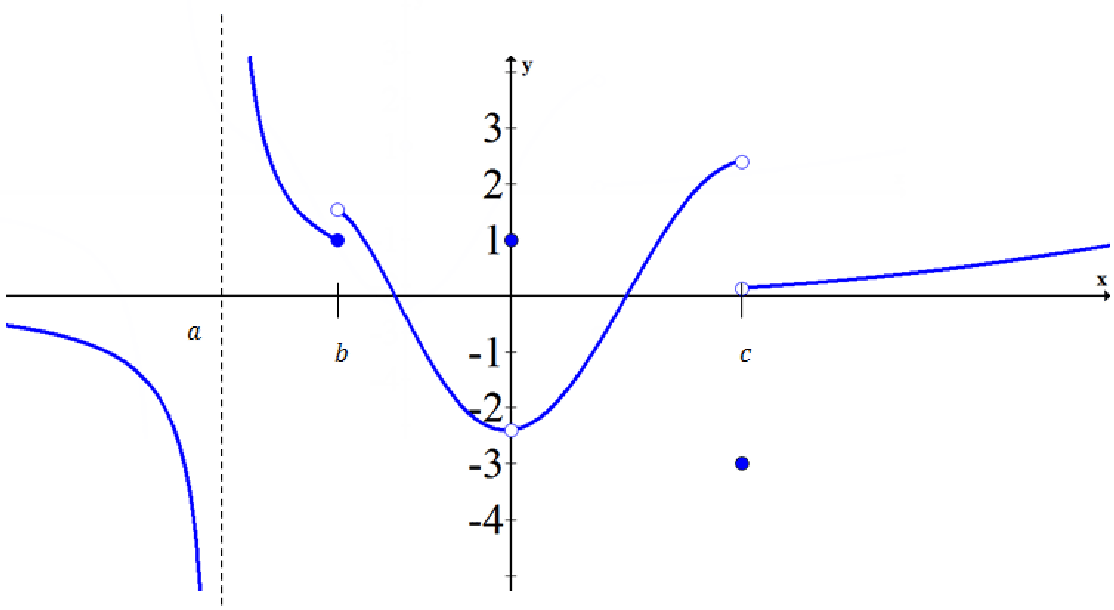
$$\lim_{x \to a^-} f(x) = L_1.$$

The limit of $f(x)$ as x approaches a from the right side is L_2:

$$\lim_{x \to a^+} f(x) = L_2.$$

Graphing to Find a Limit

One method of finding the limit of a function is by graphing. When evaluating the limit of a function from its graph, you need to distinguish between the function evaluated at the point, and the limit approaching the point.



Functions like the one above with discontinuities, asymptotes, and holes require you to have a very solid understanding of how to evaluate and interpret limits.

When you evaluate limits graphically, your main goal is to determine whether the limit exists. The limit exists only when the left and right one-sided limits are equal. The function value at that point is irrelevant in respect to the limit at that point. A function could be defined or undefined at that point, but the limit of the function at that point exists only if the one-sided limits are equal.

Existence of a Limit

The limit of $f(x)$ as x approaches a exists if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = L.$$

For instance, for the graph above, the limit of the function as x approaches 0 from the left is -2.4, and from the right is -2.4. Even though the function has a discontinuity in the form of a hole at $x = 0$, the two one-sided limits equal -2.4, so the limit of the function as x approaches 0 is -2.4. Note that when you evaluate the function at 0, the function value is the actual y-value on the graph, $(0, 1)$.

$$\lim_{x \to 0} f(x) = -2.4 \text{ and } f(0) = 1$$

At $x = b$, the limit of the function from the left is 1, and the limit of the function from the right is 1.8. Since these one-sided limits are not equal, the limit does not exist. The function value at $x = b$ is 1, because the closed dot at $(b, 1)$ indicates that the function is defined at this point.

$$\lim_{x \to b} f(x) = \text{DNE and } f(b) = 1$$

Similarly, the two one-sided limits at $x = c$ are not equal, so the limit does not exist. The function is defined at $x = c$ with a function value of -3.

$$\lim_{x \to c} f(x) = \text{DNE and } f(c) = -3$$

At $x = a$, the function is undefined because there is a vertical asymptote. Since the function approaches different values from the left and from the right of $x = a$, the limit does not exist.

$$\lim_{x \to a} f(x) = \text{DNE and } f(a) = \text{DNE}$$

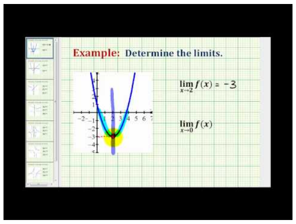
The graph of the function appears to flatten as it moves to the left. Thus, there is a horizontal asymptote at $y = 0$ as $x \to -\infty$.

$$\lim_{x \to -\infty} f(x) = 0$$

However, the graph of the function appears to continue to increase without bound as it moves to the right. Thus, the limit of the function does not exist as $x \to \infty$.

$$\lim_{x \to \infty} f(x) = \text{DNE}$$

Another example of these ideas can be found in the following video:



MEDIA

Click image to the left or use the URL below.

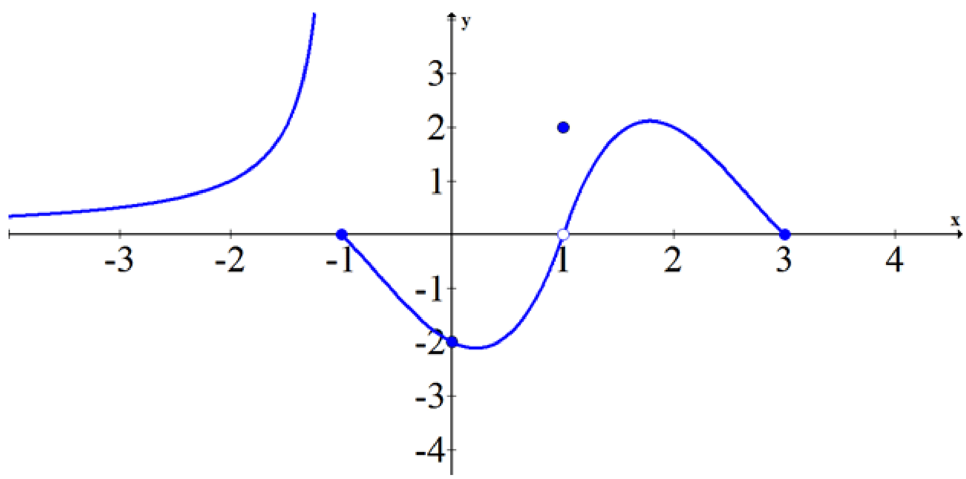
URL: http://www.ck12.org/flx/render/embeddedobject/62304

Play, Learn, and Explore One-sided Limits: www.ck12.org/a/2116523

Examples

Example 1

Evaluate the following expressions using the graph of the function $f(x)$:



a. $\lim_{x \to -\infty} f(x)$

Solution:

$\lim_{x \to -\infty} f(x) = 0$

b. $\lim_{x \to -1} f(x)$

Solution:

$\lim_{x \to -1} f(x) = DNE$

c. $\lim_{x \to 0} f(x)$

Solution:

$\lim_{x \to 0} f(x) = -2$

d. $\lim_{x \to 1} f(x)$

Solution:

$\lim_{x \to 1} f(x) = 0$

e. $\lim_{x \to 3} f(x)$

Solution:

$\lim_{x \to 3} f(x) = DNE$ because only the limit from the left side exists, and therefore the two one-sided limits do not exist.

f. $f(-1)$

Solution:

$f(-1) = 0$

g. $f(0)$

Solution:

$f(0) = -2$

h. $f(1)$

Solution:

$f(1) = 2$

i. $f(3)$

Solution:

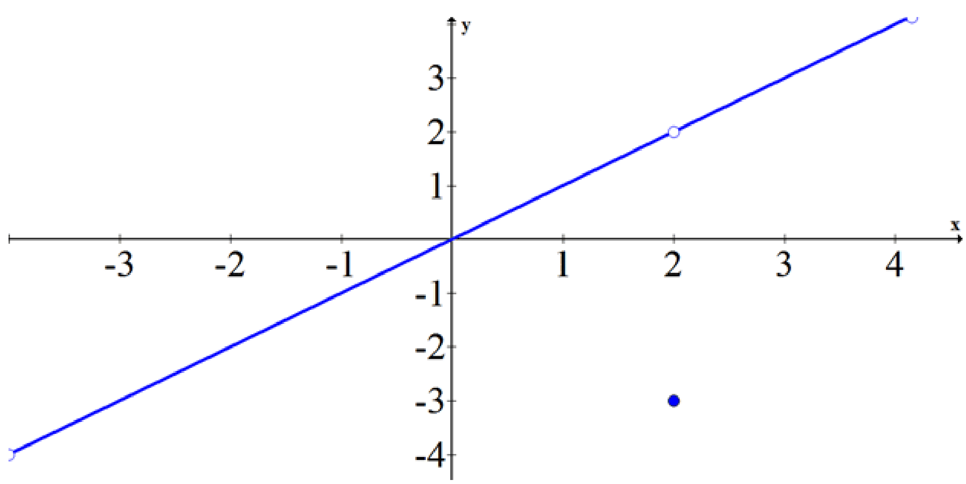
$f(3) = 0$

Example 2

Sketch a graph that has a limit at $x = 2$ where the limit value does not match the function value at that point.

Solution:

While there are an infinite number of graphs that fit this criteria, you should make sure your graph has a removable discontinuity at $x = 2$.

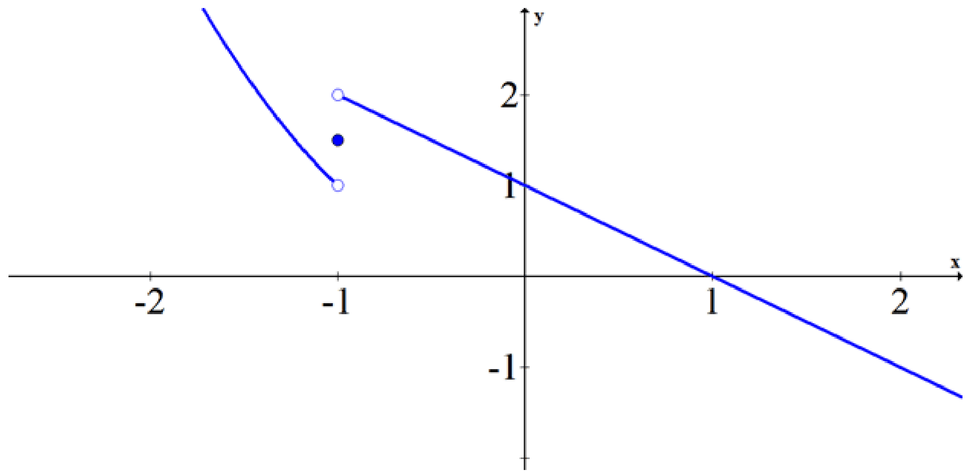


Example 3

Sketch a graph that is defined at $x = -1$, but $\lim_{x \to -1} f(x)$ does not exist.

Solution:

The graph must have either a jump or an infinite discontinuity at $x = -1$, and also have a closed hole filled in somewhere on that vertical line.



Example 4

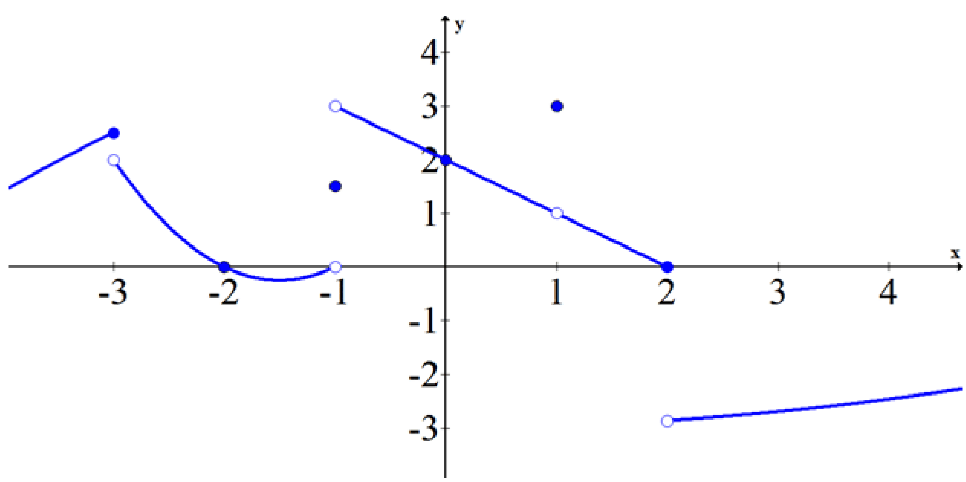
Recall the problem from the Introduction: Determine the limit of the cost function of shipping a 5-pound package overnight. The total cost is represented by $f(x) = \{\$18 \quad 02$ where x is the number of pounds of the package.

Solution:

The graph of the cost function is shown below. The limit to the left of $x = 5$ is 26, and the limit to the right of $x = 5$ is 28. Since these two one-sided limits are not equal, the limit of the function as the package weighs 5 pounds does not exist.

Example 5

Identify where the limit exists and where the limit does not exist for the following function:



Solution:

The limit does not exist at $x = -3, -1, 2, +\infty,$ and $-\infty$. The limit exists at every other point on the graph, including $x = 1$.

Example 6

Evaluate and explain how to find the limits as x approaches 0 and 1 in the previous question.

Solution:

Both of these limits exist because the two one-sided limits at these points are equal. As $x$approaches 0 from the left and from the right, the function value approaches 2. The function value at $x = 0$ is also defined at 2. As x approaches 1 from the left and from the right, the function value approaches 1. Even though the function value is defined elsewhere for $x = 1$, the limit value is still 1.

$$\lim_{x \to 0} f(x) = 2 \text{ and } \lim_{x \to 1} f(x) = 1$$

Example 7

Evaluate the limits of the following piecewise function at -2, 0, and 1:

$$f(x) = \begin{cases} 2 & x < -2 \\ -1 & x = -2 \\ -x-2 & -2 < x \leq 0 \\ x^2 & 0 < x < 1 \\ -2 & x = 1 \\ x^2 & 1 < x \end{cases}$$

Solution:

$$\lim_{x \to -2^-} f(x) = 2$$
$$\lim_{x \to -2^+} f(x) = 0$$
$$\lim_{x \to -2} f(x) = DNE$$

$$\lim_{x \to 0^-} f(x) = -2$$
$$\lim_{x \to 0^+} f(x) = 0$$
$$\lim_{x \to 0} f(x) = DNE$$

$$\lim_{x \to 1} f(x) = 1$$

Summary

- The limit of $f(x)$ as x approaches a from the left side is denoted $\lim_{x \to a^-} f(x) = L_1$.

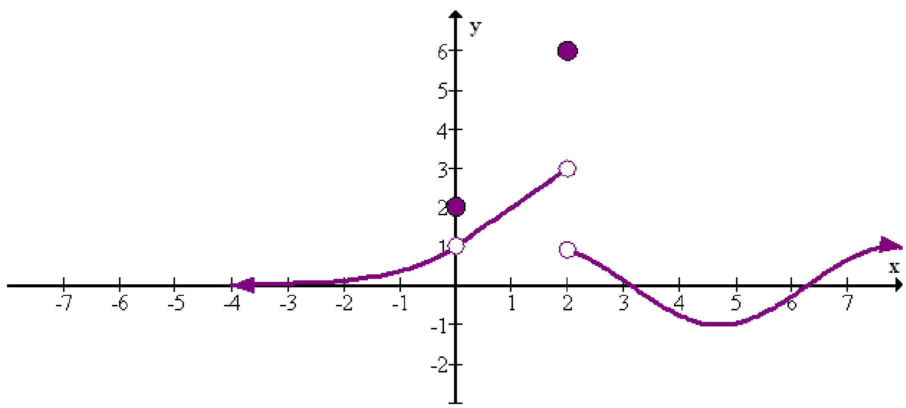
- The limit of $f(x)$ as x approaches a from the right side is denoted $\lim_{x \to a^+} f(x) = L_2$.
- The limit only exists when the left and right one-sided limits are equal.

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = L$$

- If the left and right one-sided limits are not equal, then the limit does not exist, or DNE.

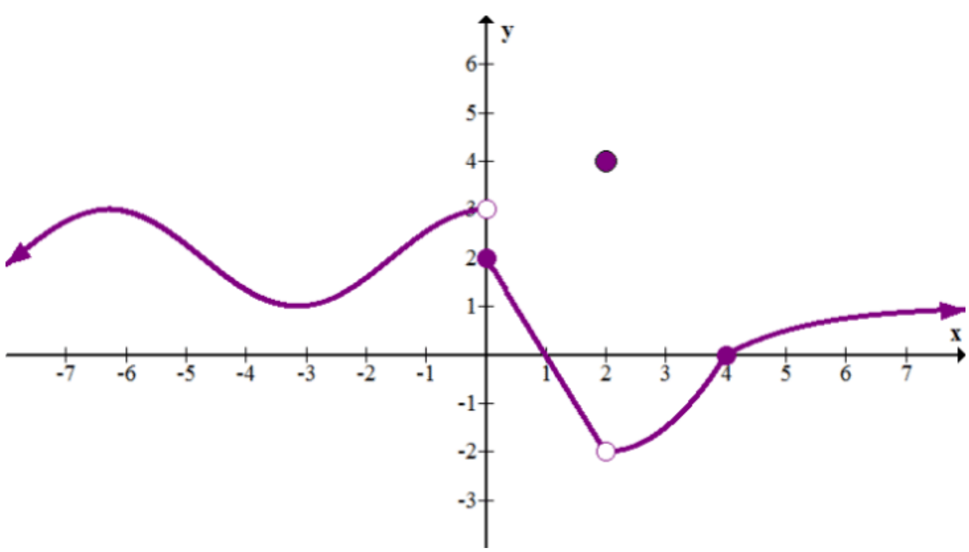
Review

Use the graph of $f(x)$ below to evaluate the expressions in 1-6.



1. $\lim_{x \to -\infty} f(x)$
2. $\lim_{x \to \infty} f(x)$
3. $\lim_{x \to 2} f(x)$
4. $\lim_{x \to 0} f(x)$
5. $f(0)$
6. $f(2)$

Use the graph of $g(x)$ below to evaluate the expressions in 7-13.



7. $\lim_{x \to -\infty} g(x)$
8. $\lim_{x \to \infty} g(x)$
9. $\lim_{x \to 2} g(x)$
10. $\lim_{x \to 0} g(x)$
11. $\lim_{x \to 4} g(x)$
12. $g(0)$
13. $g(2)$
14. Sketch a function $h(x)$ such that $h(2) = 4$, but $\lim_{x \to 2} h(x) = DNE$.
15. Sketch a function $j(x)$ such that $j(2) = 4$, but $\lim_{x \to 2} j(x) = 3$.

Review (Answers)

Please see the Appendix.

15.4 Tables to Find Limits

Learning Objectives

Learn to estimate limits using tables.

Introduction

Recall that a limit is the value the function approaches as x gets arbitrarily close to a certain number or infinity. Suppose you are 5 feet away from a fan. Then you take a step halfway closer to the fan. You repeat this process until you are nearly on top of the fan. However, you will never actually be standing in the same place as the fan. How can this situation be described using a limit and table?

Limits from Tables

A second way to solve limit problems is to create a table. The x-values in this table are numbers to the left and to the right (on the number line) of the number x in the limit. For instance, find the limit of the following:

$$\lim_{x \to 4} \frac{x^2 - 2x - 8}{x - 4}.$$

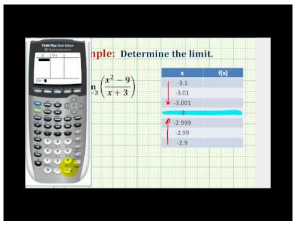
A few x-values are close to 4 on the left and on the right include 3.9, 3.99, 3.999, 4.001, 4.01, and 4.1. Plug these x-values into the function to obtain the function value at this point.

TABLE 15.1:

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$	5.9	5.99	5.999	?	6.001	6.01	6.1

Notice that the function values in the bottom row approach the number 6 from the left and the right. Using this table, you could conclude that the limit of the given function is 6 as x approaches 4.

Another example of this solving approach can be seen in the following video:



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/62306

Play, Learn, and Explore Tables to Find Limits: www.ck12.org/a/2325386

Examples

Example 1

Complete the table and use the result to estimate the limit.

$$\lim_{x \to 2} \frac{x-2}{x^2-x-2}$$

TABLE 15.2:

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

Solution:

TABLE 15.3:

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.34483	0.33445	0.33344	0.33322	0.33223	0.32258

By this table, the limit is $\frac{1}{3}$.

Example 2

Complete the table and use the result to estimate the limit.

$$\lim_{x \to 2} \frac{x-2}{x^2-4}$$

TABLE 15.4:

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

Solution:

TABLE 15.5:

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.25641	0.25063	0.25006	0.24994	0.24938	0.2439

By this table, the limit is $\frac{1}{4}$.

Example 3

Complete the table and use the result to estimate the limit.

$$\lim_{x \to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x}$$

TABLE 15.6:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

Solution:

TABLE 15.7:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.29112	0.28892	0.2887	0.28865	0.28843	0.28631

By this table, the limit is between 0.2887 and 0.28865. When you learn to find the limit analytically, you will know that the exact limit is $\frac{1}{2} \cdot 3\frac{1}{2} \approx 0.2886751346$.

Example 4

Recall the question from the Introduction: You are 5 feet away from a fan and are taking steps closer to the fan. How can this situation be described using a limit and table?

Solution:

The limit representing this situation would be $\lim_{x \to 0} f(x)$ if you define the function so the fan is at 0 and you are at 5 or -5 to start. In the table, the x-values would be to the left and to the right of 0.

Example 5

Graph the following function and the use a table to verify the limit as x approaches 1:

$$f(x) = \frac{x^3 - 1}{x - 1}, x \neq 1.$$

Solution:

TABLE 15.8:

x	$f(x)$
.75	2.3125
.9	2.71
.99	2.9701
.999	2.997
1	Error
1.001	3.003
1.01	3.0301
1.1	3.31
1.25	3.8125

By this table, the limit is 3.

Example 6

Estimate the limit numerically.

$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2}$$

Solution:

TABLE 15.9:

x	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25
$f(x)$	0.75	0.9	0.99	0.999	Error	1.001	1.01	1.1	1.25

By this table, the limit is 1.

Example 7

Estimate the limit numerically.

$$\lim_{x\to 0} \frac{\left[\frac{4}{x+2}\right]-2}{x}$$

Solution:

TABLE 15.10:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.20526	0.02005	0.002	-0.002	-0.02	-0.1952

By this table, the limit is 0.

Summary

- To solve limits numerically, use a table with x-values to the left and to the right on the number line to the number x in the limit.
- Then plug these x-values into the function to obtain the function value at this point.

Review

Estimate the following limits numerically using a table:

1. $\lim_{x\to 5} \frac{x^2-25}{x-5}$
2. $\lim_{x\to -1} \frac{x^2-3x-4}{x+1}$
3. $\lim_{x\to 2} \frac{x^3-5x^2+2x-4}{x^2-3x+2}$
4. $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$
5. $\lim_{x\to 3} \left(\frac{1}{x-3} - \frac{9}{x^2-9}\right)$
6. $\lim_{x\to 2} \frac{x^2+5x-14}{x-2}$
7. $\lim_{x\to 1} \frac{x^2-8x+7}{x-1}$
8. $\lim_{x\to 0} \frac{\sqrt{x+5}-\sqrt{5}}{x}$
9. $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$
10. $\lim_{x\to 0} \frac{x^2+5x}{x}$
11. $\lim_{x\to -3} \frac{x^2-9}{x+3}$
12. $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$
13. $\lim_{x\to 2} \frac{\sqrt{x+3}-2}{x-1}$
14. $\lim_{x\to 5} \frac{x^2-25}{x^3-125}$
15. $\lim_{x\to -1} \frac{x-2}{x+1}$

)

Please see the Appendix.

15.5 Substitution to Find Limits

Learning Objectives

Learn the basics of finding limits analytically using substitution.

Introduction

Finding limits for the vast majority of points for a given function is as simple as substituting the number that x approaches into the function. Since this turns evaluating limits into an algebra-level substitution, most questions involving limits focus on the cases where substituting does not work.

Suppose a painter working 100 feet above the ground drops his paint brush to the street. The velocity of the paint brush in feet per second at $t = 2$ is given by the following limit:

$$v(t) = \lim_{t \to 2} \frac{-16t^2 + 64}{t - 2}.$$

Determine the velocity of the paint brush at $t = 2$ given by the limit analytically.

Using Substitution to Find Limits

A 3rd approach to finding a limit is analytically using substitution. Finding a limit analytically means using algebraic approaches to find the limit.

If the function $f(x)$ has no holes or an asymptote at $x = a$, then the limit of the function is equal to the function value.

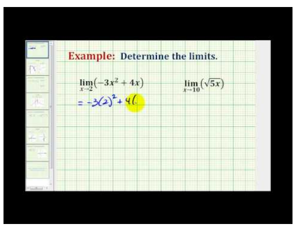
$$\lim_{x\to a} f(x) = f(a)$$

Thus, you can substitute the value that x approaches into the function and evaluate the result. This approach works perfectly when there are no holes or asymptotes at that particular x-value, and you do not divide by zero when substituting.

However, occasionally there will be a hole or asymptote at $x = a$. If the function is a rational expression with a hole, then algebraically factor the numerator and denominator. Next, cancel any common factors in the numerator and denominator. Finally, substitute the value that $x$approaches into the resulting expression. Thus, the limit in this case is the function value as if the hole did not exist.

If no factors can be canceled or the function has an asymptote, the limit likely does not exist at that point. Try another approach to confirm this conclusion.

An example of this approach can be seen in the following video:



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Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/62310

Play, Learn, and Explore to Determine Limits: www.ck12.org/a/2209436

Examples

Example 1

Which of the limits below can you determine using direct substitution? Find that limit.

$$\lim_{x\to 2} \frac{x^2-4}{x-2}, \lim_{x\to 3} \frac{x^2-4}{x-2}$$

Solution:

The limit on the right can be evaluated using direct substitution. The rational expression on the left has a hole at $x = 2$, so it would need to be simplified first.

$$\lim_{x\to 3} \frac{x^2-4}{x-2} = \frac{3^2-4}{3-2} = \frac{9-4}{1} = 5$$

Example 2

Evaluate the following limit analytically:

$$\lim_{x\to 2} \frac{x^2-4}{x-2}.$$

Solution:

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-4}{x-2} &= \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}\\ &= \lim_{x\to 2}(x+2)\\ &= 2+2\\ &= 4\end{aligned}$$

Example 3

Evaluate the following limit analytically:

$$\lim_{x\to 4}\frac{x^2-x-12}{x-4}.$$

Solution:

$$\begin{aligned}\lim_{x\to 4}\frac{x^2-x-12}{x-4} &= \lim_{x\to 4}\frac{(x-4)(x+3)}{(x-4)}\\ &= \lim_{x\to 4}(x+3)\\ &= 4+3\\ &= 7\end{aligned}$$

Example 4

Recall the question from the Introduction: A painter working 100 feet above the ground drops his paint brush to the street. The velocity of the paint brush in feet per second at $t = 2$ is given by the following limit:

$$v(t) = \lim_{t\to 2}\frac{-16t^2+64}{t-2}.$$

Determine the velocity of the paint brush at $t = 2$ given by the limit analytically.

Solution:

$$\begin{aligned}v(2) &= \lim_{t\to 2}\frac{-16t^2+64}{t-2}\\ &= \lim_{t\to 2}\frac{-16(t^2-4)}{t-2}\\ &= \lim_{t\to 2}\frac{-16(t-2)(t+2)}{t-2}\\ &= \lim_{t\to 2}-16(t+2)\\ &= -64 \text{ ft/s}\end{aligned}$$

Example 5

Evaluate the following limit analytically:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}.$$

Solution:

Example 6

Evaluate the following limit analytically:

$$\lim_{t \to 4} \sqrt{t + 32}.$$

Solution:

$$\lim_{t \to 4} \sqrt{t + 32} = \sqrt{4 + 32} = \sqrt{36} = 6$$

Example 7

Evaluate the following limit analytically:

$$\lim_{y \to 4} \frac{3|y - 1|}{y + 4}.$$

Solution:

$$\lim_{y \to 4} \frac{3|y - 1|}{y + 4} = \frac{3|4 - 1|}{4 + 4} = \frac{3 \cdot 3}{8} = \frac{9}{8}$$

Summary

- **Substitution** is a method of determining limits where the value that x is approaching is substituted into the function and the result is evaluated.
- ***If the function*** $f(x)$ ***has no holes or asymptote at*** $x = a$***, then*** $\lim_{x \to a} f(x) = f(a)$**.**
- If the function is a rational expression with a hole, then algebraically factor the numerator and denominator. Next, cancel any common factors in the numerator and denominator. Finally, substitute the value that x approaches into the resulting expression.
- If no factors can be canceled or the function has an asymptote, the limit likely does not exist at that point. Try another approach to confirm this conclusion.

Review

Evaluate the following limits analytically:

1. $\lim_{x \to 5} \frac{x^2-25}{x-5}$
2. $\lim_{x \to -1} \frac{x^2-3x-4}{x+1}$
3. $\lim_{x \to 5} \sqrt{5x}-12$
4. $\lim_{x \to 0} \frac{x^3+3x^2-x}{5x}$
5. $\lim_{x \to 1} \frac{3x|x-4|}{x+1}$
6. $\lim_{x \to 2} \frac{x^2+5x-14}{x-2}$
7. $\lim_{x \to 1} \frac{x^2-8x+7}{x-1}$
8. $\lim_{x \to 0} \frac{5x-1}{2x^2+3}$
9. $\lim_{x \to 1} 4x^2-2x+5$
10. $\lim_{x \to 0} \frac{x^2+5x}{x}$
11. $\lim_{x \to -3} \frac{x^2-9}{x+3}$
12. $\lim_{x \to 0} \frac{5x+1}{x}$
13. $\lim_{x \to 1} \frac{5x+1}{x}$
14. $\lim_{x \to 5} \frac{x^2-25}{x^3-125}$
15. $\lim_{x \to -1} \frac{x-2}{x+1}$

Review (Answers)

Please see the Appendix.

15.6 Rationalization to Find Limits

Learning Objectives

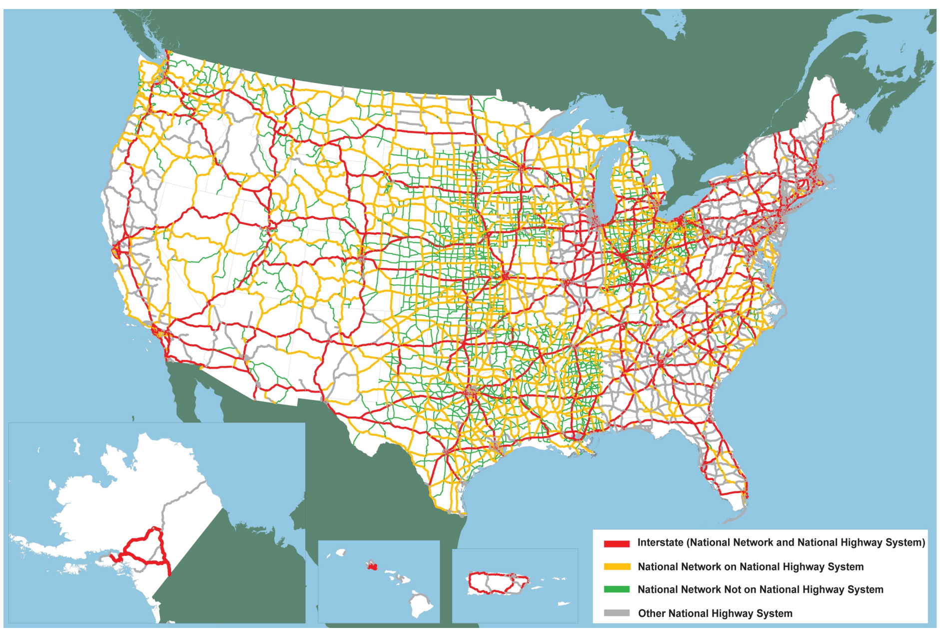
Learn to evaluate limits analytically using rationalization.

Introduction

Suppose you and a friend are driving across the country on a road trip. The amount of time the two of you can drive is modeled with the following limit, where 16 hours is the most amount of time you and your friend can drive:

$$\lim_{x \to 16} \frac{\sqrt{x}-4}{x-16}.$$

How do you evaluate the limit using rationalization?



Rationalization to Find Limits

Some limits cannot be evaluated directly by substitution, and no factors immediately cancel. In these situations, there is another algebraic technique to try called rationalization. With rationalization, you make the numerator and the denominator of an expression rational by using the properties of conjugate pairs.

Conjugates can be used to simplify expressions with a radical in the denominator:

$$\frac{5}{1+\sqrt{3}} = \frac{5}{\left(1+\sqrt{3}\right)} \cdot \frac{\left(1-\sqrt{3}\right)}{\left(1-\sqrt{3}\right)} = \frac{5-5\sqrt{3}}{1-3} = \frac{5-5\sqrt{3}}{-2}.$$

Conjugates can be used to simplify complex numbers with i in the denominator:

$$\frac{4}{2+3i} = \frac{4}{(2+3i)} \cdot \frac{(2-3i)}{(2-3i)} = \frac{8-12i}{4+9} = \frac{8-12i}{13}.$$

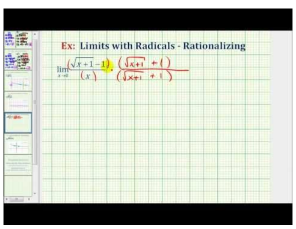
Here they can be used to transform an expression in a limit problem that does not immediately factor to one that does immediately factor:

$$\lim_{x \to 16} \frac{(\sqrt{x}-4)}{(x-16)} \cdot \frac{(\sqrt{x}+4)}{(\sqrt{x}+4)} = \lim_{x \to 16} \frac{(x-16)}{(x-16)(\sqrt{x}+4)}.$$

Now you can cancel the common factors in the numerator and denominator, and use substitution to finish evaluating the limit.

The rationalizing technique works because when you algebraically manipulate the expression in the limit to an equivalent expression, the resulting limit will be the same. Sometimes you must do a variety of different algebraic manipulations to avoid a zero in the denominator when using the substitution method.

An example of this approach can be seen in the following video:



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Examples

Example 1

Evaluate the following limit:

$$\lim_{x \to 3} \frac{x^2-9}{\sqrt{x}-\sqrt{3}}.$$

Solution:

$$\begin{aligned}\lim_{x \to 3} \frac{(x-3)(x+3)}{(\sqrt{x}-\sqrt{3})} \cdot \frac{(\sqrt{x}+\sqrt{3})}{(\sqrt{x}+\sqrt{3})} &= \lim_{x \to 3} \frac{(x-3)(x+3)(\sqrt{x}+\sqrt{3})}{(x-3)} \\ &= \lim_{x \to 3}(x+3)\left(\sqrt{x}+\sqrt{3}\right) \\ &= 6 \cdot 2\sqrt{3} \\ &= 12\sqrt{3}\end{aligned}$$

Example 2

Evaluate the following limit:

$$\lim_{x\to 25} \frac{x-25}{\sqrt{x}-5}.$$

Solution:

$$\begin{aligned}
\lim_{x\to 25} \frac{x-25}{\sqrt{x}-5} &= \lim_{x\to 25} \frac{(x-25)}{(\sqrt{x}-5)} \cdot \frac{(\sqrt{x}+5)}{(\sqrt{x}+5)} \\
&= \lim_{x\to 25} \frac{(x-25)(\sqrt{x}+5)}{(x-25)} \\
&= \lim_{x\to 25} (\sqrt{x}+5) \\
&= \sqrt{25}+5 \\
&= 10
\end{aligned}$$

Example 3

Evaluate the following limit:

$$\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7}.$$

Solution:

$$\begin{aligned}
\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} &= \lim_{x\to 7} \frac{(\sqrt{x+2}-3)}{(x-7)} \cdot \frac{(\sqrt{x+2}+3)}{(\sqrt{x+2}+3)} \\
&= \lim_{x\to 7} \frac{(x+2-9)}{(x-7)\cdot(\sqrt{x+2}+3)} \\
&= \lim_{x\to 7} \frac{(x-7)}{(x-7)\cdot(\sqrt{x+2}+3)} \\
&= \lim_{x\to 7} \frac{1}{(\sqrt{x+2}+3)} \\
&= \frac{1}{\sqrt{7+2}+3} \\
&= \frac{1}{6}
\end{aligned}$$

Example 4

Recall the question from the Introduction: The amount of time you and your friend can drive is modeled with the following limit, where 16 hours is the most amount of time you and your friend can drive:

$$\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16}$$

How do you evaluate the limit using rationalization?

Solution:

$$\begin{aligned}
\lim_{x\to 16}\frac{\sqrt{x}-4}{x-16} &= \lim_{x\to 16}\frac{(\sqrt{x}-4)}{(x-16)}\cdot\frac{(\sqrt{x}+4)}{(\sqrt{x}+4)}\\
&= \lim_{x\to 16}\frac{(x-16)}{(x-16)(\sqrt{x}+4)}\\
&= \lim_{x\to 16}\frac{1}{\sqrt{x}+4}\\
&= \frac{1}{4+4}\\
&= \frac{1}{8}
\end{aligned}$$

Example 5

Evaluate the following limit:

$$\lim_{x\to 0}\frac{(2+x)^{-1}-2^{-1}}{x}.$$

Solution:

Even though the given limit does not have radical expressions, the rationalization process learned in this concept is similar to the process used to find the given limit.

$$\begin{aligned}
\lim_{x\to 0}\frac{(2+x)^{-1}-2^{-1}}{x} &= \lim_{x\to 0}\frac{\frac{1}{x+2}-\frac{1}{2}}{x}\cdot\frac{(x+2)\cdot 2}{(x+2)\cdot 2}\\
&= \lim_{x\to 0}\frac{2-(x+2)}{2x(x+2)}\\
&= \lim_{x\to 0}\frac{-x}{2x(x+2)}\\
&= \lim_{x\to 0}\frac{-1}{2(x+2)}\\
&= -\frac{1}{2(0+2)}\\
&= -\frac{1}{4}
\end{aligned}$$

Example 6

Evaluate the following limit:

$$\lim_{x\to -3}\frac{\sqrt{x^2-5}-2}{x+3}.$$

Solution:

$$\begin{aligned}
\lim_{x\to -3}\frac{\sqrt{x^2-5}-2}{x+3} &= \lim_{x\to -3}\frac{\left(\sqrt{x^2-5}-2\right)}{(x+3)}\cdot\frac{\left(\sqrt{x^2-5}+2\right)}{\left(\sqrt{x^2-5}+2\right)}\\
&= \lim_{x\to -3}\frac{(x^2-5-4)}{(x+3)\left(\sqrt{x^2-5}+2\right)}\\
&= \lim_{x\to -3}\frac{(x^2-9)}{(x+3)\left(\sqrt{x^2-5}+2\right)}\\
&= \lim_{x\to -3}\frac{(x-3)(x+3)}{(x+3)\left(\sqrt{x^2-5}+2\right)}\\
&= \lim_{x\to -3}\frac{x-3}{\sqrt{x^2-5}+2}\\
&= \frac{-3-3}{\sqrt{9-5}+2}\\
&= \frac{-6}{2+2}\\
&= \frac{-6}{4}=\frac{-3}{2}
\end{aligned}$$

Example 7

Evaluate the following limit:

$$\lim_{x\to 0}\left(\frac{3}{x\sqrt{9-x}}-\frac{1}{x}\right).$$

Solution:

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{3}{x\sqrt{9-x}}-\frac{1}{x}\right) &= \lim_{x\to 0}\left(\frac{3}{x\sqrt{9-x}}-\frac{\sqrt{9-x}}{x\sqrt{9-x}}\right)\\
&= \lim_{x\to 0}\left(\frac{3-\sqrt{9-x}}{x\sqrt{9-x}}\right)\\
&= \lim_{x\to 0}\left(\frac{\left(3-\sqrt{9-x}\right)}{x\sqrt{9-x}}\cdot\frac{\left(3+\sqrt{9-x}\right)}{\left(3+\sqrt{9-x}\right)}\right)\\
&= \lim_{x\to 0}\left(\frac{9-(9-x)}{x\sqrt{9-x}\left(3+\sqrt{9-x}\right)}\right)\\
&= \lim_{x\to 0}\frac{x}{x\sqrt{9-x}\left(3+\sqrt{9-x}\right)}\\
&= \lim_{x\to 0}\frac{1}{\sqrt{9-x}\left(3+\sqrt{9-x}\right)}\\
&= \frac{1}{\sqrt{9}\left(3+\sqrt{9}\right)}\\
&= \frac{1}{3(3+3)}\\
&= \frac{1}{3(6)}\\
&= \frac{1}{18}
\end{aligned}$$

Summary

- **Rationalization** is a technique used to evaluate limits to avoid having a zero in the denominator when you substitute.
- With rationalization, you make the numerator and the denominator of an expression rational by using the properties of conjugate pairs.

Review

Evaluate the following limits:

1. $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$
2. $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$
3. $\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}$
4. $\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x}$
5. $\lim_{x\to 4}\frac{\sqrt{3x+4}-x}{4-x}$
6. $\lim_{x\to 0}\frac{2-\sqrt{x+4}}{x}$
7. $\lim_{x\to 0}\frac{\sqrt{x+7}-\sqrt{7}}{x}$

8. $\lim_{x\to 16} \frac{16-x}{4-\sqrt{x}}$

9. $\lim_{x\to 0} \frac{x^2}{\sqrt{x^2+12}-\sqrt{12}}$

10. $\lim_{x\to 2} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}$

11. $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$

12. $\lim_{x\to \frac{1}{9}} \frac{9x-1}{3\sqrt{x}-1}$

13. $\lim_{x\to 4} \frac{4x^2-64}{2\sqrt{x}-4}$

14. $\lim_{x\to 9} \frac{9x^2-90x+81}{9-3\sqrt{x}}$

15. When given a limit to evaluate, how do you know when to use the rationalization technique? What will the function look like?

Review (Answers)

Please see the Appendix.

15.7 Continuity

Learning Objectives

Learn to determine one sided limits graphically, numerically, and algebraically and to use the concept of a one sided limit to define continuity.

Introduction

Suppose Jack is traveling around his neighborhood. First he is walking, but then switches to riding his bike. His velocity is modeled by the piecewise function below. Is the piecewise function continuous on the interval $x > 0$?

$$f(x) = \begin{cases} -x-2 & 0 < 1 \\ -3 & x = 1 \\ x^2-4 & 1 < x \end{cases}$$

Continuity

In the past, you may have defined continuity as the ability to draw a function completely without lifting your pencil off the paper. If a function could be drawn without lifting your pencil, then it was continuous. If a function could not be drawn without lifting your pencil, then it had some sort of discontinuity and was not a continuous function. You can now define continuity with a more rigorous definition using limits.

Continuity

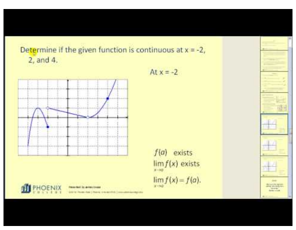
A function $f(x)$ is continuous at $x = a$ if:

1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. The limit of the function is equal to the function value at $x = a$.

$$\lim_{x \to a} f(x) = f(a)$$

A function is **continuous over an interval** if it is continuous at each point in that interval.

The following video further discusses continuity with limits:



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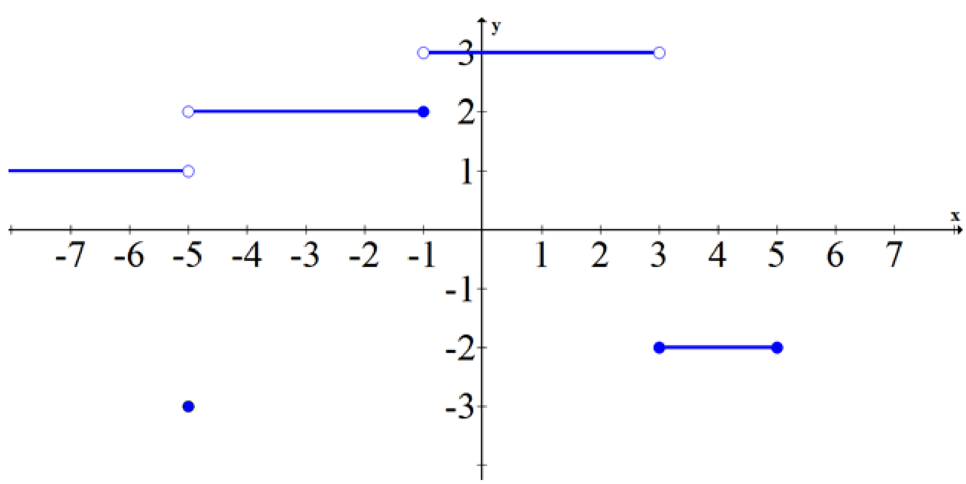
URL: http://www.ck12.org/flx/render/embeddedobject/79236

Play, Learn, and Explore Continuity: www.ck12.org/a/1824114

Examples

Example 1

Where is the function graphed below discontinuous?



Solution:

The function is discontinuous at $x = -5, -1, 3,$ and 5 because the limit does not exist at each of these points.

Example 2

Is the following function continuous?

$$f(x) = \begin{cases} x^2 - 1 & x < -1 \\ 3 & x = -1 \\ -x + 3 & -1 < x \end{cases}$$

Solution:

The piecewise function is continuous if every point of the function is continuous. All x-values must then be checked where the piecewise function changes from one piece to another. For the given function, the x-value where the piecewise function changes from one piece to another is $x = -1$.

Using the definition of continuity:

$$f(-1) = 3$$

$$\lim_{x \to 1^-} f(x) = (-1)^2 - 1 = 1 - 1 = 0$$

$$\lim_{x \to -1^+} f(x) = -1 + 3 = 2.$$

Since the two one-sided limits are not equal, the limit of the function does not exist.

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

Thus, this function is discontinuous at $x = -1$.

Example 3

On what intervals is the function in Example 2 continuous?

Solution:

The function is continuous on the intervals $x < -1$ and $x > -1$.

Example 4

Recall the problem from the Introduction: Jack is first walking around his neighborhood, and then switches to riding his bike. Is the piecewise function continuous on the interval $x > 0$?

$$f(x) = \begin{cases} -x - 2 & 0 < 1 \\ -3 & x = 1 \\ x^2 - 4 & 1 < x \end{cases}$$

Solution:

The piecewise function is continuous on the interval if every point of the function on the interval is continuous. All x-values must then be checked where the piecewise function changes from one piece to another. For the given function, the x-value where the piecewise function changes from one piece to another is $x=1$.

Using the definition of continuity:

$$f(1)=-3$$

$$\lim_{x\to 1^-} f(x)=-1-2=-3$$

$$\lim_{x\to 1^+} f(x)=1^2-4=-3.$$

Since the two one-sided limits are equal, the limit of the function at $x=1$ exists.

$$\lim_{x\to 1} f(x)=-3$$

Since the limit value and the function value at $x=1$ are equal, both are equal to -3, the function is continuous at $x=1$. Also, the function is continuous to the left and right of $x=1$, so the piecewise function is continuous on the interval.

Example 5

Megan argues that according to the definition of continuity, the function below is continuous. She says

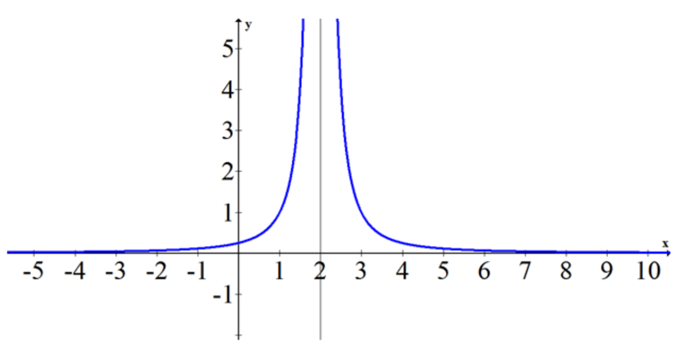
$$\lim_{x\to 2^-} f(x)=\infty$$

$$\lim_{x\to 2^+} f(x)=\infty$$

$$f(2)=\infty.$$

Since $\lim_{x\to 2^-} f(x)=f(2)=\lim_{x\to 2^+} f(x)$, it meets the definition of continuous.

How could you use the graph below to clarify Megan's reasoning?



Solution:

Megan is being extremely liberal with the idea of $= \infty$. Remember that when a limit approaches infinity, the limit does not exist. Two limits that do not exist cannot be equal to one another. Also, the function is undefined at $x = 2$.

Example 6

The function $f(x) = \tan x$ is not continuous. However, determine an interval in which $f(x)$ is continuous.

Solution:

While $\tan x$ is not continuous everywhere, it is continuous over certain intervals. One such interval is $-\frac{\pi}{2} < \frac{\pi}{2}$. Another possible interval is $\frac{\pi}{2} < \frac{3\pi}{2}$.

Example 7

What must k be equal to in order for the following piecewise function to be continuous?

$$f(x) = \begin{cases} 4x-5 & x < 2 \\ k & x = 2 \\ -2x+7 & x > 2 \end{cases}$$

Solution:

Using the definition of continuity:

$$f(2) = k$$

$$\lim_{x \to 2^-} f(x) = 4(2) - 5 = 3$$

$$\lim_{x \to 2^+} f(x) = -2(2) + 7 = 3$$

Since the two one-sided limits equal, the limit of the function at $x = 2$ exists.

$$\lim_{x \to 2} f(x) = 3$$

In order for the piecewise function to be continuous, it must be continuous at every point, including $x = 2$. Thus, the limit value and the function value at $x = 2$ must be equal.

$$\lim_{x \to 2} f(x) = f(2)$$

So, $k = 3$.

Summary

- A function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$, where both $f(a)$ and $\lim_{x\to a} f(x)$ exist.
- A function is **continuous over an interval** if it is continuous at each point in that interval.

Review

Consider

$$f(x) = \begin{cases} 2x^2 - 1 & x < 1 \\ 1 & x = 1 \\ -x + 2 & 1 < x \end{cases}$$

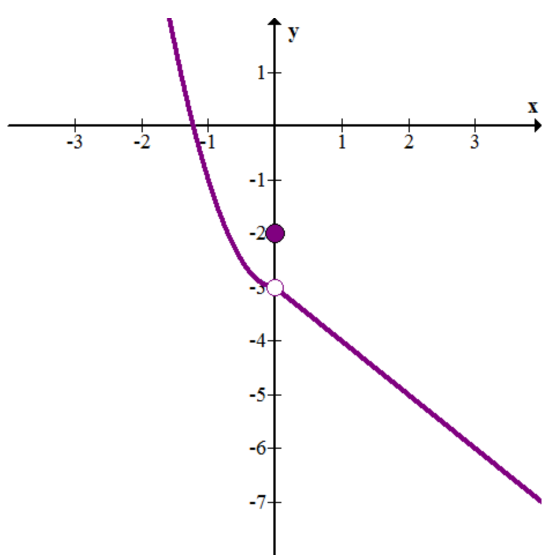
1. What is $\lim_{x\to 1^-} f(x)$?
2. What is $\lim_{x\to 1^+} f(x)$?
3. Is $f(x)$ continuous at $x = 1$?

Consider

$$g(x) = \begin{cases} 4x^2 + 2x - 1 & x < -2 \\ 8 & x = -2 \\ -3x + 5 & -2 < x \end{cases}$$

4. What is $\lim_{x\to -2^-} g(x)$?
5. What is $\lim_{x\to -2^+} g(x)$?
6. Is $g(x)$ continuous at $x = -2$?

Consider $h(x)$ shown in the graph below.



7. What is $\lim_{x\to 0^-} h(x)$?

8. What is $\lim_{x \to 0^+} h(x)$?

9. What is $h(0)$?

10. Is $h(x)$ continuous at $x = 0$?

11. The function $f(x) = \frac{1}{x}$ is not continuous. However, determine an interval in which $f(x)$ is continuous.

12. The function $g(x) = \sec x$ is not continuous. However, determine an interval in which $g(x)$ is continuous.

What must k be equal to in order for the following piecewise function to be continuous?

13. $f(x) = \begin{cases} 3x+1 & x < -2 \\ k & x = -2 \\ -2x-1 & x > -2 \end{cases}$

14. $g(x) = \begin{cases} x^2-3 & x < 2 \\ k & x = 2 \\ -2x+5 & x > 2 \end{cases}$

15. $k(x) = \begin{cases} 3x-6 & x < 1 \\ k & x = 1 \\ x^3-4 & x > 1 \end{cases}$

Please see the Appendix.

15.8 Intermediate and Extreme Value Theorems

Learning Objectives

Learn to use continuity to explore the Intermediate and Extreme Value Theorems.

Introduction

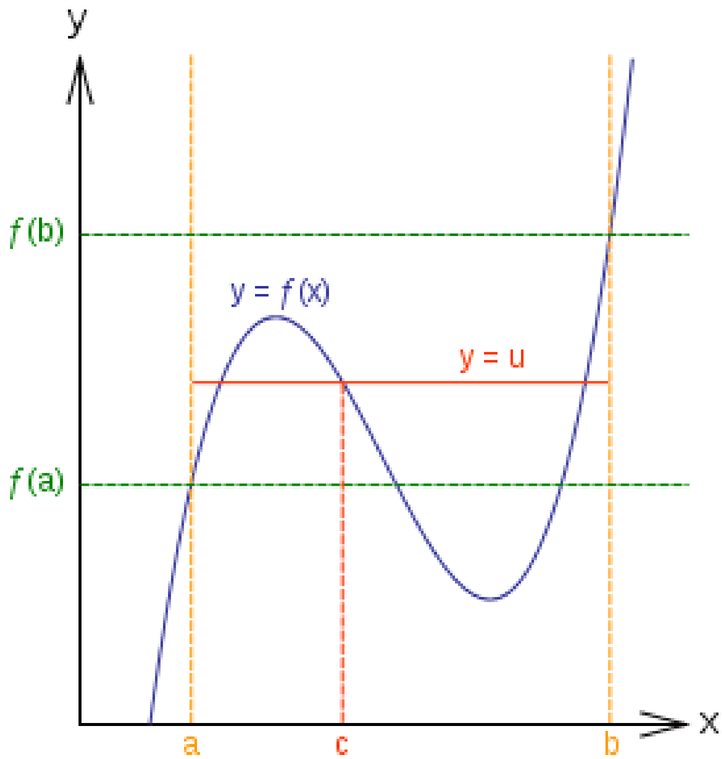
While the idea of continuity may seem somewhat basic, when a function is continuous over a closed interval like $x \in [0,20]$, you can actually draw some major conclusions. For instance, April takes a run through the Central Park for 20 minutes, and then returns to where she started. Suppose the path of April's runs is modeled by a continuous function. What can be concluded regarding the Intermediate Value Theorem and Extreme Value Theorem about April's run?

Intermediate Value Theorem

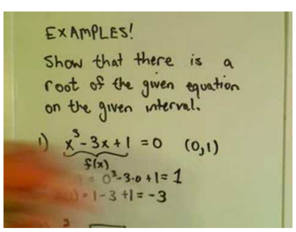
Intermediate Value Theorem

If a function $f(x)$ is continuous on a closed interval $[a,b]$, and u is a value between $f(a)$ and $f(b)$, then there exists a $c \in [a,b]$ such that $f(c) = u$.

Simply stated, if a function is continuous between a low point and a high point, then there must be a y-value in between the low and high points that has a corresponding x-value that is between x-values of the low and high points.



The following video provides examples of the Intermediate Value Theorem:



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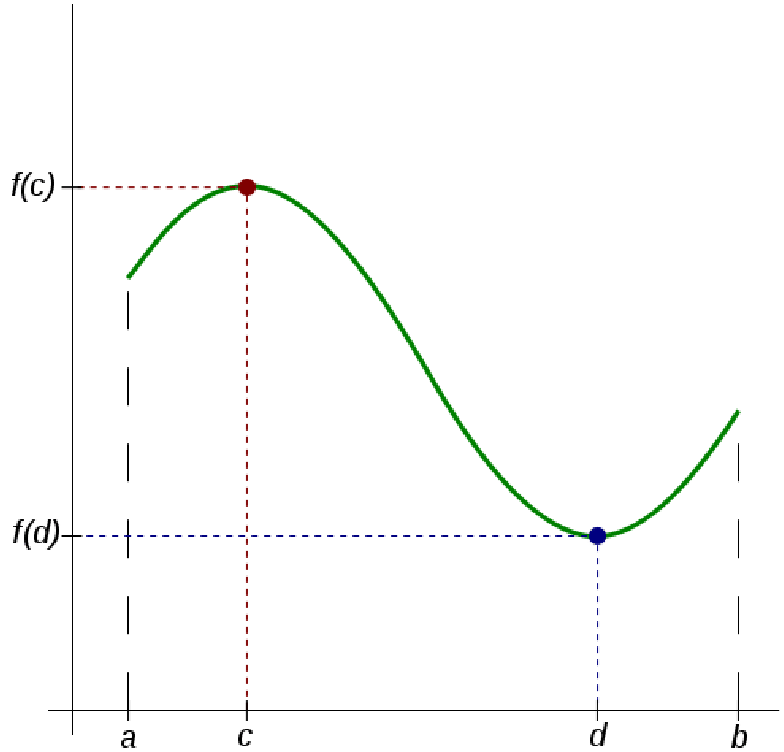
URL: http://www.ck12.org/flx/render/embeddedobject/62353

Play, Learn, and Explore the Intermediate Value Theorem: www.ck12.org/a/2152207

Extreme Value Theorem

Extreme Value Theorem

If a function $f(x)$ is continuous on a closed interval $[a,b]$, then there is at least one maximum and one minimum value. That is, there exist numbers c and d in $[a,b]$, such that $f(d) \leq f(x) \leq f(c)$ for all $x \in [a,b]$.



Examples

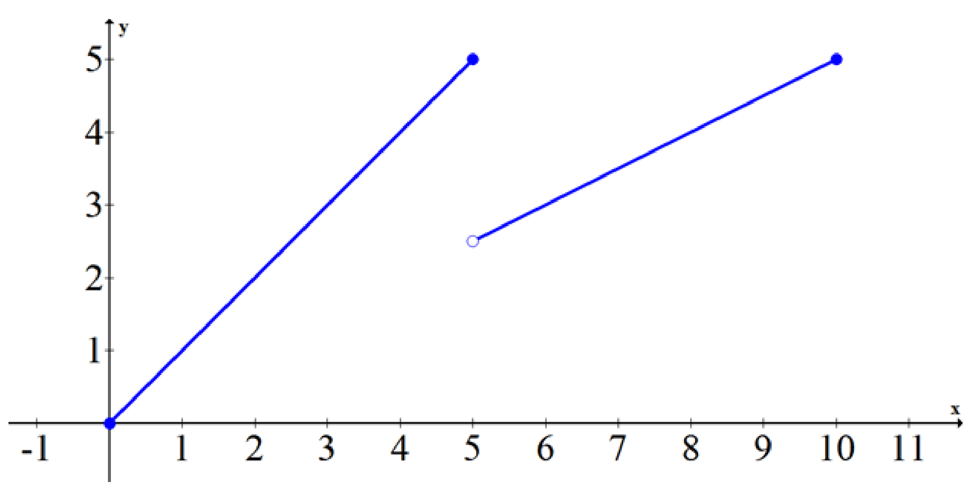
Example 1

Show that the converse of the Intermediate Value Theorem is false.

Solution:

The converse of the Intermediate Value Theorem is: If there exists a value $c \in [a,b]$ such that $f(c) = u$ for every u between $f(a)$ and $f(b)$, then the function is continuous.

To show the statement is false, all you need is one counterexample where every intermediate value is hit, and the function is discontinuous.



This function is discontinuous on the interval $[0,10]$, but every intermediate value between the first height at $(0,0)$ and the height of the last point $(10,5)$ is hit.

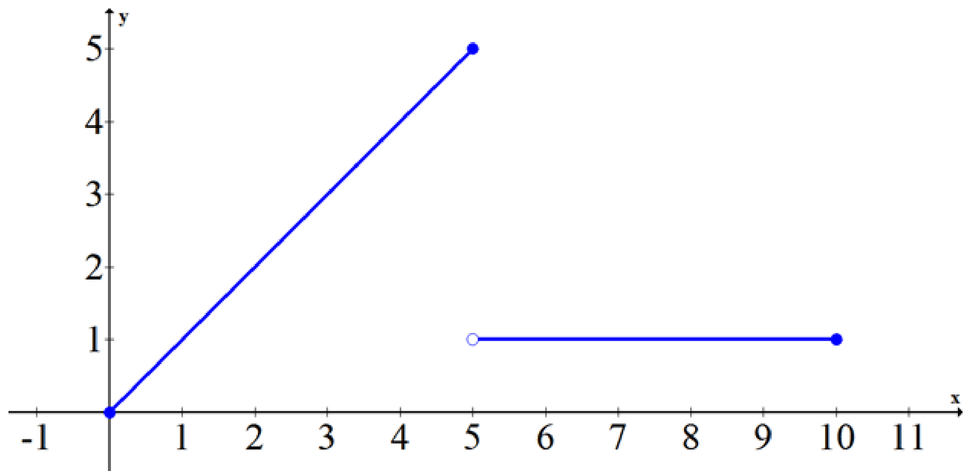
Example 2

Show that the converse of the Extreme Value Theorem is false.

Solution:

The converse of the Extreme Value Theorem is: If there is at least one maximum and one minimum in the closed interval $[a,b]$, then the function is continuous on $[a,b]$.

To show the statement is false, all you need is one counterexample. The goal is to find a function on a closed interval $[a,b]$ that has at least one maximum and one minimum and is also discontinuous.



On the interval $[0,10]$, the function attains a maximum at $(5,5)$ and a minimum at $(0,0)$, but is still discontinuous.

Example 3

Use the Intermediate Value Theorem to show that the function $f(x) = (x+1)^3 - 4$ has a zero on the interval $[0,3]$.

Solution:

First note that the function is a cubic, and therefore is continuous everywhere.

$$f(0) = (0+1)^3 - 4 = 1^3 - 4 = -3$$

$$f(3) = (3+1)^3 - 4 = 4^3 - 4 = 60$$

By the Intermediate Value Theorem, there must exist a $c \in [0,3]$ such that $f(c) = 0$, since 0 is between -3 and 60.

Example 4

Recall the question from the Introduction: April takes a run through the Central Park for 20 minutes and then returns to where she started. What can be concluded regarding the Intermediate Value Theorem and Extreme Value Theorem about April's run?

Solution:

April's run can be represented by a continuous function on the interval $x \in [0,20]$. By the Intermediate Value Theorem, it can be concluded that at some point during the 20-minute run, the path was either higher or lower than April's starting and ending position. At that height, there is some corresponding time value between 0 and 20 minutes. By the Extreme Value Theorem, it can be concluded that there was at least one maximum height and one minimum height along the path during the 20-minute run.

Example 5

Use the Intermediate Value Theorem to show that the equation below has at least one real solution.

$$x^8 = 2^x$$

Solution:

First, write the equation as a continuous function.

$$f(x) = x^8 - 2^x$$

This function is continuous, because the difference of two continuous functions is continuous.

Then apply the Intermediate Value Theorem.

$$f(0) = 0^8 - 2^0 = 0 - 1 = -1$$

$$f(2) = 2^8 - 2^2 = 256 - 4 = 252$$

By the Intermediate Value Theorem, there must exist a c such that $f(c) = 0$ because $-1 < 0 < 252$. The number c is one solution to the initial equation.

Example 6

Show that there is at least one solution to the equation below.

$$\sin x = x + 2$$

Solution:

First, write the equation as a continuous function.

$$f(x) = \sin x - x - 2$$

This function is continuous because the difference of two continuous functions is continuous.

Then apply the Intermediate Value Theorem.

$$f(0) = \sin 0 - 0 - 2 = -2$$

$$f(-\pi) = \sin(-\pi) + \pi - 2 = 0 + \pi - 2 > 0$$

By the Intermediate Value Theorem, there must exist a c such that $f(c) = 0$ because $-2 < 0 < \pi - 2$. The number c is one solution to the initial equation.

Example 7

When are you not allowed to use the Intermediate Value Theorem?

Solution:

The Intermediate Value Theorem should not be applied when the function is not continuous over the interval.

Summary

- The **Intermediate Value Theorem** states: If a function is continuous on a closed interval $[a,b]$ and u is a value between $f(a)$ and $f(b)$, then there exists a $c \in [a,b]$ such that $f(c) = u$.
- The **Extreme Value Theorem** states: If a function is continuous on a closed interval $[a,b]$, then there is at least one maximum and one minimum value.

Review

Use the Intermediate Value Theorem to show that each equation has at least one real solution.

1. $\cos x = -x$

2. $\ln(x) = e^{-x} + 1$

3. $2x^3 - 5x^2 = 10x - 5$

4. $x^3 + 1 = x$

5. $x^2 = \cos x$

6. $x^5 = 2x^3 + 2$

7. $3x^2 + 4x - 11 = 0$

8. $5x^4 = 6x^2 + 1$

9. $7x^3 - 18x^2 - 4x + 1 = 0$

10. Show that $f(x) = \frac{2x-3}{2x-5}$ has a real root on the interval $[1,2]$.

11. Show that $f(x) = \frac{3x+1}{2x+4}$ has a real root on the interval $[-1,0]$.

12. True or false: A function has a maximum and a minimum in the closed interval $[a,b]$; therefore, the function is continuous.

13. True or false: A function is continuous over the interval $[a,b]$; therefore, the function has a maximum and a minimum in the closed interval.

14. True or false: If a function is continuous over the interval $[a,b]$, then it is possible for the function to have more than one relative maximum in the interval $[a,b]$.

15. What do the Intermediate Value and Extreme Value Theorems have to do with continuity?

Review (Answers)

Please see the Appendix.

15.9 Instantaneous Rate of Change

Learning Objectives

Learn about instantaneous rate of change and the concept of a derivative.

Introduction

Consider a car driving down the highway and think about its speed. You are probably thinking about speed in terms of going a given distance in a given amount of time. The units could be miles per hour or feet per second, but the units always have time in the denominator. What happens when you consider the instantaneous speed of the car at one instant of time? Wouldn't the denominator be zero?

Instantaneous Rate of Change

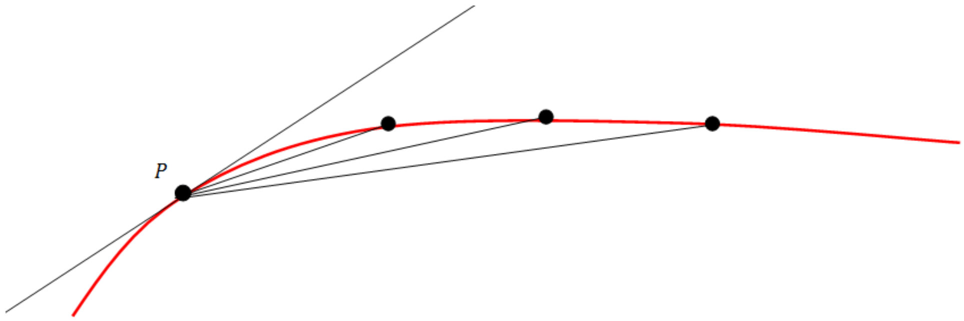
When you first learned about slope, you learned the mnemonic device "rise over run" to help you remember that to calculate the slope between two points, you use the following formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

In calculus, you learn that for curved functions, it makes more sense to discuss the slope at one precise point rather than between two points. The slope at one point is called the slope of the tangent line, and the slope between two separate points is called a secant line.

The **average rate of change** of a function is the slope of the secant line through two points. The **instantaneous rate of change** is the slope of the tangent line at a point.

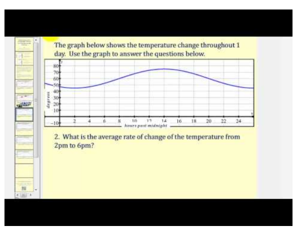
The slope at a point P (the slope of the tangent line) can be approximated by the slope of secant lines as the "run" of each secant line approaches zero.



Because you are interested in the slope as the "run" approaches zero, this is a limit question. One of the main reasons you study limits in calculus is so you can determine the slope of a curve at a point (the slope of a tangent line).

A **derivative function** is the function of the slopes of the tangent lines of the original function.

The following video discusses the graphical approach to average and instantaneous rate of change:



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Click image to the left or use the URL below.

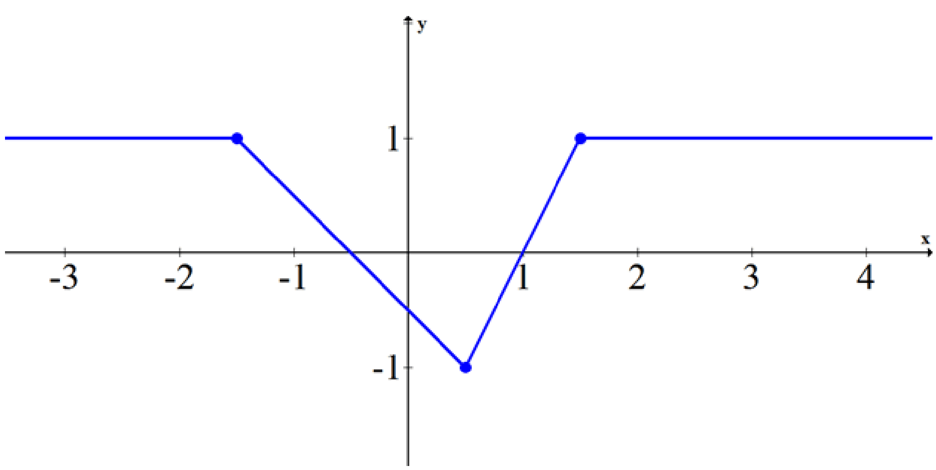
URL: http://www.ck12.org/flx/render/embeddedobject/195477

Play, Learn, and Explore Average and Instantaneous Rate of Change: www.ck12.org/a/2547409

Examples

Example 1

Estimate the slope of the following function at -3, -2, -1, 0, 1, 2, and 3. Organize the slopes in a table.



Solution:

By mentally drawing a tangent line at the following *x*-values, you can estimate the following slopes:

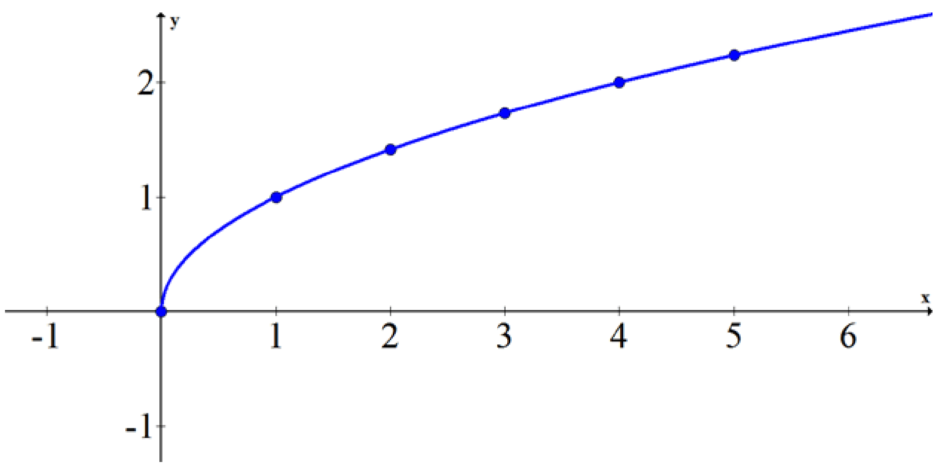
TABLE 15.11:

x	***slope***
-3	0
-2	0
-1	-1
0	-1
1	2
2	0
3	0

If you graph these points, you will produce a graph of what's known as the derivative of the original function.

Example 2

Estimate the slope of the function $f(x) = \sqrt{x}$ at the point $(1,1)$ by calculating four successively close secant lines.



Solution:

Calculate the slope between $(1,1)$ and four other points on the curve:

The slope of the line between $\left(5,\ \sqrt{5}\right)$ and $(1,1)$ is $m_1 = \frac{\sqrt{5}-1}{5-1} \approx 0.309$.

The slope of the line between $(4,\ 2)$ and $(1,1)$ is $m_2 = \frac{2-1}{4-1} \approx 0.333$.

The slope of the line between $\left(3,\ \sqrt{3}\right)$ and $(1,1)$ is $m_3 = \frac{\sqrt{3}-1}{3-1} \approx 0.366$.

The slope of the line between $\left(2,\ \sqrt{2}\right)$ and $(1,1)$ is $m_1 = \frac{0.7^3-1}{0.7-1} \approx 2.19$.

An estimate for the slope of the function $f(x) = \sqrt{x}$ at the point $(1,1)$ would be ≈ 0.45.

Example 3

Evaluate the following limit and explain its connection with Example 2:

$$\lim_{x \to 1} \left(\frac{\sqrt{x}-1}{x-1} \right)$$

Solution:

Notice that the pattern in the previous problem is leading up to $\frac{\sqrt{1}-1}{1-1}$. Unfortunately, this cannot be computed directly because there is a zero in the denominator. Luckily, you know how to evaluate using limits.

$$\begin{aligned}
m &= \lim_{x\to 1}\left(\frac{(\sqrt{x}-1)}{(x-1)}\cdot\frac{(\sqrt{x}+1)}{(\sqrt{x}+1)}\right)\\
&= \lim_{x\to 1}\left(\frac{(x-1)}{(x-1)(\sqrt{x}+1)}\right)\\
&= \lim_{x\to 1}\left(\frac{1}{(\sqrt{x}+1)}\right)\\
&= \frac{1}{\sqrt{1}+1}\\
&= \frac{1}{2}\\
&= 0.5
\end{aligned}$$

The slope of the function $f(x)=\sqrt{x}$ at the point $(1,1)$ is exactly $m=\frac{1}{2}$.

Example 4

Recall the question from the Introduction: What happens when you consider the instantaneous speed of a car driving down the highway at one instant of time?

Solution:

Write the ratio of distance to time and use limit notation to allow time to go to zero. Notice that you get a zero in the denominator.

$$\lim_{\text{time}\to 0}\left(\frac{\text{distance}}{\text{time}}\right)$$

The great thing about limits is that you have learned techniques for finding a limit even when the denominator goes to zero. Instantaneous speed for a car essentially means the number that the speedometer reads at that precise moment in time. You are no longer restricted to finding slope from two separate points.

Example 5

Sketch a complete cycle of a sine function. Estimate the slopes at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2},$ and 2π.

Solution:

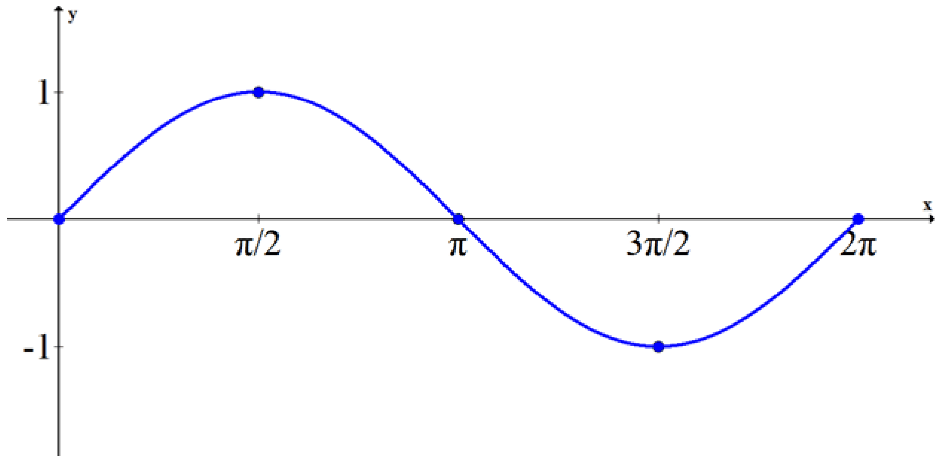


TABLE 15.12:

x	***Slope***
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1

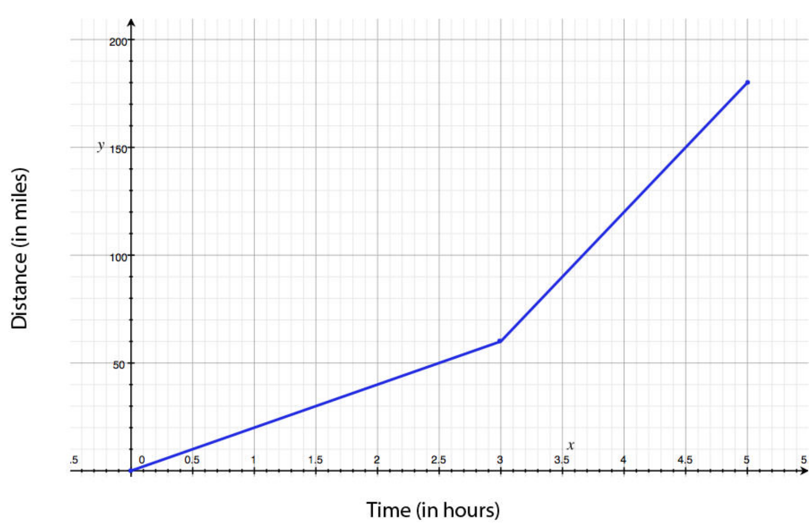
Notice that these are the exact values of cosine evaluated at those points.

Example 6

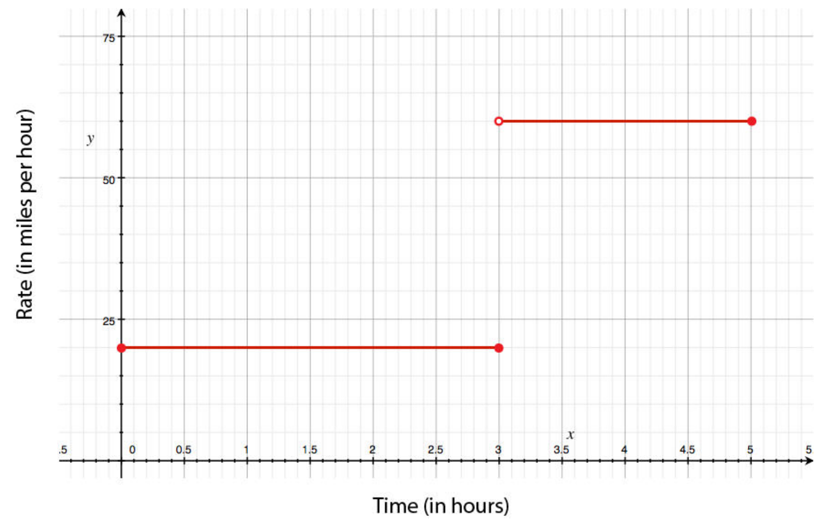
Logan travels by bike at 20 mph for 3 hours. Then she gets in a car and drives 60 mph for 2 hours. Sketch both the distance vs. time graph and the rate vs. time graph.

Solution:

Distance vs. Time:



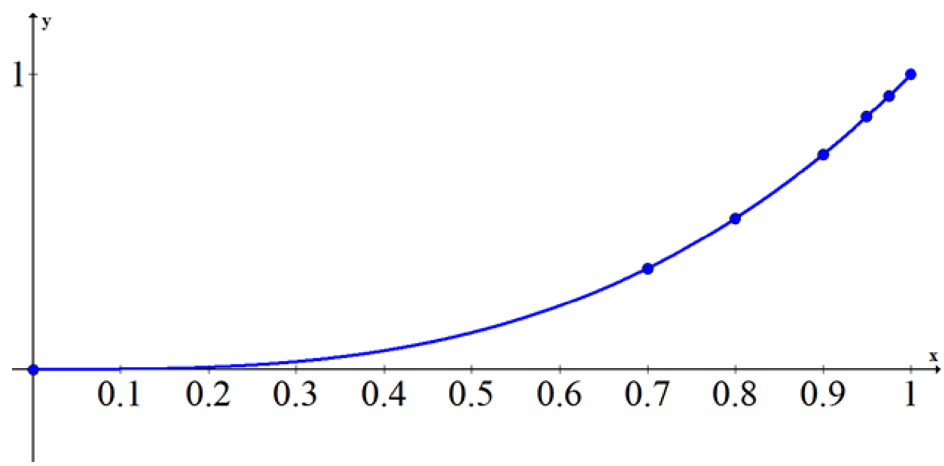
Rate vs. Time: (This is the graph of the derivative of the original function shown above.)



Example 7

Approximate the slope of $y = x^3$ at $(1,1)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

Solution:



The slope of the line between $(0.7, 0.7^3)$ and $(1,1)$ is $m_1 = \frac{0.7^3-1}{0.7-1} \approx 2.19$.

The slope of the line between $(0.8, 0.8^3)$ and $(1,1)$ is $m_2 = \frac{0.8^3-1}{0.8-1} \approx 2.44$.

The slope of the line between $(0.9, 0.9^3)$ and $(1,1)$ is $m_3 = \frac{0.9^3-1}{0.9-1} \approx 2.71$.

The slope of the line between $(0.95, 0.95^3)$ and $(1,1)$ is $m_1 = \frac{0.95^3-1}{0.95-1} \approx 2.8525$.

The slope of the line between $(0.975, 0.975^3)$ and $(1,1)$ is $m_1 = \frac{0.975^3-1}{0.975-1} \approx 2.925625$.

The slope at $(1,1)$ will be slightly greater than the estimates because of the way the slope curves. The slope at $(1,1)$ appears to be about 3.

Summary

- A **secant line** is a line that passes through two distinct points on a function.

- The **average rate of change** of a function is the slope of the secant line through two points.
- A **tangent line** is a line that passes through one point on a function.
- The **instantaneous rate of change** is the slope of the tangent line at a point.
- A **derivative function** is a function of the slopes of the original function.

Review

1. Approximate the slope of $y = x^2$ at $(1, 1)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

2. Evaluate the following limit and explain how it confirms your answer to Number 1:

$$\lim_{x \to 1} \left(\frac{x^2 - 1}{x - 1} \right)$$

3. Approximate the slope of $y = 3x^2 + 1$ at $(1, 4)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

4. Evaluate the following limit and explain how it confirms your answer to Number 3:

$$\lim_{x \to 1} \left(\frac{3x^2 + 1 - 4}{x - 1} \right)$$

5. Approximate the slope of $y = x^3 - 2$ at $(1, -1)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

6. Evaluate the following limit and explain how it confirms your answer to Number 5:

$$\lim_{x \to 1} \left(\frac{x^3 - 2 - (-1)}{x - 1} \right)$$

7. Approximate the slope of $y = 2x^3 - 1$ at $(1, 1)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

8. What limit could you evaluate to confirm your answer to Number 7?

9. Sketch a complete cycle of a cosine function. Estimate the slopes at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2},$ and 2π.

10. How do the slopes found in the previous question relate to the sine function? What function do you think is the derivative of the cosine function?

11. Sketch the line $y = 2x + 1$. What is the slope at each point on this line? What is the derivative of this function?

12. Logan travels by bike at 30 mph for 2 hours. Then she gets in a car and drives 65 mph for 3 hours. Sketch both the distance vs. time graph and the rate vs. time graph.

13. Explain what a tangent line is and how it relates to derivatives.

14. Why is finding the slope of a tangent line for a point on a function the same as the instantaneous rate of change at that point?

15. What do limits have to do with finding the slopes of tangent lines?

Review (Answers)

Please see the Appendix.

15.10 Area Under a Curve

Learning Objectives

Learn to estimate the area under a curve and interpret its meaning in context.

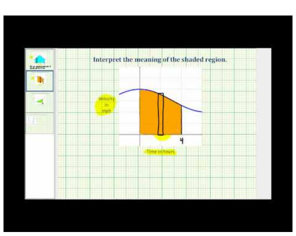
Introduction

Suppose a person is driving on the highway. The velocity of the car in meters per second can be modeled by a quadratic for the 1st 8 seconds of acceleration, $v(t) = t^2$. How far has the car traveled in those 8 seconds?

Area Under a Curve

The area under a curve refers to the area between a curve and the x-axis. This area could be entirely above the x-axis, entirely below the x-axis, or a combination of above and below the x-axis. In calculus, the area under a curve is a visual representation of an integral. Often the area under a curve can be interpreted as the accumulated amount of whatever the function is modeling.

For instance, suppose a curve represents the velocity of a moving object over 4 hours. The amount of time in hours the object moves is the x variable, and the velocity in miles per hour is the y variable. The area under the curve represents the total distance traveled over 4 hours. Two additional examples are shown in the following video:



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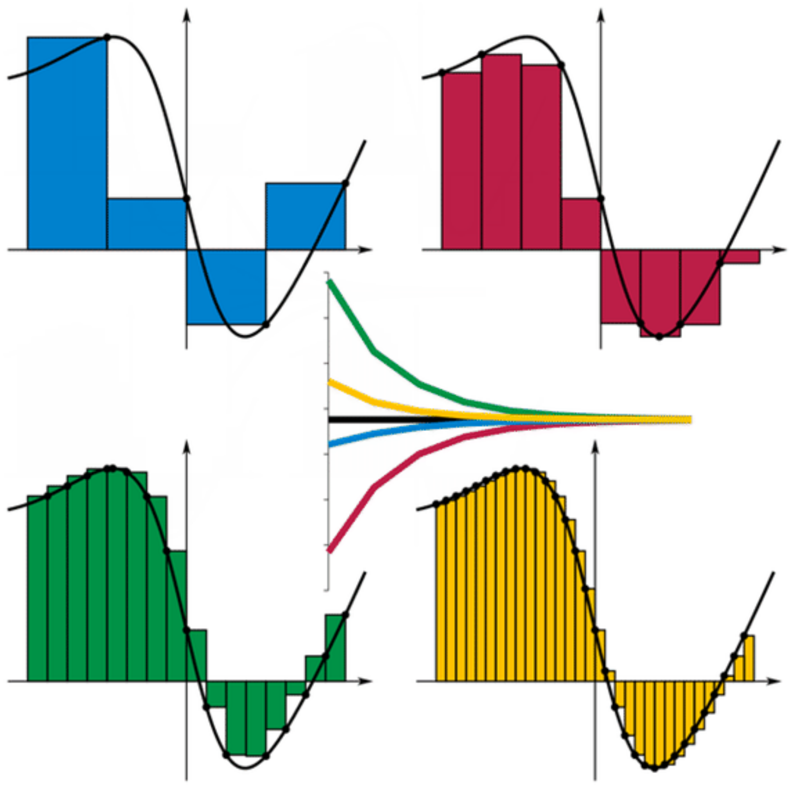
Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/195456

When the area under a curve is a simple shape, such as the area under a straight line, the area can be calculated using geometry. However, if the area under a curve is not a simple shape, then the area can either be approximated using rectangles or trapezoids, or the area can be calculated directly using integration methods taught in calculus. The rest of this section will focus on approximating the area under a curve using rectangles.

As mentioned, the area under a curve can be approximated with rectangles equally spaced under a curve as shown below. This approximation method for the area under a curve is called Riemann Sums.



For consistency, you can choose whether the boxes should hit the curve on the left-hand corner, the right-hand corner, the maximum value, or the minimum value. The blue approximation uses right-handed boxes. The red approximation assigns the height of the box to be the minimum value of the function in each subinterval. The green approximation assigns the height of the box to be the maximum value of the function in each subinterval. The yellow approximation uses left-handed boxes.

For each of these examples, the area of each rectangle is calculated using the equationArea $= b \cdot h$. Rectangles above the x-axis will have positive area because the height (or y-value) is positive. Rectangles below the x-axis will have negative area because the height (or y-value) is negative.

The more boxes you use, the narrower the boxes will be, and thus the more accurate the approximation will be to the actual area. In fact, the limit of the area approximation as the number of boxes increases to infinity is the precise area under the curve.

Relationship Between an Integral and the Area Under a Curve

$$\int_1^\infty f(x) = \lim_{n\to\infty} \sum_{i=1}^{n} (\text{Area of each box})$$

In general, the area under a curve using Riemann Sums can be calculated by adding the areas of all the rectangles where the bases are equally spaced.

Riemann Sum

The area under a curve is calculated by

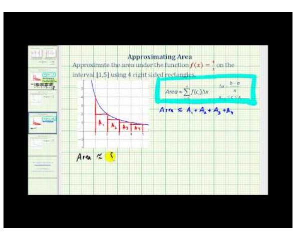
$$\text{Area} = \sum_{i=1}^{n} f(c_i) \cdot \Delta x,$$

where c_i is the point of the box that hits the curve, $f(c_i)$ is the function value at that point, and Δx is the width of the base of each rectangle.
For a curve defined on the interval $[a,b]$,

$$\Delta x = \frac{b-a}{n},$$

where n is the number of rectangles or boxes.

An example using this formula can be seen in the following video:



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URL: http://www.ck12.org/flx/render/embeddedobject/62357

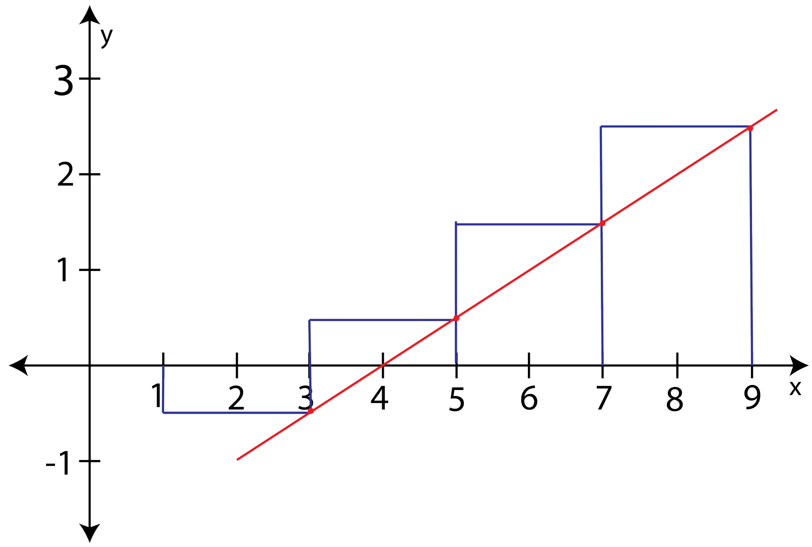
Play, Learn, and Explore Area Under a Curve with Area Sums: www.ck12.org/a/2112821

Examples

Example 1

Use four right-handed boxes to approximate the area between 1 and 9 of the function $f(x) = \frac{1}{2}x - 2$.

Solution:



Step 1: For the interval $[1,9]$ with 4 rectangles, the width of the base of each rectangle is

$$\Delta x = \frac{9-1}{4} = \frac{8}{4} = 2.$$

Since the rectangles hit the curve on the right-hand corner, the 1st intersection point will be 2 units (the base width) from the start of the interval, 1. Thus, the 1st x-value the rectangle hits the curve is at $x = 3$. The next x-values are 5, 7, and 9.

Step 2: Calculate each corresponding function value for these x-values. Each function value is the height of its corresponding rectangle.

$$f(3) = \frac{1}{2}(3) - 2 = -\frac{1}{2}$$
$$f(5) = \frac{1}{2}(5) - 2 = \frac{1}{2}$$
$$f(7) = \frac{1}{2}(7) - 2 = \frac{3}{2}$$
$$f(9) = \frac{1}{2}(9) - 2 = \frac{5}{2}$$

Step 3: Calculate the area of each rectangle by multiplying each height by the base width.

$$2 \cdot -\frac{1}{2} = -1$$
$$2 \cdot \frac{1}{2} = 1$$
$$2 \cdot \frac{3}{2} = 3$$
$$2 \cdot \frac{5}{2} = 5$$

Step 4: Add the four areas together to get the total approximation for the area under the given curve.

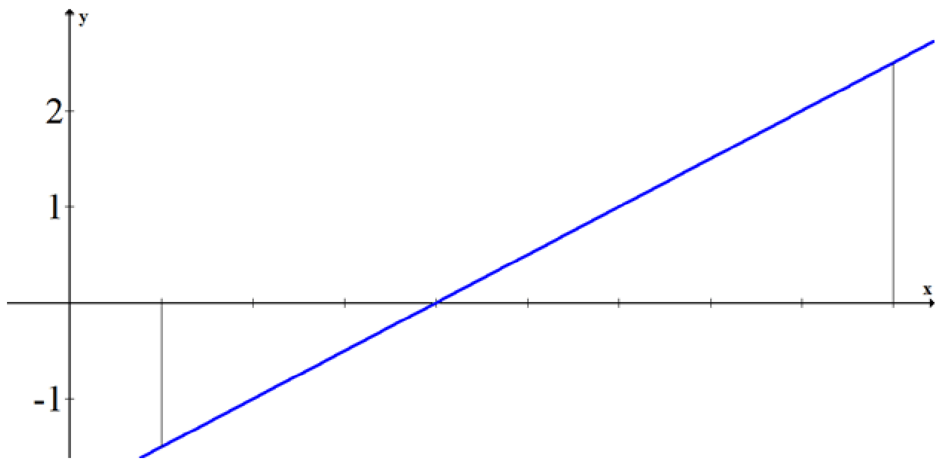
$$\text{Area} = -1+1+3+5 = 8 \text{ square units}$$

Example 2

Evaluate the exact area under the curve in Example 1 using the area formula for a triangle.

Solution:

Remember that the area below the x-axis is negative, while the area above the x-axis is positive.



Below the x-axis: The base of the triangle is 3 units, and the height is -1.5 units.

$$\frac{1}{2} \cdot 3 \cdot -1.5 = -\frac{9}{4}$$

Above the x-axis: The base of the triangle is 5 units, and the height is 2.5 units.

$$\frac{1}{2} \cdot 5 \cdot 2.5 = \frac{25}{4}$$

The total area under the curve between 1 and 9 is

$$-\frac{9}{4} + \frac{25}{4} = \frac{16}{4} = 4 \text{ square units.}$$

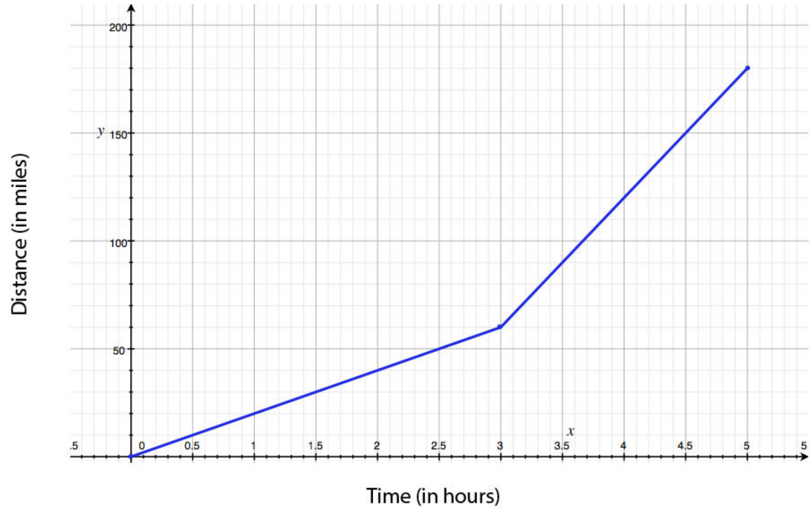
As expected, the approximated area under the curve is not as accurate the the exact area.

Example 3

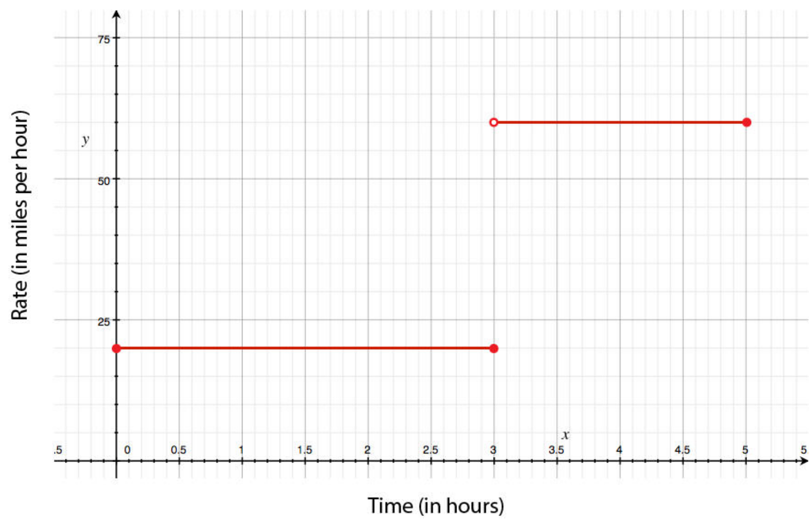
Logan travels by bike at 20 mph for 3 hours. Then she gets in a car and drives 60 mph for 2 hours. Sketch both the distance vs. time graph and the rate vs. time graph. Use an area under the curve argument to connect the two graphs.

Solution:

Distance vs. Time:



Rate vs. Time:



The slope of the 1st graph is 20 from 0 to 3, and then 60 from 3 to 5. The 2nd graph is a graph of the slopes from the 1st graph. If you calculate the area of the 2nd graph at the key points 0, 1, 2, 3, 4, and 5, you will see that they align perfectly with the points on the 1st graph.

TABLE 15.13:

x	***Area under curve from 0 to*** x
0	0
1	20
2	40
3	60
4	120
5	180

Example 4

Recall the problem from the Introduction: The velocity of the car in meters per second can be modeled by a quadratic for the 1st 8 seconds of acceleration $v(t) = t^2$.

Solution:

The area under the curve is equal to the total distance traveled in the 1st 8 seconds. Since the quadratic is a curve, you must choose the number of subintervals you want to use and whether you want right- or left-handed boxes for estimating. Suppose you choose 8 left-handed boxes of width equal to 1.

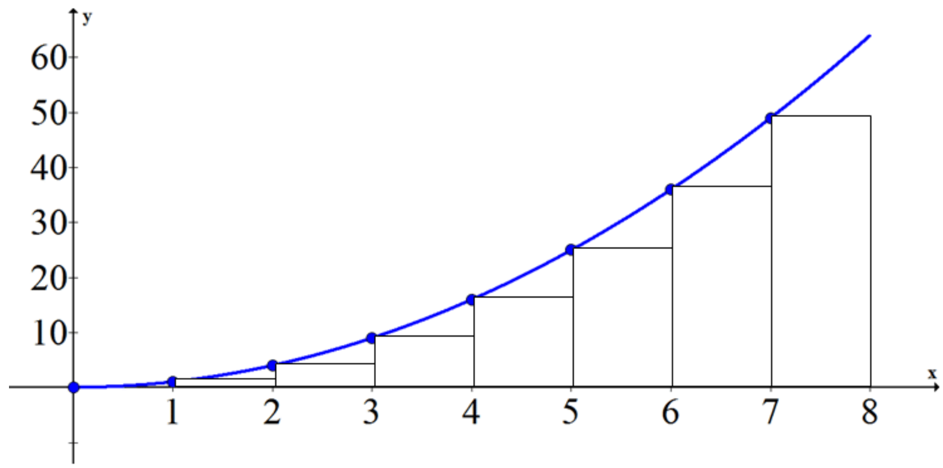


TABLE 15.14:

x	0	1	2	3	4	5	6	7
Area of box to the right	1·0=0	1·1=1	1·4=4	1·9=9	1·16=16	1·25=25	1·36=36	1·49=49

The approximate sum is $1+4+9+16+25+36+49 = 140$. This means that the car traveled approximately 140 meters in the 1st 8 seconds.

Example 5

Approximate the area under the curve using 8 subintervals and right endpoints for the function $f(x) = 3x^2 - 1, \ -1 \leq x \leq 7$.

Solution:

While a graph is helpful to visualize the problem, and drawing each box can help give meaning to each summand, this is not always necessary.

Step 1: Since there are going to be 8 subintervals over the total interval of $-1 \leq x \leq 7$, each interval is going to have a width of 1.

$$\Delta x = \frac{7-(-1)}{8} = \frac{8}{8} = 1$$

The right-hand endpoint x-values are 0, 1, 2, 3, 4, 5, 6, and 7.

Step 2: The corresponding function value for these x-values are:

$$\begin{aligned}
f(0) &= 3\cdot 0^2 - 1 = -1\\
f(1) &= 3\cdot 1^2 - 1 = 2\\
f(2) &= 3\cdot 2^2 - 1 = 11\\
f(3) &= 3\cdot 3^2 - 1 = 26\\
f(4) &= 3\cdot 4^2 - 1 = 47\\
f(5) &= 3\cdot 5^2 - 1 = 74\\
f(6) &= 3\cdot 6^2 - 1 = 107\\
f(7) &= 3\cdot 7^2 - 1 = 146
\end{aligned}$$

Step 3: The area of each rectangle is -1, 2, 11, 26, 47, 74, 107, and 146.

Step 4: The area under the curve is

$$\text{Area} = \sum_{i=1}^{8} f(c_i)\cdot \Delta x = -1+2+11+26+47+74+107+146 = 412 \text{ square units.}$$

Example 6

Approximate the area under the curve using 8 subintervals and left endpoints for the function: $f(x) = \frac{4}{x}+3,\ 2 \le x \le 6$.

Solution:

Step 1: Since there are going to be 8 subintervals over the total interval of $2 \le x \le 6$, each interval is going to have a width of $\frac{1}{2}$.

$$\Delta x = \frac{6-2}{8} = \frac{4}{8} = \frac{1}{2}$$

Since the rectangles hit the curve on the left-hand corner, the 1st intersection point will be at the start of the interval, 2. The left-hand endpoint x-values are 2, 2.5, 3, 3.5, 4, 4.5, 5, and 5.5.

Step 2: The corresponding function value for these x-values are:

$$f(2) = \frac{4}{2} + 3 = 5$$
$$f(2.5) = \frac{4}{2.5} + 3 = \frac{8}{5}$$
$$f(3) = \frac{4}{3} + 3 = \frac{13}{3}$$
$$f(3.5) = \frac{4}{3.5} + 3 = \frac{29}{7}$$
$$f(4) = \frac{4}{4} + 3 = 4$$
$$f(4.5) = \frac{4}{4.5} + 3 = \frac{35}{9}$$
$$f(5) = \frac{4}{5} + 3 = \frac{19}{5}$$
$$f(5.5) = \frac{4}{5.5} + 3 = \frac{41}{11}$$

Step 3: The area of each rectangle is $\frac{5}{2}, \frac{4}{5}, \frac{13}{6}, \frac{29}{14}, 2, \frac{35}{18}, \frac{19}{10},$ and $\frac{41}{22}$.

Step 4: The area under the curve is:

$$\begin{aligned} \text{Area} = \sum_{1}^{20} f(c_i) \cdot \Delta x &= f(1) \cdot 0.1 + f(1.1) \cdot 0.1 + f(1.2) \cdot 0.1 + \cdots + f(2.9) \cdot 0.1 \\ &= 0.1\,(f(1) + f(1.1) + \cdots f(2.9)) \\ &\approx 12.471 \text{ square units.} \end{aligned}$$

Example 7

Approximate the area under the curve using 20 subintervals and left endpoints for the function $f(x) = x^x,\ 1 \leq x \leq 3$.

Solution:

When the number of subintervals gets large and the subintervals get extremely narrow, it will be impossible to draw an accurate picture. This is why using summation notation and thinking through what the indices and the argument will be is incredibly important.

Step 1: Since there are going to be 20 subintervals over the total interval of $1 \leq x \leq 3$, each interval is going to have a width of 0.1.

$$\Delta x = \frac{3-1}{20} = \frac{2}{20} = 0.1$$

The left-hand endpoint x-values will consist of 1, 1.1, 1.2, 1.3, 1.4, ...

Step 2: The area under the curve is

$$\begin{aligned}\text{Area} = \sum_{1}^{20} f(c_i)\cdot\Delta x &= f(1)\cdot 0.1 + f(1.1)\cdot 0.1 + f(1.2)\cdot 0.1 + \cdots + f(2.9)\cdot 0.1\\ &= 0.1(f(1) + f(1.1) + \cdots f(2.9))\\ &\approx 12.471 \text{ square units.}\end{aligned}$$

Summary

- **Riemann Sums** can be used to approximate the area under a curve.
- The **area under a curve** is calculated by Area $= \sum_{i=1}^{n} f(c_i)\cdot\Delta x$, where c_i is the x-value of the point of the box that hits the curve, $f(c_i)$ is the y-value at that point, and Δx is the width of the box.
- For a curve defined on the interval $[a,b]$, $\Delta x = \frac{b-a}{n}$, where n is the number of rectangles or boxes.
- A **definite integral** is the limit of a sum as the number of summands increases to infinity.

Review

1. Approximate the area under the curve using 8 subintervals and right endpoints for the function $f(x) = x^2 - x + 1,\ 0 \le x \le 8$.

2. Approximate the area under the curve using 8 subintervals and left endpoints for the function $f(x) = x^2 - 2x + 1,\ -4 \le x \le 4$.

3. Approximate the area under the curve using 20 subintervals and left endpoints for the function $f(x) = \sqrt{x+3},\ 0 \le x \le 4$.

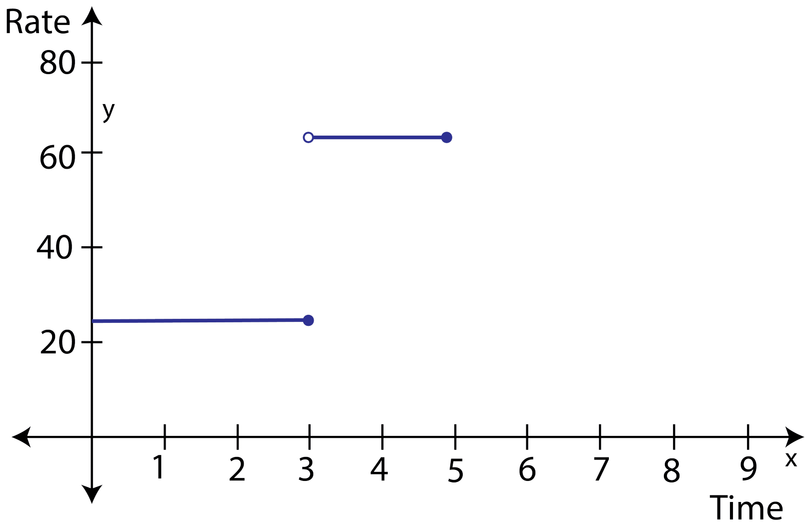
4. Approximate the area under the curve using 100 subintervals and left endpoints for the function $f(x) = \sqrt{x+3},\ 0 \le x \le 4$. Compare to your answer from Number 3.

5. Approximate the area under the curve using 8 subintervals and left endpoints for the function $f(x) = \cos(x),\ 0 \le x \le 4$.

6. Approximate the area under the curve using 20 subintervals and left endpoints for the function $f(x) = \cos(x),\ 0 \le x \le 4$.

7. Approximate the area under the curve using 100 subintervals and left endpoints for the function $f(x) = \cos(x),\ 0 \le x \le 4$.

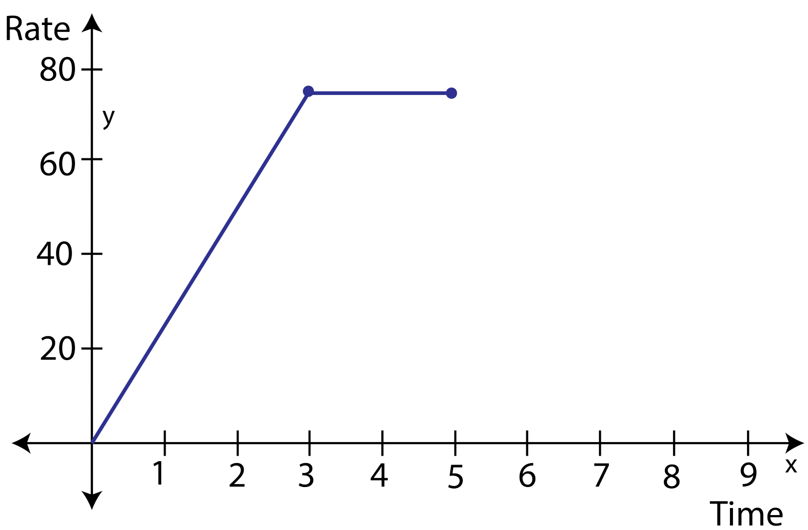
The following graph shows the rate (in miles per hour) vs. time (in hours) for a car:



8. Describe what is happening with the car.

9. How far did the car travel in 5 hours?

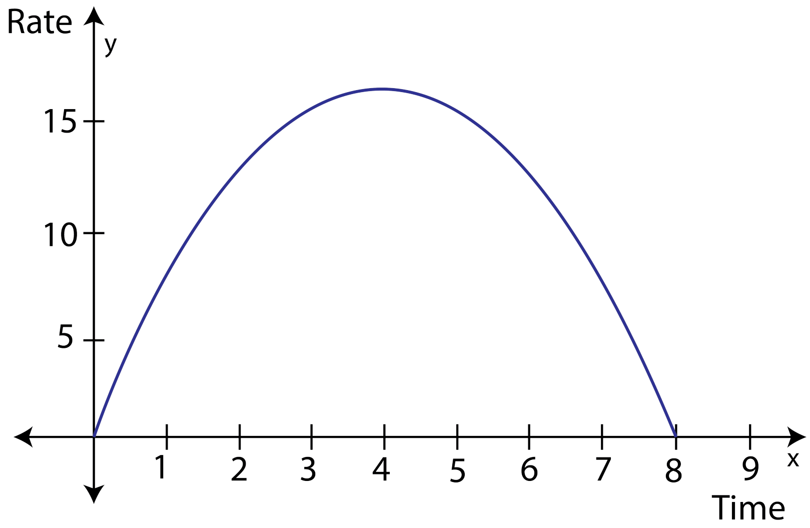
The following graph shows the rate (in feet per second) vs. time (in seconds) for a car:



10. Describe what is happening with the car. In particular, what is happening in the 1st 3 seconds?

11. How far did the car travel in 5 seconds?

The following graph shows the function $f(x) = -(x-4)^2 + 16$, which represents the rate (in feet per second) vs. time (in seconds) for a runner:



12. Describe what is happening with the runner. In particular, what happens after 4 seconds?

13. Use rectangles to approximate the total distance (in feet) that the runner traveled in the 8 seconds. Try to get as good an approximation as possible.

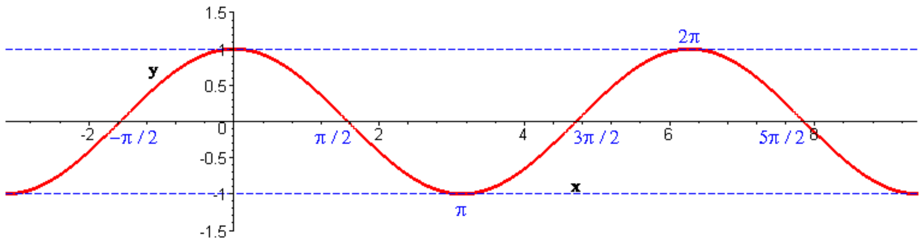
14. How do integrals relate to sums?

Review (Answers)

Please see the Appendix.

15.11 Project: A Preview of Calculus

Sound waves can be represented by sine and cosine curves. In this project, you will develop a formula for the slope of the tangent line to the cosine curve at an arbitrary point (x, y). The 1st approach will be graphical, the 2nd will be numerical, and the 3rd will be analytical.



1. **Graphical Approach:** Use the graph shown above, or graph using Desmos or a graphing calculator, to estimate the slope of this curve at 15 different x-values. For instance, the slope at $x = \frac{3\pi}{2}$ is 1. Once you have determined 15 slope values, plot the points and connect them with a continuous curve. Do you recognize this curve?
2. **Numerical Approach:** The slope of the tangent line is given by the following formula:

$$m = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

This formula is a good approximation of the slope when the change in x, Δx, is small. Complete the table below when $\Delta x = 0.001$, such that

$$m \approx \frac{\cos(x+0.001) - \cos x}{0.001}.$$

TABLE 15.15:

x	-2	-1.5	-1.0	-0.5	0	0.5	1.0	1.5	2.0
m									

3. **Analytical Approach:** To calculate the slope of the tangent line to the cosine curve analytically, you will need two trigonometric limits:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

Use these limits to find a formula for the slope of the cosine curve.

15.12 Summary: A Preview of Calculus

Chapter Summary

In this chapter, we learned about:

Limits:

- To solve limits graphically, graph the function and determine if the limit exists.
  - The limit exists only when the left and right one-sided limits are equal.

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x) = L$$

  - If the left and right one-sided limits are not equal, then the limit does not exist, or DNE.
- To solve limits numerically, use a table with x-values to the left and to the right on the number line to the number in the limit. Then plug these values into the function to obtain the function value at this point.
- To solve limits analytically, substitute the value that x approaches into the function.
  - If the function is a rational expression with a hole, then algebraically factor the numerator and denominator. Next, cancel any common factors in the numerator and denominator. Finally, substitute the value that x approaches into the resulting expression.
  - Rationalization may also be used to help simply the rational expression.

Continuity:

- A function is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$, where both $f(a)$ and $\lim_{x\to a} f(x)$ exist.

Intermediate and Extreme Value Theorem:

- The Intermediate Value Theorem states: If a function is continuous on a closed interval $[a,b]$ and u is a value between $f(a)$ and $f(b)$, then there exists a $c \in [a,b]$ such that $f(c) = u$.
- The Extreme Value Theorem states: If a function is continuous on a closed interval $[a,b]$, then there is at least one maximum and one minimum value.

Derivative:

- A secant line is a line that passes through two distinct points on a function.
- The average rate of change of a function is the slope of the secant line through two points.
- A tangent line is a line that passes through one point on a function.
- The instantaneous rate of change is the slope of the tangent line at a point.
- A derivative function is a function of the slopes of the original function.

Area Under a Curve:

- The area under a curve is calculated by Area $= \sum_{i=1}^{n} f(c_i) \cdot \Delta x$, where c_i is the point of the box that hits the curve, $f(c_i)$ is the function value at that point, and Δx is the width of the base of each rectangle.
- For a curve defined on the interval $[a,b]$, $\Delta x = \frac{b-a}{n}$, where n is the number of rectangles or boxes.

Review

Try the following cumulative review problems to practice the concepts in this chapter:

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/196135

15.13 References

1. H.F. Helmolt. https://fr.wikipedia.org/wiki/Fichier:Hw-newton.jpg .
2. Christoph Bernhard Francke. https://commons.wikimedia.org/wiki/File:Gottfried_Wilhelm_von_Leibniz.jpg .
3. Leo Hidalgo. https://www.flickr.com/photos/ileohidalgo/11991219445/in/photolist-jgC8LR-84r6hq-Enc5s-4cyLxP-7tKBAB-et4s5E-57rBfi-a4XvjS-7tPuBf-7tPBVh-5VMLZU-7tPALL-jR9w3r-7tPCrC-7tKG2F-5VHqLn-7tPgyJ-isuMff-91MEoN-o3ciio-btJh64-9qaWCG-o6gP-nqq7EM-mmjGJo-2VfN4F-hXRmdo-iydUNZ-9cuhQG-7tP7rS-oWwBCQ-fXWso7-2VkgT5-7tPk29-7TPc5U-7tPbR1-dDaXp9-6PAVS8-aEKVcN-cjZBGN-7tK5BT-3bPPRY-9ETSiP-fbhRWm-2VkjYS-fEVic-d4ABhW-5GNc7j-dcRNMH-f9aJHW .
4. David. https://commons.wikimedia.org/wiki/File:Amazon_Box.JPG .
5. ChiemseeMan. https://commons.wikimedia.org/wiki/File:VentilatorSiemsens1920.JPG .
6. idleman. https://www.flickr.com/photos/idleman/3439242266/in/photolist-6eV1Us-K3frr-dYZzMe-T7uouG-c3kk3-hQZ5MG-aw8wzd-jpzbsj-gyfEuj-3ihJqx-qp3YR9-pJwAnQ-TFSzFR-84E9oJ-CSwix-qHm2tL-mVB78C-mYjHid-bcsLgZ-6i41Vu-7JTC8k-4pUVjR-54QeHk-dekdVU-4HLZb1-r73D6D-rBYzCe-jXv2KV-g2ioyG-cM5gab-2Zk3Y-rnBDYN-pKhWpb-rkwDZH-9SUbpG-pJzNB5-dAmFx6-eFYsUp-nVzv2R-mYhJx2-knvn1L-mVBbwh-meHxmm-9Rn3hz-7HMTi4-QvFtbb-2ZjZe-dNWLND-mo4V-rF7we2 .
7. U.S. Department of Transportation. https://commons.wikimedia.org/wiki/File:National_Network.jpg .
8. Chris Dlugosz. https://www.flickr.com/photos/chrisdlugosz/2609920304 .
9. en:User:RoySmith. https://commons.wikimedia.org/wiki/File:CentralParkFromAboveCropped.jpg .
10. Shadowlink1014. https://en.wikipedia.org/wiki/File:Us127_east_lansing.jpg .
11. CK-12 Foundation. .
12. CK-12 Foundation. .
13. Erdenebayar. https://pixabay.com/en/street-traffic-high-way-automobile-1721451/ .
14. KSmrq. https://commons.wikimedia.org/wiki/File:Riemann_sum_convergence.png .
15. CK-12 Foundation. .
16. CK-12 Foundation. .
17. CK-12 Foundation. .
18. CK-12 Foundation. .
19. CK-12 Foundation. .
20. CK-12 Foundation. .

CHAPTER 16 Appendix A: Answers

Chapter Outline

Please note that this Answer Key is based on the published version of the CK-12 College Precalculus book. If you customize this book, please update the matching answer keys in your customized version.

16.1 Answers - Ch 1: Prerequisites

Section 1.2: Real Numbers

1. Real, Rational, Integer
2. Real, Rational
3. Real, Rational, Integer, Whole
4. Real, Irrational
5. Real, Rational
6. Real, Rational
7. Real, Irrational
8. Real, Rational, Integer, Whole, Natural
9. Real, Irrational
10. Real, Rational, Integer, Whole, Natural
11. $\frac{61}{50}, \frac{\sqrt{6}}{2} = \sqrt{1.5}, \frac{16}{13}$ ($\frac{\sqrt{6}}{2}$ and $\sqrt{1.5}$ are equal and between $\frac{61}{50}$ and $\frac{16}{13}$)
12. -7$\frac{1}{2}$, -3, -1.5, 5.5, 7$\frac{1}{2}$
13. -2.2, -0.3, $\frac{9}{10}$, $\sqrt{4}$
14. $\frac{3}{4}$, 1, $\sqrt{3}$, 1.85, $\frac{26}{10}$
15. $a = \frac{2}{3},\ b = 1\frac{4}{5} = 1.8,\ c = 2\frac{1}{4} = 2.25,\ d = 3\frac{7}{9} = 3.7\overline{7}$

Section 1.3: Exponents

1. -$\frac{1}{27}$
2. 24
3. 1
4. $\frac{4}{5}$
5. -729
6. 9
7. $\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2}$
8. $\frac{64w^3y^9}{x^6z^{12}}$
9. $a^2b^2c^4d^2$
10. $m^6 + m^5 - 5m^2$
11. x^9y^6
12. x^2y^{10}
13. $\frac{8x^{30}}{27y^{60}}$
14. $5x^4$
15. -$64x^7$
16. $12x^{15}$
17. x^8
18. 3^{15}
19. 3^8
20. 3^8

Section 1.4: Scientific Notation

1. 4.2×10^4
2. 8.7×10^{-4}
3. 1.5064×10^2
4. 5.6789×10^4
5. 9.47×10^{-3}
6. 426,000
7. 80,000
8. 59,670,000,000
9. 0.000001482
10. 0.00764
11. 5.19×10^5
12. 8.05×10^{-4}
13. 4.187×10^{12}
14. $\approx 1.392857 \times 10^2$
15. 3.8906125×10^{15} or 3,890,612,500,000,000
16. $\approx 3.0715 \times 10^{-13}$ or 0.00000000000030715

Section 1.5: Radicals

1. 5
2. $2\sqrt{6}$
3. $20\sqrt{5}$
4. $\frac{1}{2}$
5. $\frac{3}{2}$ or $1\frac{1}{2}$
6. 0.4
7. 0.1
8. 3.61
9. 9.95
10. 1.41
11. 44.72
12. 0.5 (or, when set to 2 decimal places, 0.50)
13. 1.16
14. 0.61
15. 0.1 (or, when set to 2 decimal places, 0.10)
16. $45^{\frac{1}{5}} \approx 2.14$
17. $140^{\frac{1}{9}} \approx 1.73$
18. $50^{\frac{3}{8}} \approx 4.34$
19. $\sqrt[3]{72}^{5} \approx 1{,}246.10$
20. $\sqrt[4]{125^3} \approx 37.38$
21. $2\sqrt{15} + 4\sqrt{3}$
22. $a - b$
23. $4x + 20\sqrt{x} + 25$
24. $\frac{7\sqrt{15}}{15}$
25. $\frac{9\sqrt{10}}{10}$
26. $\frac{2\sqrt{5}}{5}$
27. 16
28. 81

29. 32
30. 125
31. 243
32. 4
33. $\left(\frac{2}{3}\right)^{\frac{8}{4}} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$
34. $\left(\frac{7}{2}\right)^{\frac{6}{3}} = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$ or $12\frac{1}{4}$
35. $\left(16^{\frac{1}{2}}\right)^{\frac{6}{2}} = 16^{\frac{6}{4}} = 64$
36. $2ab^2\sqrt[3]{6b}$
37. $\frac{2x}{3y}\sqrt[3]{\frac{2x^2}{5y}}$
38. False
39. -18$\sqrt{2}$
40. $7\sqrt{6}+9\sqrt{3}$
41. -26$x\sqrt{2x}$
42. $7\sqrt{3a}$
43. $5x\sqrt[3]{4}$
44. $r \approx 6.098\ cm$
45. $r \approx 3.0456\ cm$

Section 1.6: Expressions with One or More Variables

1. $22x$
2. $1.35y$
3. $\frac{3}{4}$
4. $\frac{z}{4}$
5. 22
6. 21
7. 4
8. -5
9. 16
10. -79
11. -$\frac{3}{2}$ or -1$\frac{1}{2}$
12. -$\frac{11}{7}$ or -1$\frac{4}{7}$
13. $\frac{3}{10}$
14. -8
15. -53
16. -$\frac{37}{91}$
17. -10
18. $3\frac{1}{9}$ or $\frac{28}{9}$
19. $c \approx 7.85\ inches$

20. $A = 93.5$ *square inches*
21. \$15.84
22. Approximately 26.1 hours
23. $A = 110.25$ *square miles*

Section 1.7: Factoring Polynomials

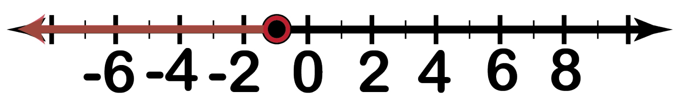
1. $7(x-2)$
2. $3x(1+3y)$
3. $yz(x+yz^2)$
4. $2(x+5)(x+3)$
5. $5(x-7)(x-7)$
6. $-x(x-7)(x-10)$
7. $2(x-4)(x+4)(x^2+16)$
8. $x^2(5x-2)(5x-2)$
9. $3x(2x+1)(2x+1)$
10. $3(2c-5)(2c+5)$
11. $6(x+10)(x-10)$
12. $-5(t+2)(t+2)$
13. $6(x+4)(x-1)$
14. $-(n-3)(n-7)$
15. $2(a+1)(a-8)$

Section 1.8: Solving Multi-Step Equations

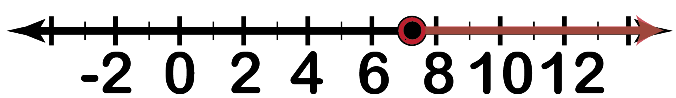
1. $p = 15$
2. $j = 18\frac{2}{3}$
3. $b = 1$
4. $x = 4.5$
5. $x = -6$
6. $k = \frac{1}{6}$
7. $c = -14$
8. $x = 4$
9. $x = \frac{6}{5}$ or $1\frac{1}{5}$
10. $t = -4$
11. $x = \frac{12}{7}$ or $1\frac{5}{7}$
12. $d = \frac{5}{9}$
13. $x = -1$
14. $x = -\frac{6}{5}$ or $-1\frac{1}{5}$
15. $j = -10$

Section 1.9: Inequalities

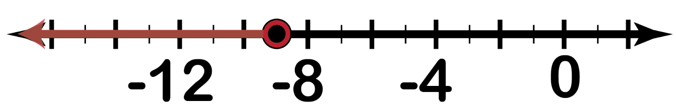
1. $x < -3$ $(-\infty, -3)$
2. $x \geq 17$ $[17, \infty)$
3. $x > -\frac{1}{12}$ $\left(-\frac{1}{12}, \infty\right)$
4. $x \geq -75$ $[-75, \infty)$
5. $x > \frac{120}{31}$ $\left(\frac{120}{31}, \infty\right)$
6. $x \geq -\frac{7}{2}$ $\left[-\frac{7}{2}, \infty\right)$
7. $x > 1$ $(1, \infty)$
8. $-\infty < x < \infty$ $(-\infty, \infty)$
9. $x < -14$ $\{x \mid x < -14\}$
10. $x \geq 92$ $\{x \mid x \geq 92\}$
11. $x \geq -15$ $\{x \mid x \geq -15\}$
12. $x \geq 1\frac{2}{3}$ $\left\{x \mid x \geq 1\frac{2}{3}\right\}$
13. $x < \frac{10}{3}$ $\left\{x \mid x < \frac{10}{3}\right\}$
14. $x > -120$ $\{x \mid x > -120\}$
15. $x > 5$ $\{x \mid x > 5\}$
16. $-\infty < x < \infty$ $\{x \mid -\infty < x < \infty\}$
17. $\{x \mid x < -1\}$



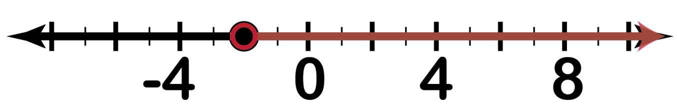
18. $\left\{x \mid x > \frac{36}{5}\right\}$



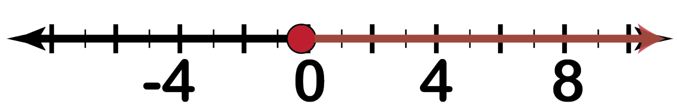
19. $\{x \mid x < -9\}$



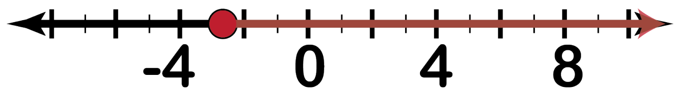
20. $\{x \mid x > -2\}$



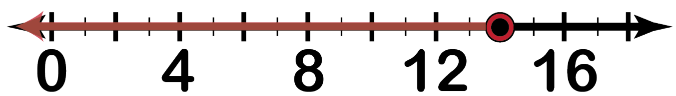
21. $\left\{x \mid x \geq -\frac{1}{5}\right\}$



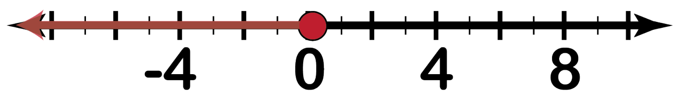
22. $\{x \mid x \geq -\frac{11}{4}\}$



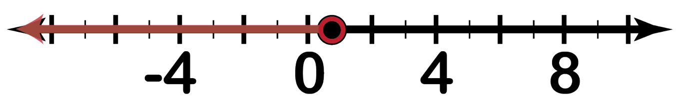
23. $\{x \mid x < 14\}$



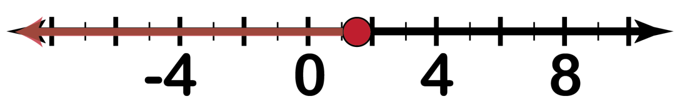
24. $\{x \mid x \leq \frac{1}{10}\}$



25. $\{x \mid x < \frac{3}{5}\}$



26. $\{x \mid x \leq \frac{3}{2}\}$

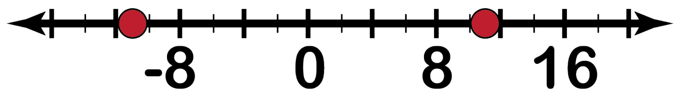


27. a) 3 times b) Finitely many
28. 99

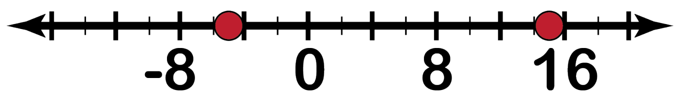
Section 1.10: Absolute Value

1. 250
2. 12
3. 0.003
4. $\frac{2}{5}$
5. $\frac{1}{10}$
6. 23
7. 17
8. 9
9. 5

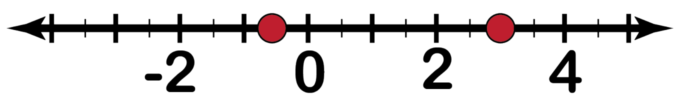
10. 1.079
11. $\frac{37}{24}$
12. $u = 11$ or $u =$ -11



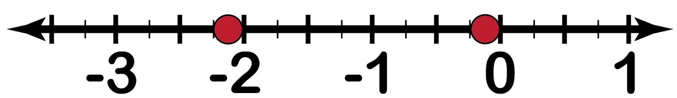
13. $x = 15$ or $x =$ -5



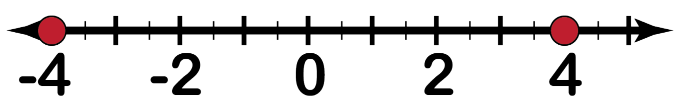
14. $r = 3$ or $r =$ -$\frac{3}{5}$



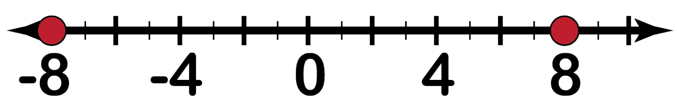
15. $z =$ -$\frac{1}{5}$ or $z =$ -$\frac{11}{5}$



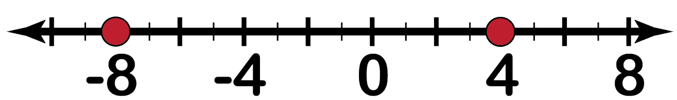
16. $x = 4$ or $x =$ -4



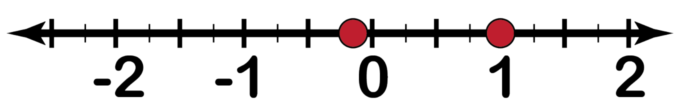
17. $m = 8$ or $m =$ -8



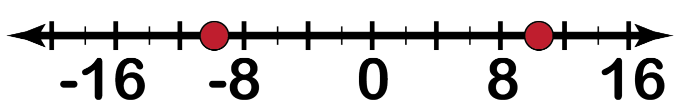
18. $x = 4$ or $x =$ -8



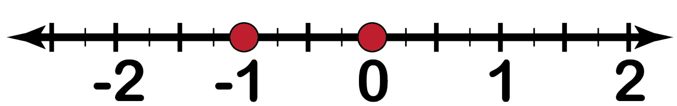
19. $x = 1$ or $x =$ -$\frac{1}{5}$



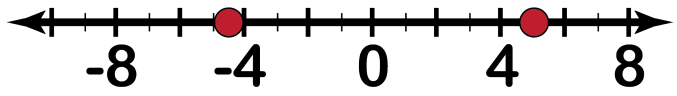
20. $b =$ -10 or $b = 10\frac{2}{5}$



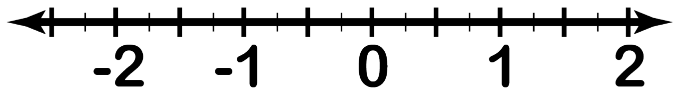
21. $y = 0$ or $y =$ -1



22. $x = 5$ or $x =$ -4.5



23. No solutions

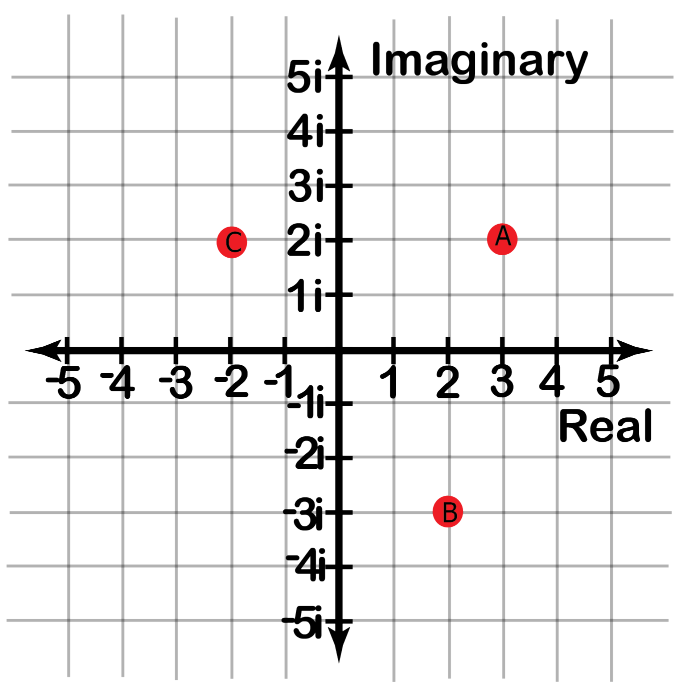


24. Longest length = $12\frac{1}{32}$ *inches*

    Shortest length = $11\frac{31}{32}$ *inches*
25. 20 feet
26. $16,150

Section 1.11: Imaginary and Complex Numbers

1. $3i$
2. $i\sqrt{17}$
3. $4i\sqrt{2}$
4. -12
5. $12i$
6. -4$\sqrt{3}$
7. -2$\sqrt{15}+11i$
8. $0+i$
9.



10. $18i$
11. $11i$

12. -4i
13. -i
14. 1.1i
15. -i
16. 24
17. i
18. 1
19. -i
20. -25
21. -225
22. $720i\sqrt{15}$
23. A real number and an imaginary number

24. $13-7i$
25. $10-\frac{1}{3}i$
26. $4+5\sqrt{10}i$
27. $0.27i$
28. $3\sqrt{3}+0.4i$
29. $15+18j^4i\sqrt{2j}$
30. -2 + 2i and 4 − 2i

Section 1.12: Coordinate Geometry

1. a. (5,6)

   b. (-5,5)

   c. (-2,3)

   d. (-2,-2)

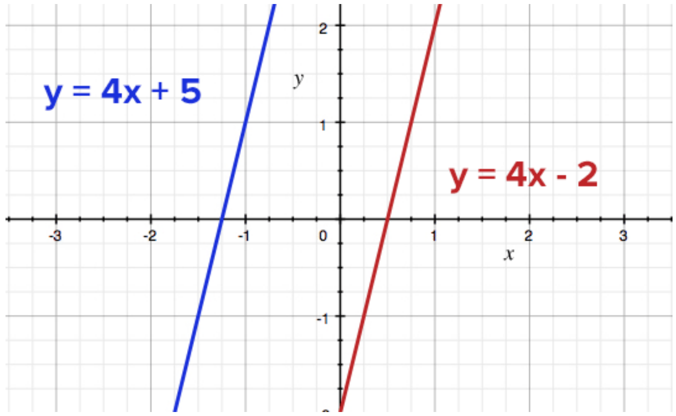
   e. (3,-4)

   f. (2,-6)
2. (-2,2)

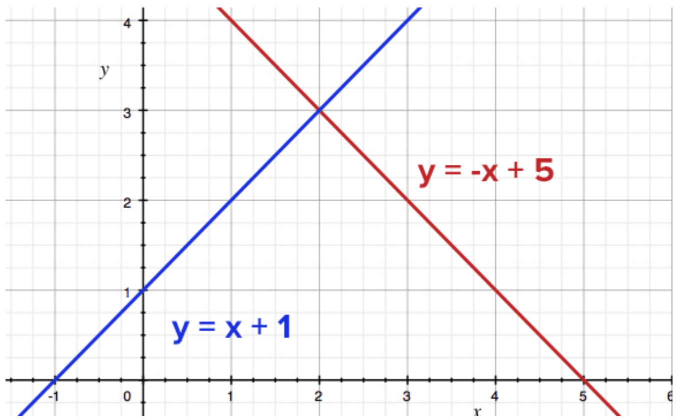
3. Quadrant I
4. Quadrant II
5. Quadrant IV
6. Quadrant III

7. -1
8. -2
9. 4
10. Undefined
11. 0
12. $-\frac{1}{18}$
13. $-\frac{5}{6}$

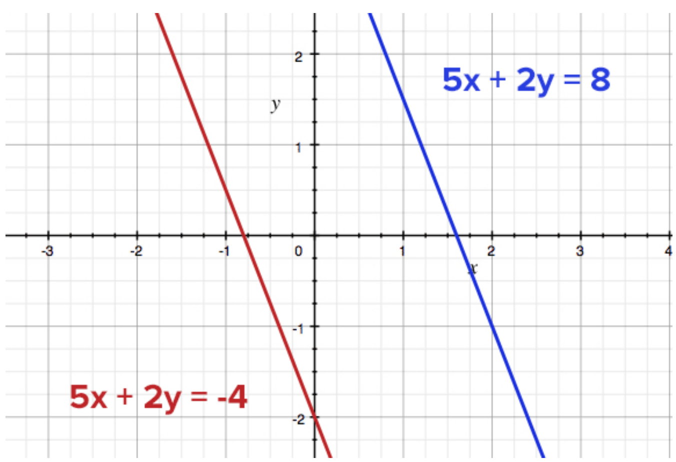
14. Parallel



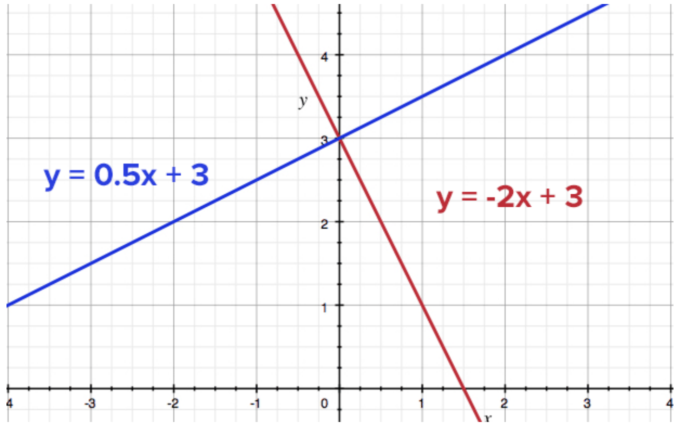
15. Perpendicular



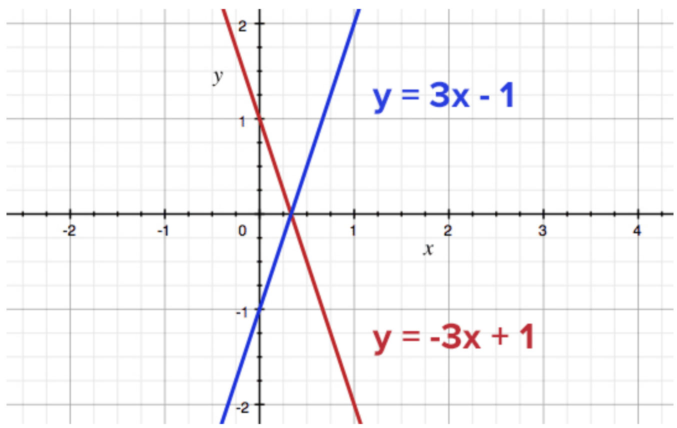
16. Parallel



17. Perpendicular



18. Neither

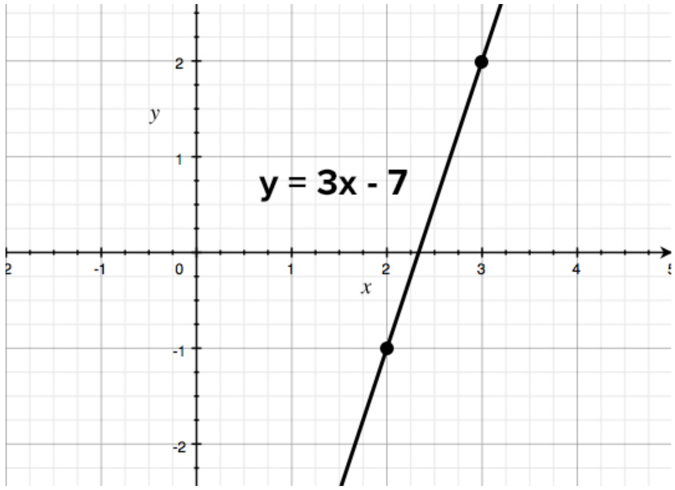


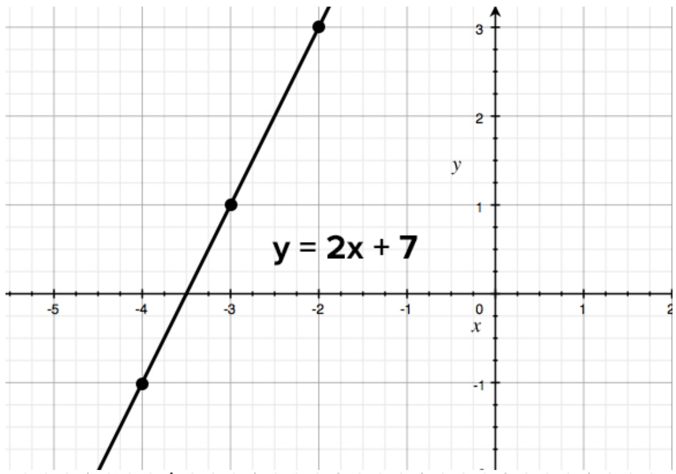
19. $y = -5x - 7$
20. $y = \frac{2}{3}x - 5$
21. $y = \frac{1}{4}x + 2$
22. $y = -x - 4$
23. $y = -\frac{1}{3}x - 4$
24. $y = -\frac{2}{5}x + 7$
25. $x = -1$

Section 1.13: Linear Equations

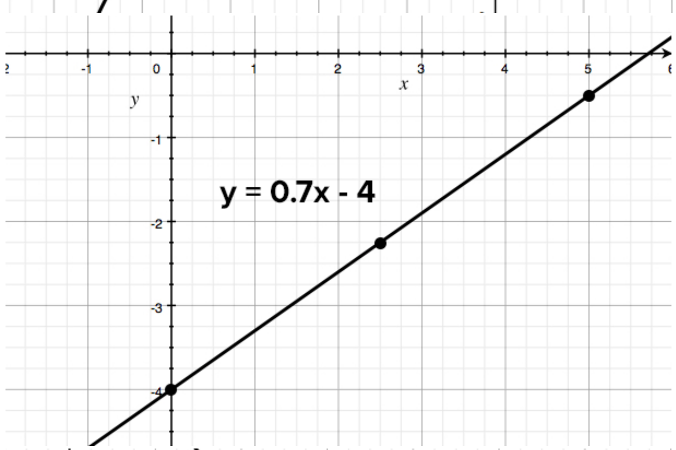
1. The solutions are points on the coordinate plane, as opposed to values on a number line.
2. (Table values may vary.)

x	y
1	-4
2	-1
3	2

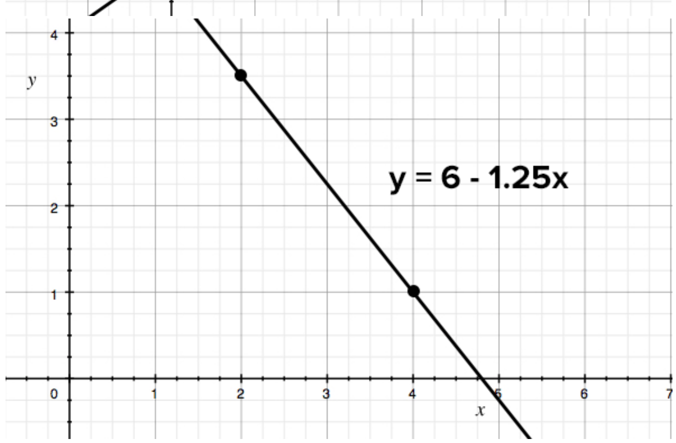




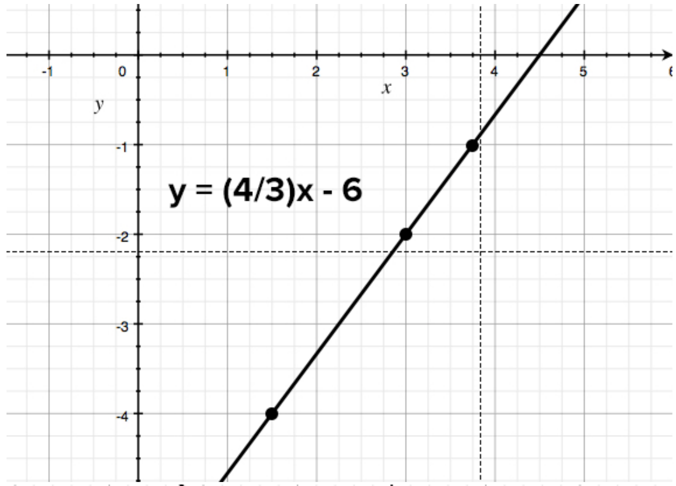
3.

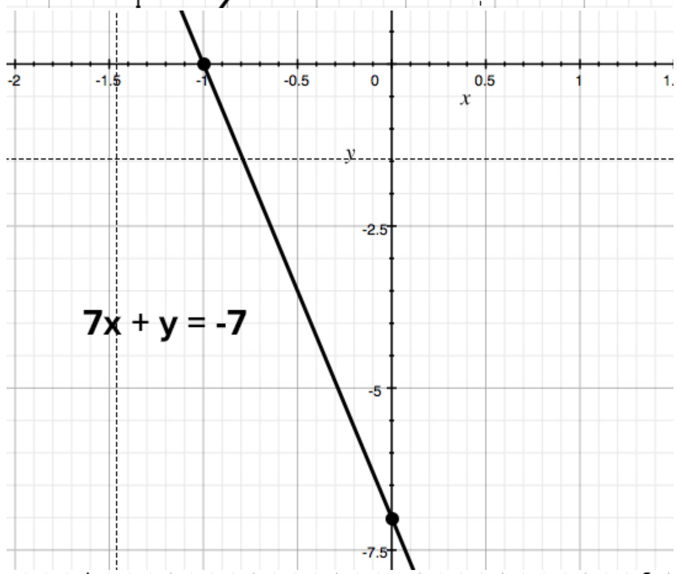


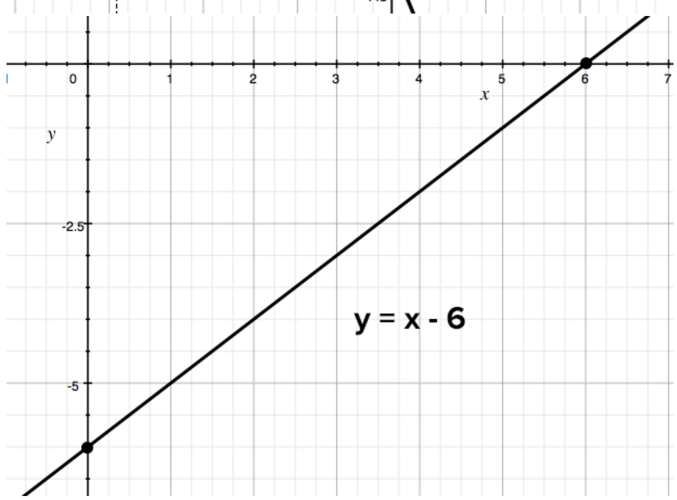
4.

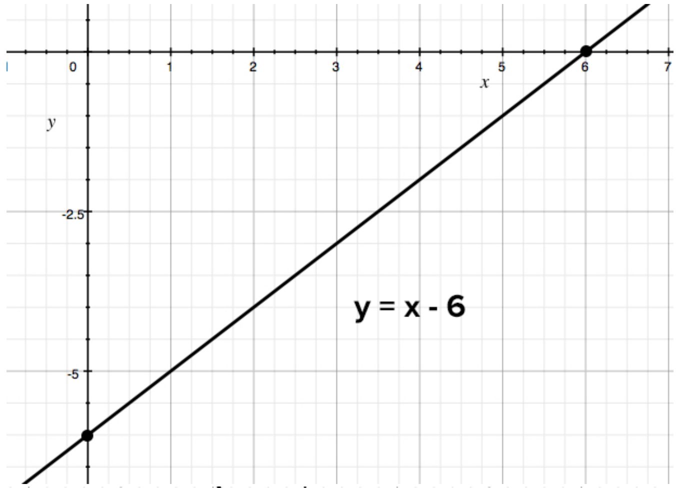


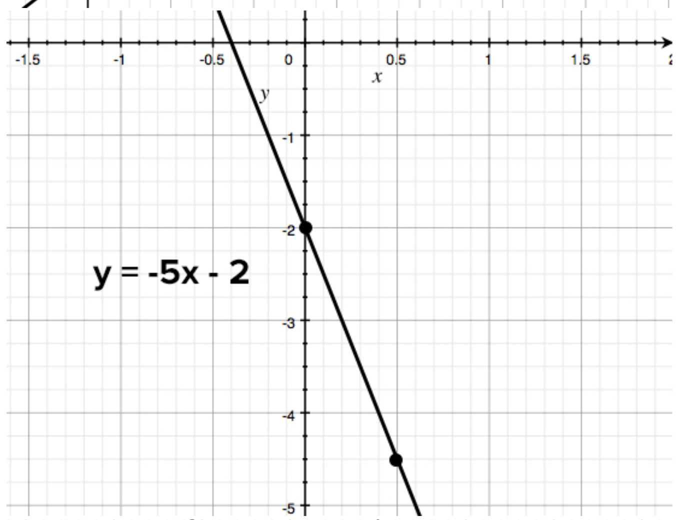
5.

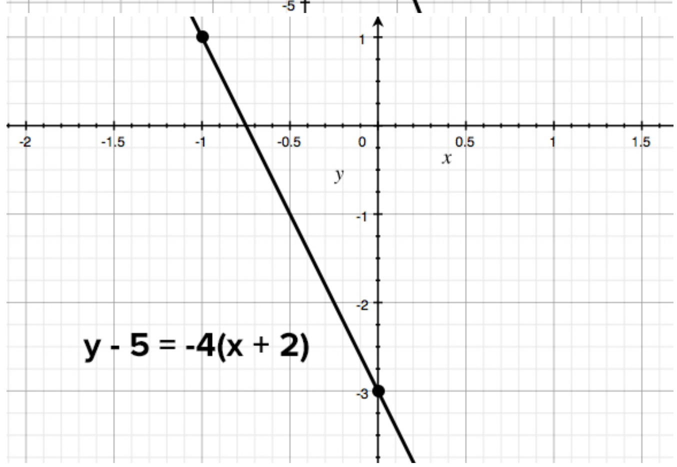
6. 
y = (4/3)x - 6
x
y
-1
0
1
2
3
4
5
-1
-2
-3
-4

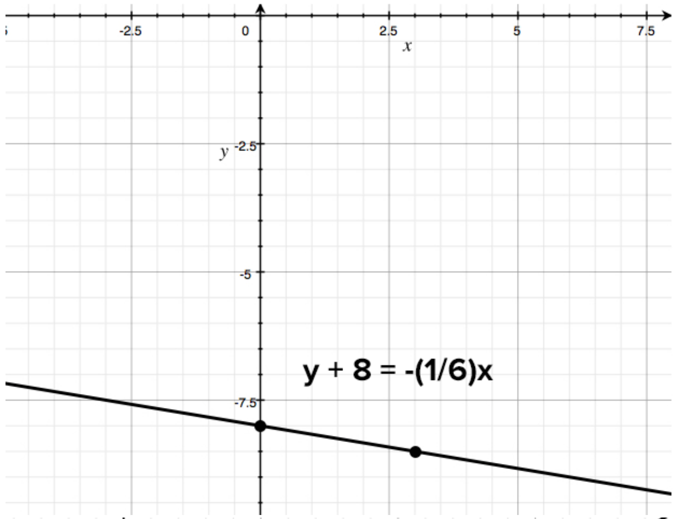
7. 
7x + y = -7
x
y
-2
-1.5
-1
-0.5
0
0.5
1
-2.5
-5
-7.5

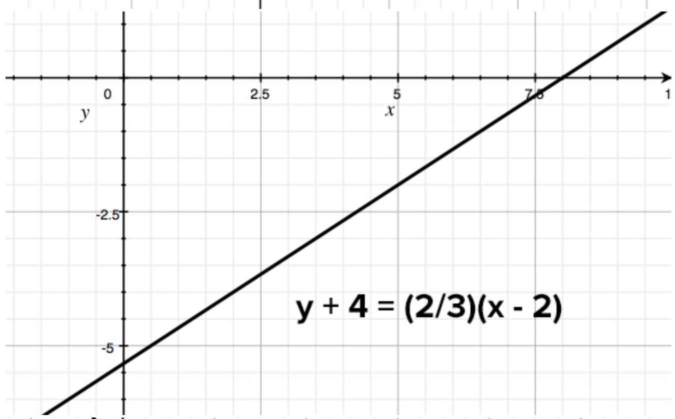
8. 
y = x - 6
x
y
0
1
2
3
4
5
6
7
-2.5
-5

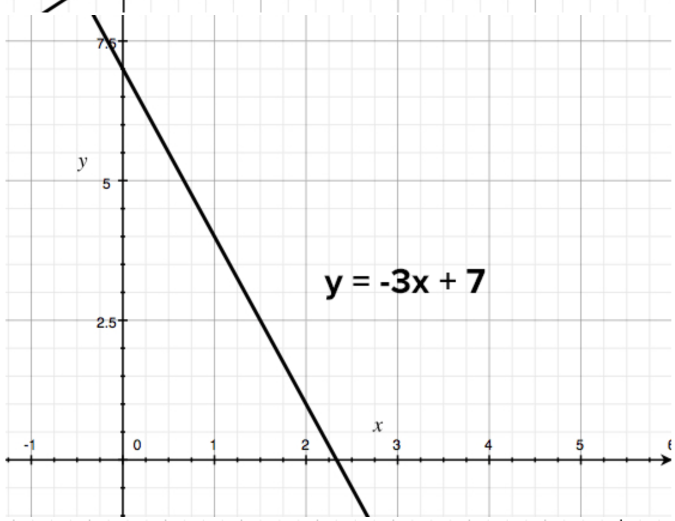
9. 
0
1
2
3
4
5
6
7
x
y
-2.5
-5
y = x - 6

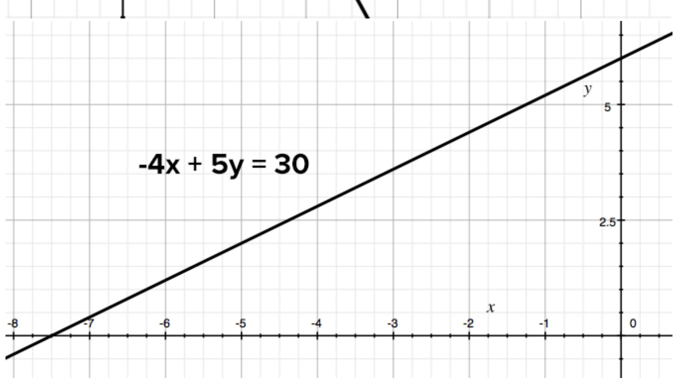
10. 
-1.5
-1
-0.5
0
0.5
1
1.5
x
y
-1
-2
-3
-4
-5
y = -5x - 2

11. 
1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
x
y
-1
-2
-3
y - 5 = -4(x + 2)

12. 
y + 8 = -(1/6)x
x
y
-2.5
0
2.5
5
7.5
-5
-7.5

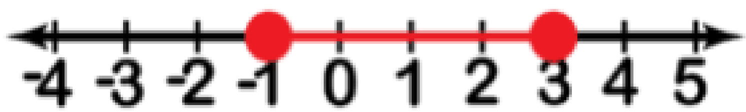
13. 
y + 4 = (2/3)(x - 2)
x
y
0
2.5
5
-2.5
-5

14. 
y = -3x + 7
x
y
-1
0
1
2
3
4
5
2.5
5

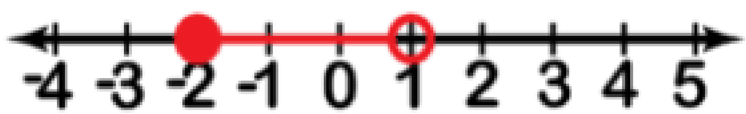
15. 
-4x + 5y = 30
x
y
-8
-7
-6
-5
-4
-3
-2
-1
0
2.5
5

Section 1.14: Intervals and Interval Notation

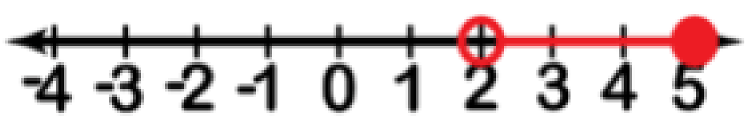
1. [-3, 1)
2. (0, 2)
3. (-3, +∞)
4. (-∞, 2]
5. $x > 1 \therefore (1, +\infty)$
6. $x \leq -2 \therefore (-\infty, 2]$
7. (-∞, -3) (0, +∞)
8. (-∞, -5)
9. (-3, 5]
10. (-∞, 2) (2, +∞)
11. *Domain*: (-4, 6) *Range*: (-2, 7)
12. *Domain*: (-5, 7] *Range*: [-1.25, 8), [-1, 8) is also acceptable
13. [-1, 3]



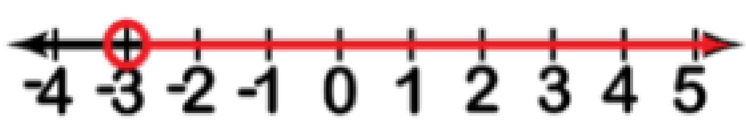
14. [-2, 1)



15. (2, 5]



16. (-3, +∞)



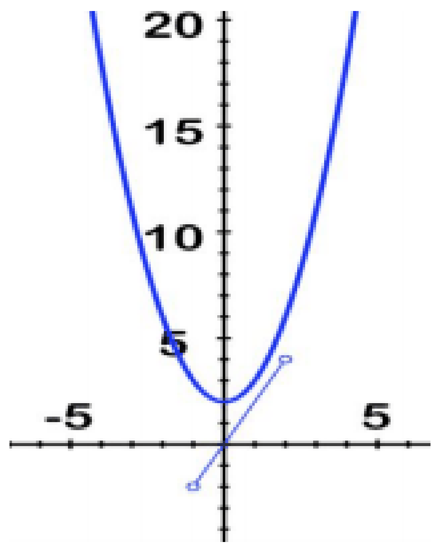
Section 1.15: Average Rate of Change

1. At $x = 1, y = 1$ and at $x = 3, y = 27$, so the average rate of change is $\frac{26}{2} = 13$ or $\frac{13}{1}$.
2. At $x = -4,\ y = -64$, and at $x = -1, y = -1$, so the average rate of change is $\frac{63}{3} = 21$ or $\frac{21}{1}$.
3. If $f(1) = 2$ and the average rate of change of f between 1 and 5 is 3, then $f(2) = 5 : f(3) = 8 : f(4) = 11$, and $f(5) = 14$.

4. Jamie's average speed during the 1st half of the trip was: $\frac{21km}{1.5hr} = 14\ km/hr$. During the last half it was $\frac{15km}{1.5hr} = 10\ km/hr$. His entire trip was $\frac{36km}{3hr} = 12\ km/hr$. Jamie averaged more than 11.5 km/hr.
5. The difference in price is \$0.35 total, the time change was 5 days, therefore the gas price averaged a change of \$0.07 per day.
6. The population difference was 1,317 over a period of 21 years. The average rate of change was $\frac{1{,}317}{21} \approx 62.7$ persons per year.
7. The change in price is \$335 over an increase of 18 yd^3. The average change in length is $\frac{335}{18} = \$18.61/yd^3$.
8. The increase in length is 2 inches, the increase in weight is 4 lbs. The average change in length is $\frac{2in}{4lbs} = \frac{1in}{2lbs} = \frac{.5in}{1lb}$.
9. The change in speed is 15 mph, the increase in efficiency is 5 mpg. The average increase in efficiency is $\frac{15mph}{5mpg} = 3\frac{mph}{mpg}$ or $\frac{1mpg}{3mph}$.
10. If $y = f(x) = x^2 + x + 2$, then when $x =$ -1, $y =$ -2, and when $x = 2$, $y = 4$.

    a) Average rate of change: $\frac{\Delta y=6}{\Delta x=3} = 2$

    b) Graph of f and the graph of the line through (-1,-2) and (2,4):



    c) $m = 2$

    d) Answers will vary, but should note that the slope is the slope of the line.
11. a) Amy's average *driving* speed for the trip was $42.5mph$ or $\frac{85mi}{120min}$.

    b) Her average speed for the *entire* trip (including the stop at the diner) was $\frac{85mi}{130min} = 39.2mph$.
12. If the weight $w(t)$ (in grams) of tumor, t weeks after it forms is given by $w(t) = \frac{t^2}{15}$, then the average rate at which the tumor is growing during the 5th week after it was formed is the increase in weight "during" week 5. Subtract the weight at week 6 from the weight at week 5: $\frac{36}{15} - \frac{25}{15} \approx 0.75g$.
13. \$10,833
14. $\frac{5}{12}$
15. \$200
16. 5 mph

Section 1.16: Relations and Functions

1. A function is a statement defining a single result for each question, or a single output of each input.
2. Yes
3. Yes
4. Yes
5. Answers vary, but should mention how the function does not always have the same output for a given input.
6. Function
7. Not a function
8. Not a function
9. Function
10. Function
11. Not a function
12. Not a function
13. Function
14. Function
15. Yes, this is a function. One boy, one corsage/date.
16. It does change the answer; now it is not a function. Cory is one boy with two corsages/dates.

16.2 Answers - Ch 2: Functions and Graphs

Section 2.2: Domain and Range

1. Domain: $x \in (-\infty, \infty)$ Range: $y \in [-1, 1]$
2. Domain: $x \in (-\infty, 2) \cup [3, \infty)$ Range: $y \in [-8, \infty)$
3. Domain: $x \in (-\infty, \infty)$ Range: $y \in [0, 2]$
4. Domain: $x \in (-\infty, \infty)$ Range: $y \in [-3, \infty)$
5. Domain: $x \in (-\infty, \infty)$ Range: $y \in (-\infty, 2]$
6. Domain: $x \in (-\infty, \infty)$ Range: $y \in (-1, \infty)$
7. Domain: $x \in (-3, \infty)$ Range: $y \in (-\infty, \infty)$
8. Domain: $x \in \left\{-2, \frac{3}{4}, \frac{\pi}{2}, 2, 3\right\}$ Range: $y \in \{1, \pi, 5, 7\}$
9. Domain: $x \in (-\infty, \infty)$ Range: $y \in (-\infty, 4]$
10. Domain: $x \in \left(\frac{1}{2}, \infty\right)$ Range: $y \in (-\infty, \infty)$
11. Domain: $x \in (-\infty, 1) \cup (1, \infty)$ Range: $y \in (-\infty, 0) \cup (0, \infty)$
12. Domain: $x \in [-4, \infty)$ Range: $y \in (-\infty, -1]$
13. Domain: $x \in (-\infty, -6) \cup (-6, \infty)$ Range: $y \in (-\infty, -1) \cup (-1, \infty)$
14. Domain: $x \in (-\infty, -1) \cup (1, \infty)$ Range: $y \in (-\infty, \infty)$
15. Domain: $x \in (-\infty, -1.5] \cup [1.5, \infty)$ Range: $y \in [6, \infty)$
16. The independent variable is h, the hours he worked. Domain: $x \in [20, 25]$ Range: $y \in [200, 250]$
17. Domain: $x \in [10, 12]$ Range: $y \in [300, 360]$ She can drive between 300 and 360 miles.
18. Domain: $x \in [4, 8]$ Range: $y \in [11, 22]$ The evening cost between \$11 and \$22.

Section 2.3: Maximums and Minimums

1. There is a global minimum at (3, 0).
2. There is a local minimum at (3, 0).
3. Global minimum at $\left(-\frac{\pi}{2}, -1\right)$, and global maximum at $\left(\frac{\pi}{2}, 1\right)$
4. Local minimum at $\left(-\frac{\pi}{2}, -1\right)$, and local maximum at $\left(\frac{\pi}{2}, 1\right)$
5. There are no global extrema.
6. There are no local extrema.
7. There are no global extrema.
8. Local minimums: (0.4, -1), (2.5, -13). Local maximums: (-1.5, 22), (1, 0). [Note: Points are approximate.]
9. There are no global extrema.
10. Local minimum: (3, 0). Local maximum: (0.5, 9.5). [Note: Points are approximate.]
11. A global maximum is the overall highest point on the graph, while the local maximum is the highest point within a certain neighborhood of the graph.
12. Answers vary. Graph should show a global minimum, a local maximum, and no global maximum. (There can be a local minimum.)
13. Answers vary. Graph should have no global extrema, but both types of local extrema.
14. Local maximum: (-1.16, 36.24). Local minimum: (-4, 0). No global maximum. Global minimum: (2,16, -18.49).
15. Local maximum: (-1,0). Local minimum: (0.22, -3.23). Global maximum: (2.28, 9.91). No global minimum.
16. A length and width of approximately 4.472 will minimize the perimeter. The perimeter would be approximately 17.889 inches.

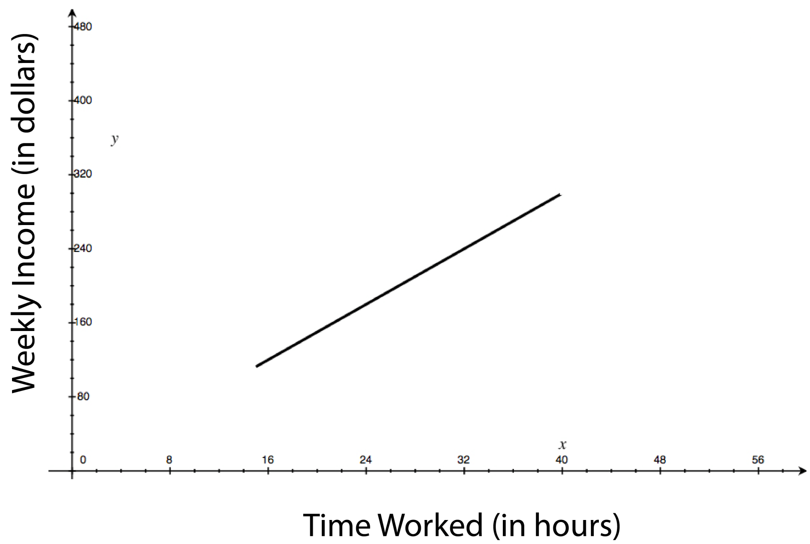
17. $A = w \cdot \left(\frac{P-2w}{2}\right)$ or $A = -w^2 + \frac{1}{2}Pw$. The rectangle with the maximum area would be the one where the width is 1/4 of the perimeter.
18. 800 square feet

Section 2.4: Symmetry

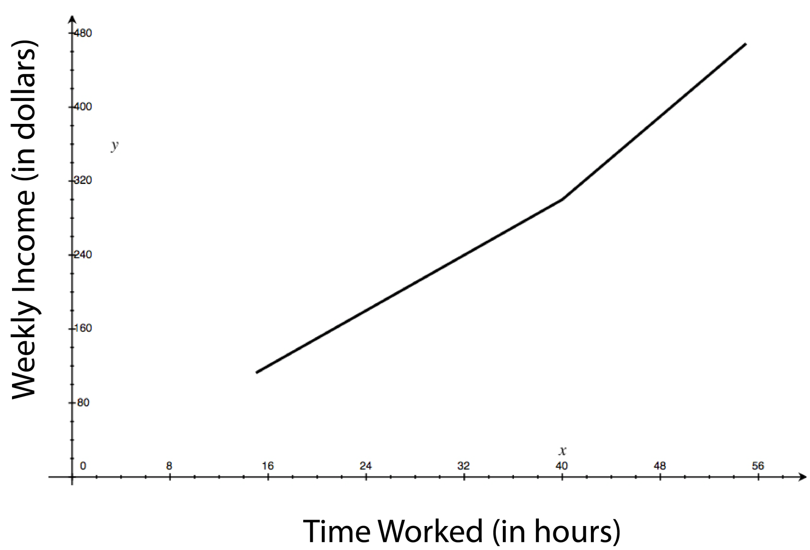
1. Even
2. Odd
3. Neither
4. Neither
5. Odd
6. Neither
7. Neither
8. Even
9. $f(-x) = h(-x) - g(-x) = h(x) + g(x) \neq f(x)$ or $-f(x)$
10. $f(-x) = \frac{h(-x)}{g(-x)} = -\frac{h(x)}{g(x)} = -f(x)$
11. $f(-x) = h(-x)g(-x) = -h(x)g(x) = -f(x)$
12. Yes. If $h(x)$ and $g(x)$ are both even and $f(x) = h(x) + g(x)$, then $f(-x) = h(-x) + g(-x) = h(x) + g(x) = f(x)$.
13. Yes. If $h(x)$ and $g(x)$ are both odd and $f(x) = h(x) + g(x)$, then $f(-x) = h(-x) + g(-x) = -h(x) - g(x) = -[h(x) + g(x)] = -f(x)$
14. There are some functions that do not have reflection symmetry across the y-axis, or rotation symmetry about the origin.
15. If a function is even, then it is symmetrical across the y-axis. If a function is odd, then it has rotation symmetry about the origin.

Section 2.5: Increasing and Decreasing

1. Increasing: $x \in (3,\infty)$
2. Decreasing: $x \in (-\infty,3)$
3. Increasing: $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$
4. Decreasing: $x \in (-\pi,\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$
5. Increasing: $x \in (-\infty,\infty)$
6. None
7. Increasing: $x \in (-\infty, -1.4) \cup (0.3, 1) \cup (2.5, \infty)$ [Note: Points are approximate.]
8. Decreasing: $x \in (-1.4, 0.3) \cup (1, 2.5)$ [Note: Points are approximate.]
9. Increasing: $x \in (-\infty, 0.3) \cup (3, \infty)$
10. Decreasing: $x \in (0.3, 3)$
11. Answers vary. [Possible answer: A line with a positive slope.]
12. Answers vary. [Possible answer: A line with a negative slope.]
13. Increasing: $x \in (-\infty, 1) \cup (3, \infty)$ Decreasing: $x \in (1, 3)$
14. Increasing: $x \in (-\infty, 1)$ Decreasing: $x \in (1, \infty)$
15. Increasing: $x \in (5, \infty)$ Decreasing: $x \in (-\infty, 5)$
16. Extrema are at (15, 112.5) and (40, 300).



17. Max value changes if he works overtime. The new extrema are (15, 112.5) and (55, 468.75).
18. Extrema are (15, 112.5) and (55, 468.75).



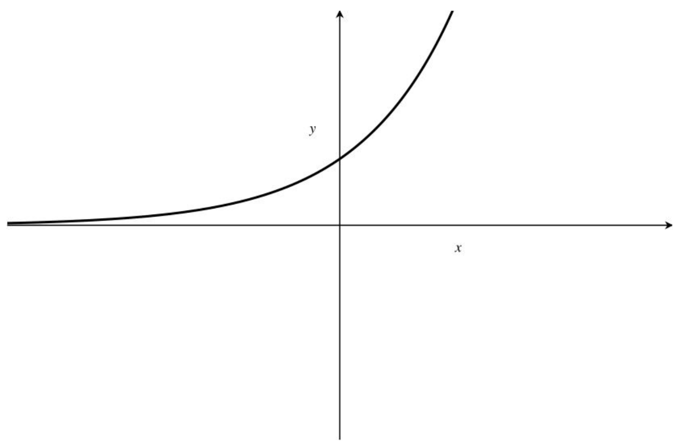
Section 2.6: Intercepts of Graphs of Functions

1. y-intercept: (0, -4); Zeroes: (-1,0) and (4, 0)
2. y-intercept: (0, -12); Roots: (-3,0), (1,0), and (2, 0)
3. y-intercept is approximately (0, 6), x-intercepts are (-2,0) and (1, 0)
4. Both x- and y -intercepts are at (0,0).
5. Both x- and y-intercepts are at (0,0).
6. No y-intercept; x-intercept is (1, 0).
7. No x- or y-intercepts
8. y-intercept is (0, 1); no x-intercept
9. Both x- and y-intercepts are (0,0).
10. Yes, because there are functions that are undefined when $x = 0$.
11. Yes, because there are functions with no real solutions when $y = 0$.

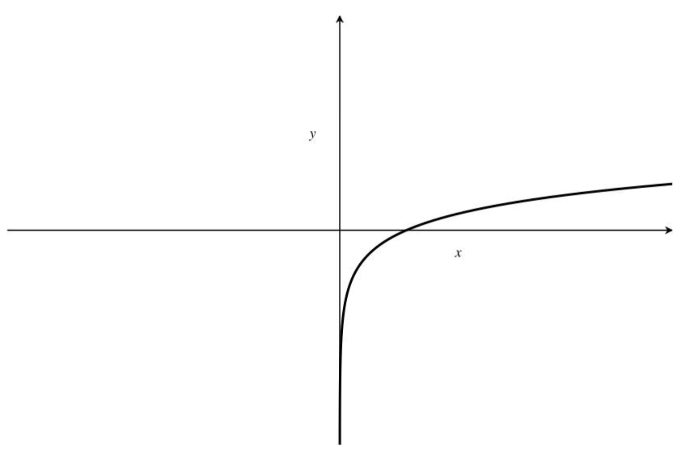
12. The x-intercept of $f(x)$ is called a zero because it is the solution to $f(x) = 0$.
13. y-intercept: (0, 10); x-intercepts: (2,0), (-1,0), (5,0)
14. y-intercept: (0, -7); x-intercepts: (-1,0), (7,0)
15. y-intercept: (0, 5); x-intercepts: (5,0), (-1/2, 0), (1,0)

Section 2.7: Function Families

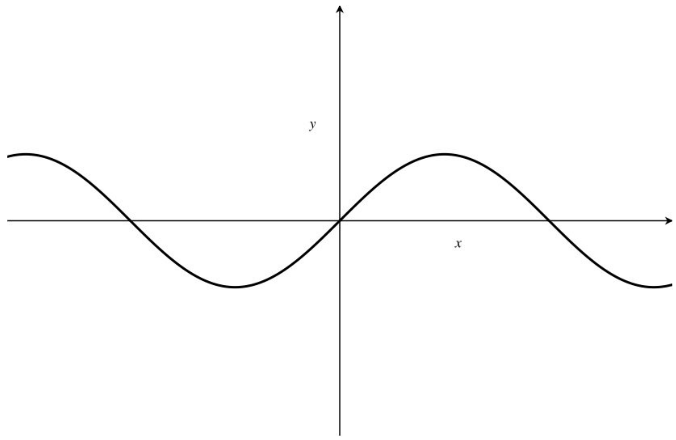
1. $y = b^x$



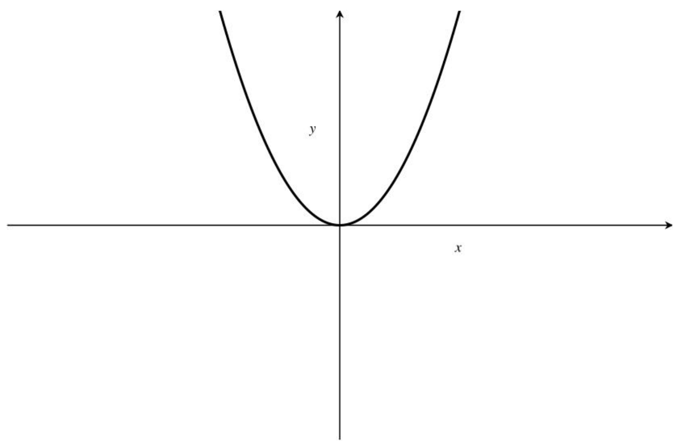
2. $y = \log_b x$



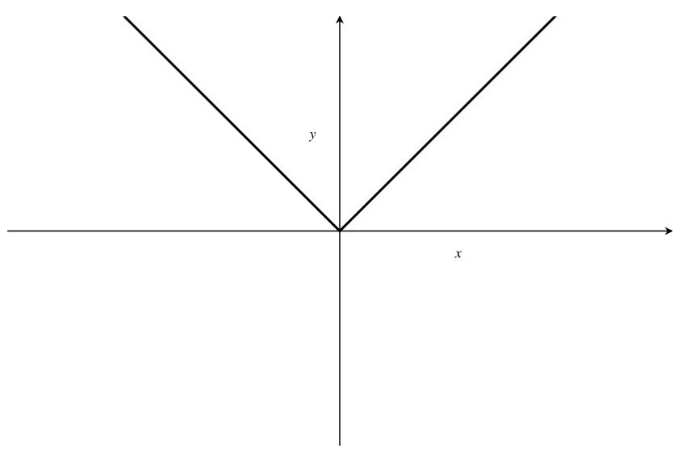
3. $y = \sin x$



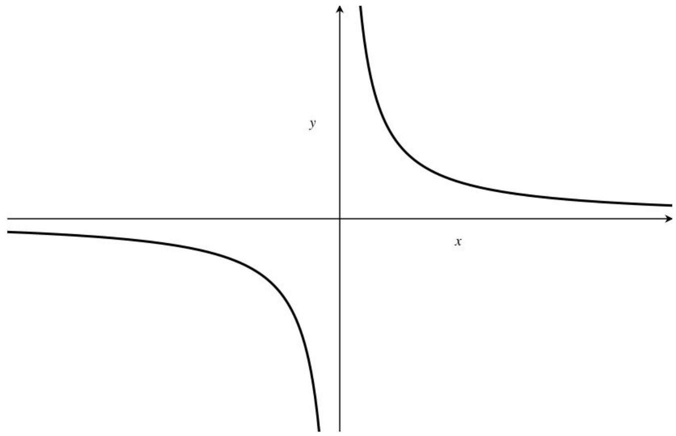
4. $y = x^2$



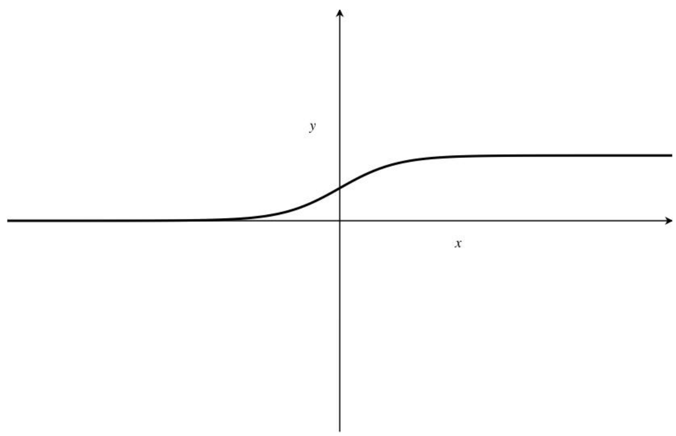
5. $y = |x|$



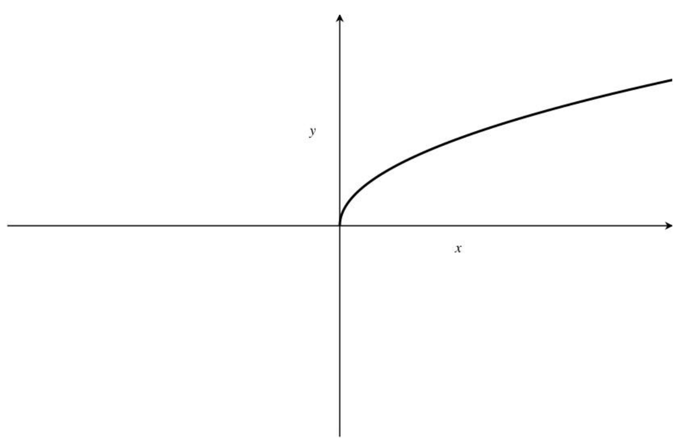
6. $y = \frac{1}{x}$



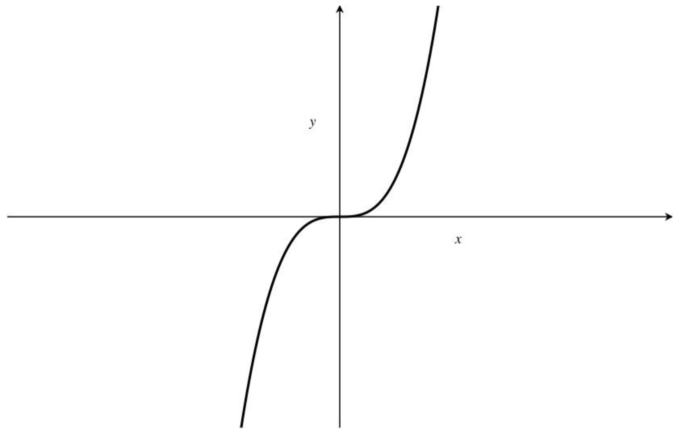
7. $y = \frac{1}{1+b^{-x}}$



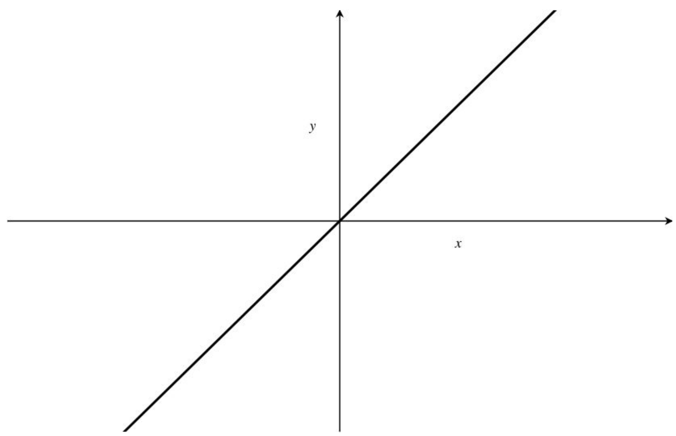
8. $y = \sqrt{x}$



9. $y = x^3$



10. $y = x$



11. $y = \frac{1}{x}$ because $\frac{1}{0}$ is undefined.
12. $y = e^x, y = x^2, \ y = \sqrt{x}, \ y = |x|$
13. One difference is $y = x^2$ has a minimum value, while $y = x^3$ doesn't.
14. The two graphs are reflections of one another across the line $y = x$.
15. $y = \sqrt{x}$ is not defined for all values of x because the square root of any negative number is not a real number.

Section 2.8: Graphical Transformations

1. Reflection across the x-axis and reflection across the y-axis.
2. Reflection across the x-axis and a horizontal shift left 3 units.
3. Horizontal shift left 1 unit and vertical shift down 2 units.
4. Reflection across the y-axis and horizontal shift right 3 units.
5. Reflection across the x-axis and horizontal compression by a factor of 2.
6. Vertical stretch by a factor of 4, horizontal stretch by a factor of 2, and horizontal shift left 2 units.
7. A reflection across the x-axis, a horizontal shift right 2 units, vertical shift down 2 units, and a vertical stretch by a factor of 3.
8. Vertical stretch by a factor of 5 and a horizontal shift left 1 unit.
9. $2h(x-2)+3$
10. $-f(x+2)-1$
11. $\frac{1}{4}g(-x)$

12. $3j(x-2)+3$
13. $k(\frac{1}{4}(x+1))+3$
14. $\frac{1}{2}h(-(x-3))$
15. $-5f(x)$

Section 2.9: Transforming Functions Defined by Data

1. Vertical reflection across the x-axis, vertical compression by a factor of 2, horizontal shift 1 unit left.

$(x,y) \rightarrow (x-1, -\frac{y}{2})$

x	y
0	5
1	6
2	7

$\rightarrow$

x	y
-1	$-\frac{5}{2}$
0	$-\frac{6}{2}$
1	$-\frac{7}{2}$

2. Vertical stretch by a factor of 2, horizontal compression by a factor of 3, and vertical shift up 2 units.

$(x,y) \rightarrow (\frac{x}{3}, 2y+2)$

x	y
0	5
1	6
2	7

$\rightarrow$

x	y
0	12
$\frac{1}{3}$	14
$\frac{2}{3}$	16

3. Reflection across the x-axis, horizontal shift 4 units to the right, vertical shift 3 units down.

$(x,y) \rightarrow (x+4, -y-3)$

x	y
0	5
1	6
2	7

$\rightarrow$

x	y
4	-8
5	-9
6	-10

4. Vertical stretch by a factor of 3, horizontal compression by a factor of 2, horizontal shift 2 units to the right, and vertical shift up 1 unit.

$(x,y) \rightarrow (\frac{x}{2}+2, 3y+1)$

x	y
0	5
1	6
2	7

$\rightarrow$

x	y
2	16
$2\frac{1}{2}$	19
3	22

5. Reflection across the x-axis, horizontal shift right 3 units.

$(x,y) \rightarrow (x+3, -y)$

x	y
0	5
1	6
2	7

$\rightarrow$

x	y
3	-5
4	-6
5	-7

6. $f(x) \rightarrow f(2x-6)-4$
7. $f(x) \rightarrow -f(\frac{x}{2}-2)+1$
8. $f(x) \rightarrow 3f(\frac{x}{4})-5$
9. $f(x) \rightarrow -f(\frac{x}{2})+1$
10. $f(x) \rightarrow -f(\frac{x}{3})+1$

11. $(x,y) \rightarrow (x+2, 3y+1)$
12. $(x,y) \rightarrow (x+1,\text{-}4y+3)$
13. $(x,y) \rightarrow (\frac{x}{2}-1, \frac{y}{2}-5)$
14. $(x,y) \rightarrow (2x+4, 5y-1)$
15. $(x,y) \rightarrow (\frac{x}{2}+2, \frac{y}{4})$

Section 2.10: Asymptotes and End Behavior

1. There are no asymptotes. As x approaches positive infinity, y approaches positive infinity. As x approaches negative infinity, y approaches negative infinity.
2. There are no asymptotes. As x approaches both positive and negative infinity, y approaches positive infinity.
3. There are no asymptotes. As x approaches positive infinity, y approaches positive infinity. As x approaches negative infinity, y approaches negative infinity.
4. There are no asymptotes. As x approaches positive infinity, y approaches positive infinity.
5. There is a horizontal asymptote at $y=0$ and a vertical asymptote at $x=0$. As x approaches both positive and negative infinity, y approaches 0.
6. As x approaches negative infinity, there is a horizontal asymptote at $y=0$. As x approaches positive infinity, y approaches positive infinity. There is no vertical asymptote.
7. There is a vertical asymptote at $x=0$. As x approaches positive infinity, y approaches positive infinity. As x approaches 0, y approaches negative infinity. There is no horizontal asymptote.
8. As x approaches negative infinity, there is a horizontal asymptote at $y=0$. As x approaches positive infinity, there is a horizontal asymptote at $y=1$. There is no vertical asymptote.
9. There is a vertical asymptote at $x=0$. As x approaches positive infinity, there is a horizontal asymptote at $y=0$. As x approaches negative infinity, there is a horizontal asymptote at $y=2$.
10. There is a vertical asymptote at $x=1$. As x approaches both positive and negative infinity, there is a horizontal asymptote at $y=2$.
11. There is a vertical asymptote at $x=4$. As x approaches both positive and negative infinity, there is a horizontal asymptote at $y=1$.
12. Because when $x=0$, $y=\frac{1}{0}$, which is undefined.
13. Because when $x=-3$, $y=\frac{1}{0}$, which is undefined.
14. $x=2$
15. $x=\text{-}4$

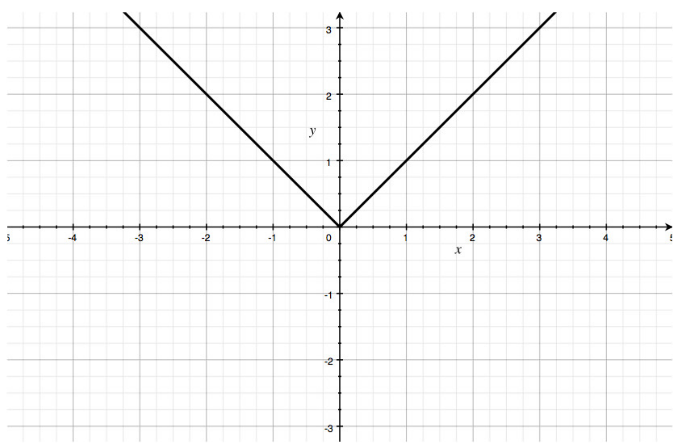
Section 2.11: Continuity and Discontinuity

1. This function is continuous.
2. This function is continuous.
3. This function is continuous.
4. This function is continuous on its domain.

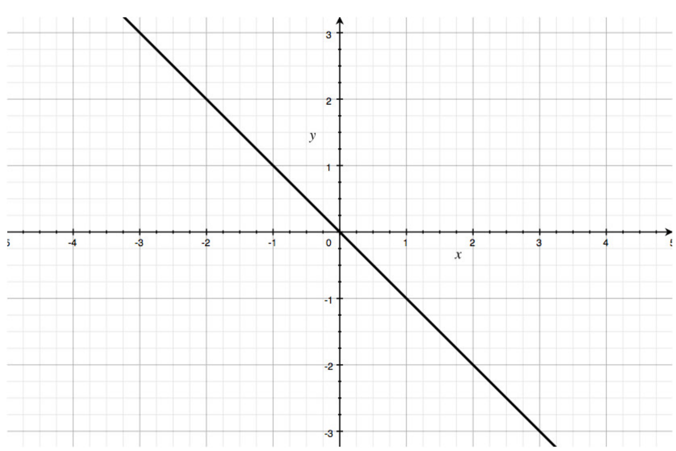
5. Infinite discontinuity at $x = 0$.
6. This function is continuous.
7. This function is continuous on its domain.
8. This function is continuous.
9. There is a removable discontinuity at $x =$ -2, infinite discontinuity at $x = 0$, and a jump discontinuity at $x = 4$.
10. There is a removable discontinuity at $x = 2$.
11. There is a jump discontinuity at $x = 0.3$.
12. Answers vary, but should show $f(x)$ has a jump discontinuity at $x = 3$, a removable discontinuity at $x = 5$, and another jump discontinuity at $x = 6$.
13. Answers vary, but should show $g(x)$ has a jump discontinuity at $x =$ -2, an infinite discontinuity at $x = 1$, and another jump discontinuity at $x = 3$.
14. Answers vary, but should show $h(x)$ has a removable discontinuity at $x =$ -4, a jump discontinuity at $x = 1$, and another jump discontinuity at $x = 7$.
15. Answers vary, but should show $j(x)$ has an infinite discontinuity at $x = 0$, a removable discontinuity at $x = 1$, and a jump discontinuity at $x = 4$.

Section 2.12: Function Combinations and Composition

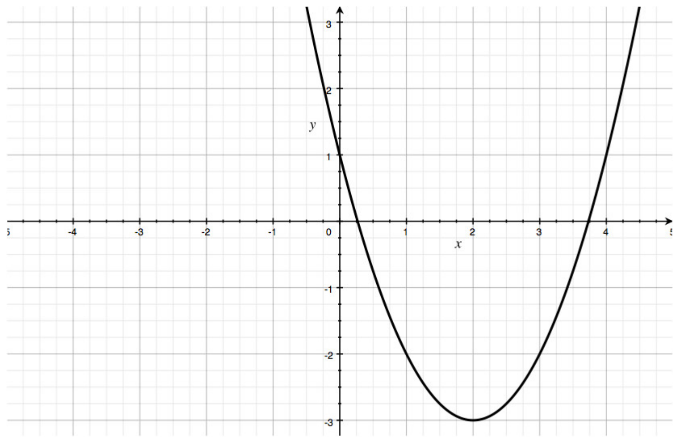
1. $g(x) - h(x) = (x-2)^2 - 3 - (-x) = x^2 - 3x + 1$
2. $f(x) = |x|$



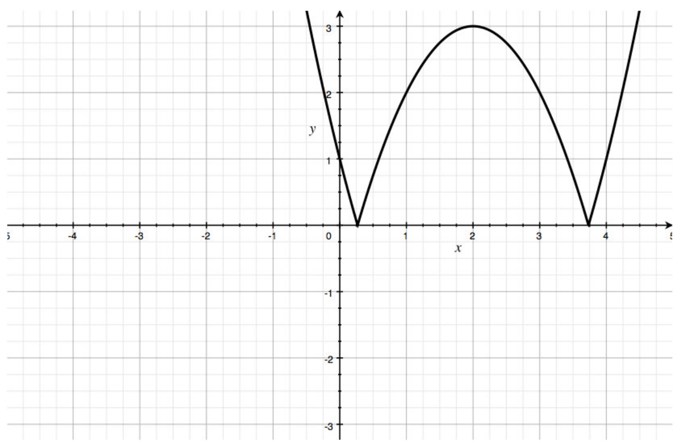
$h(x) =$ -x



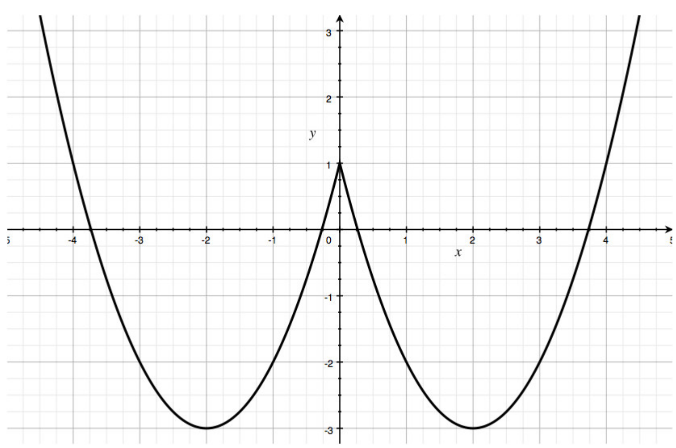
$g(x) = (x-2)^2 - 3$



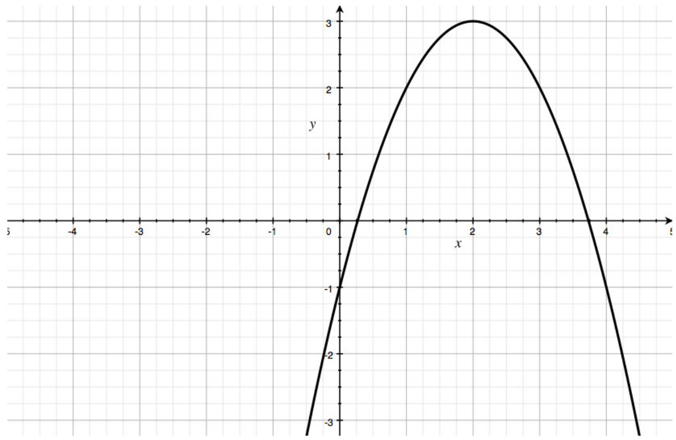
3. $f(g(x)) = \left|(x-2)^2 - 3\right| = \left|x^2 - 4x + 1\right|$
4. The absolute value from $f(x)$ made all negative y values for $g(x)$ positive.



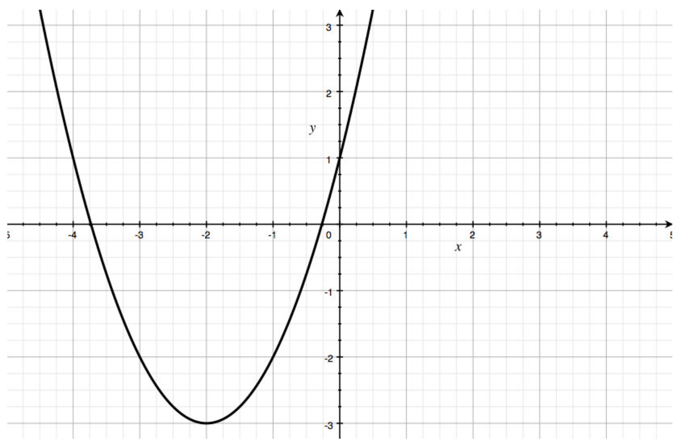
5. $g(f(x)) = (|x| - 2)^2 - 3$
6. Since the absolute value function is even, it created a similar reflection in $g(x)$.



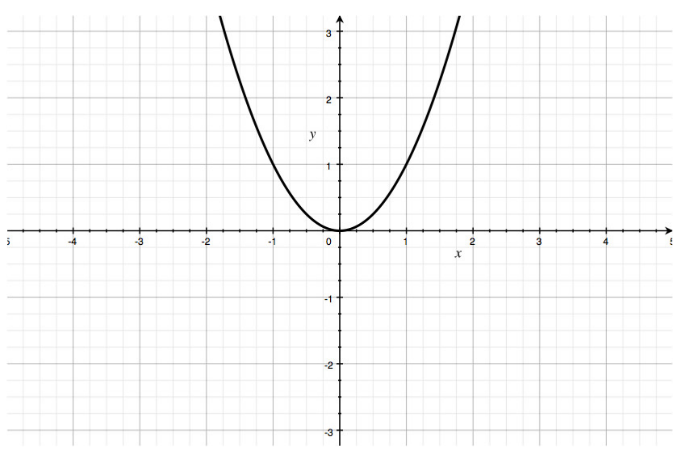
7. $h(g(x)) = \text{-}[(x-2)^2 - 3] = \text{-}(x-2)^2 + 3$
8. The negative from $h(x)$ reflects $g(x)$ across the x-axis.



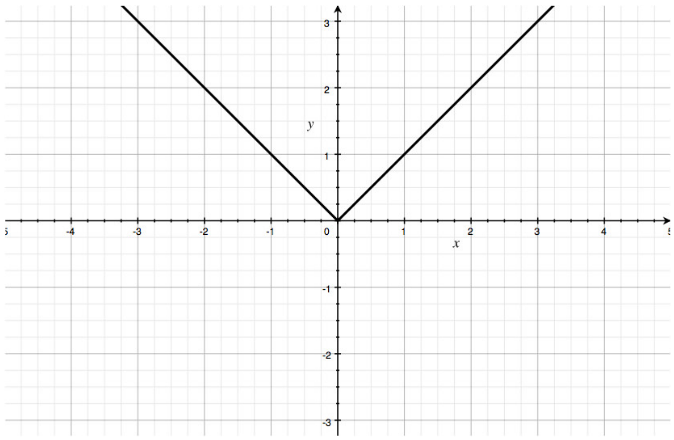
9. $g(h(x)) = x^2 + 4x + 1$
10. The negative from $h(x)$ reflects $g(x)$ across the y-axis.



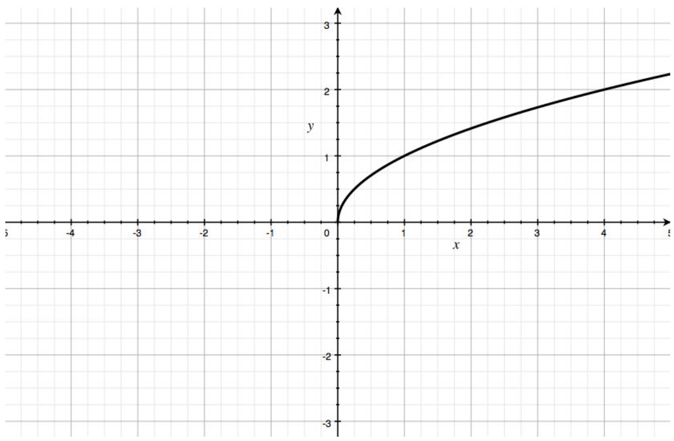
11. $j(x) + m(x) = x^2 + \sqrt{x}$
12. $j(x) = x^2$



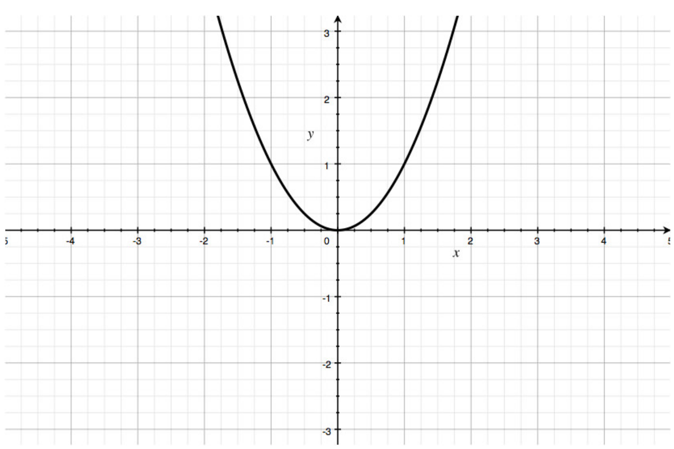
$k(x) = |x|$



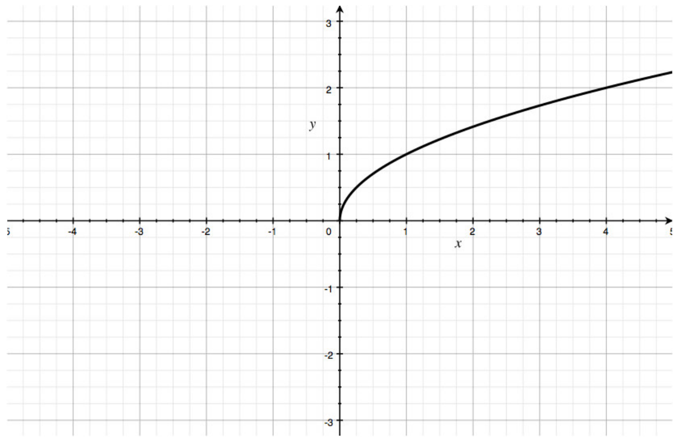
$m(x) = \sqrt{x}$



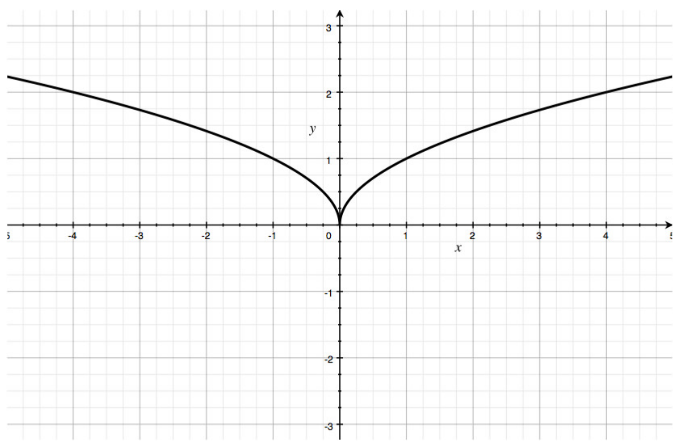
13. $j(k(x)) = |x|^2$
14. Since squaring a number automatically makes it positive, there is no change to the graph of $j(x)$.



15. $k(m(x)) = \left|\sqrt{x}\right|$
16. The graph looks the same as $m(x)$.



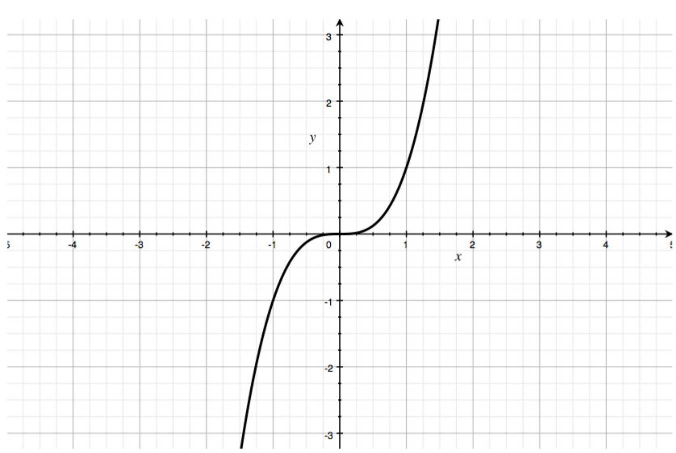
17. $m(k(x)) = \sqrt{|x|}$
18. The original square root graph is there, as well as its reflection across the y-axis.



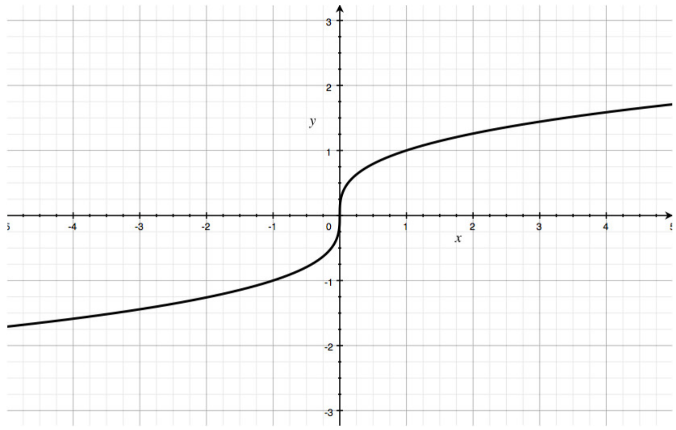
19. $r(p) = 1{,}000 - \frac{1}{4}(30 - 25p)^2 = \text{-}156.25p^2 + 375p + 775$

Section 2.13: Inverses of Functions

1. $f(x) = x^3$



$f^{-1}(x) = \sqrt[3]{x}$



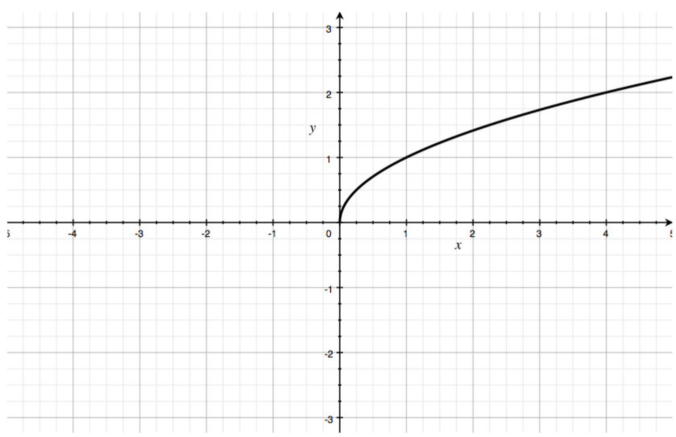
2. $f^{-1}(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$

   It is a function.

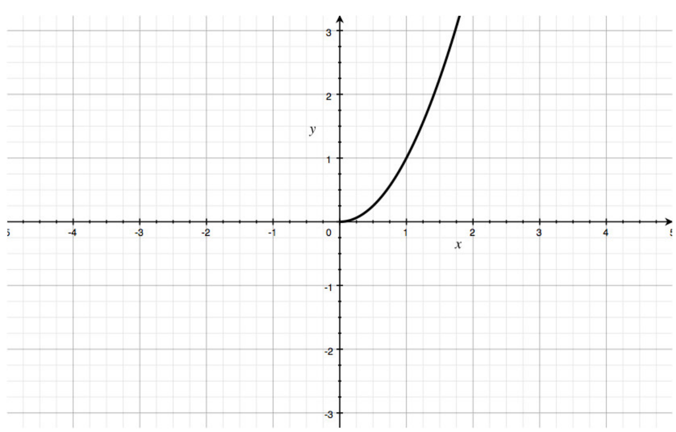
3. $f(f^{-1}(x)) = x^{\frac{1}{3}(3)} = x$

   $f^{-1}(f(x)) = x^{3(\frac{1}{3})} = x$

4. $g(x) = \sqrt{x},\ x \geq 0$



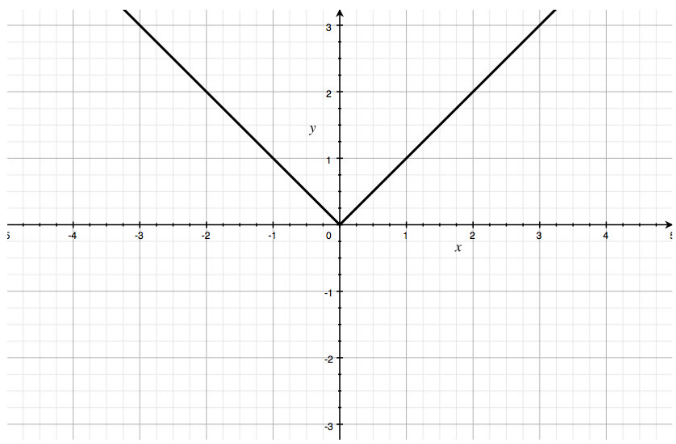
$g^{-1}(x) = x^2, x \geq 0$



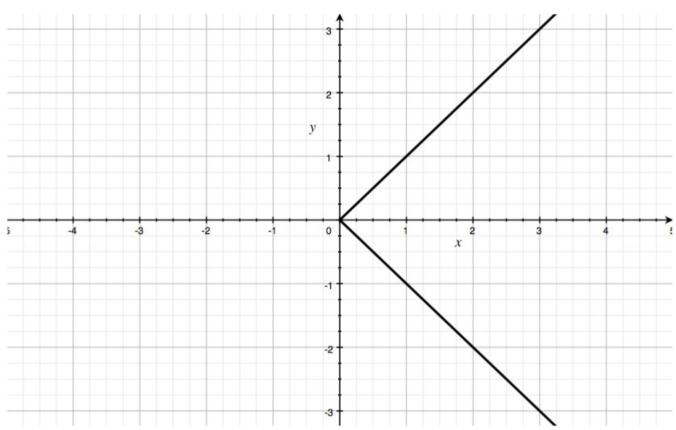
5. $g^{-1}(x) = x^2, x \geq 0$
It is a function.
6. $g(g^{-1}(x)) = (\sqrt{x})^2 = x$

$$g^{-1}(g(x)) = \sqrt{x^2} = x$$

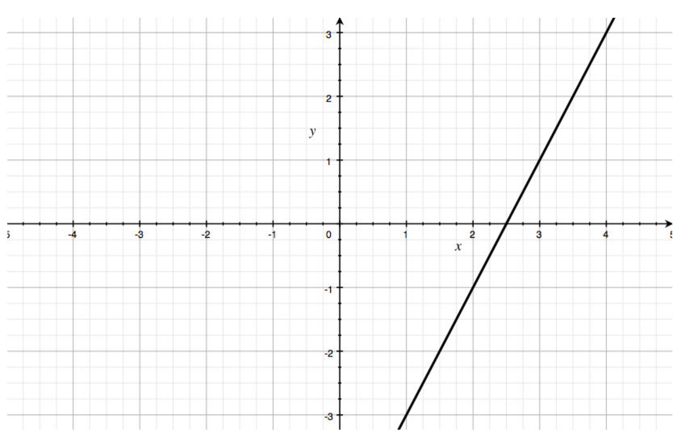
7. $h(x) = |x|$



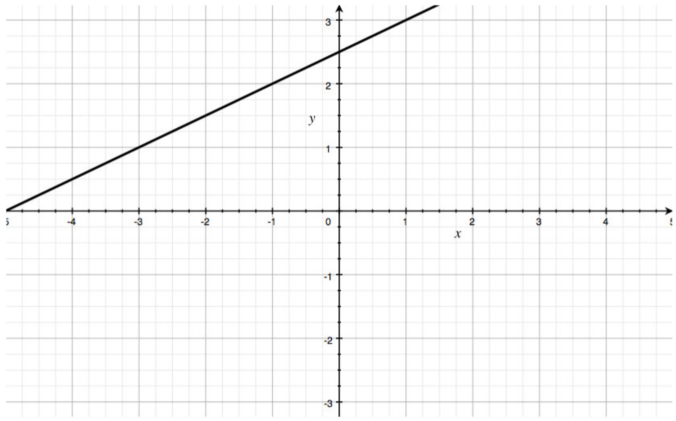
$h^{-1}(x)$



8. The inverse is $x = |y|$ and is not a function.
9. You can see from the graphs that they are inverses because they are symmetrical across the line $y = x$.
10. $j(x) = 2x - 5$



$j^{-1}(x)$



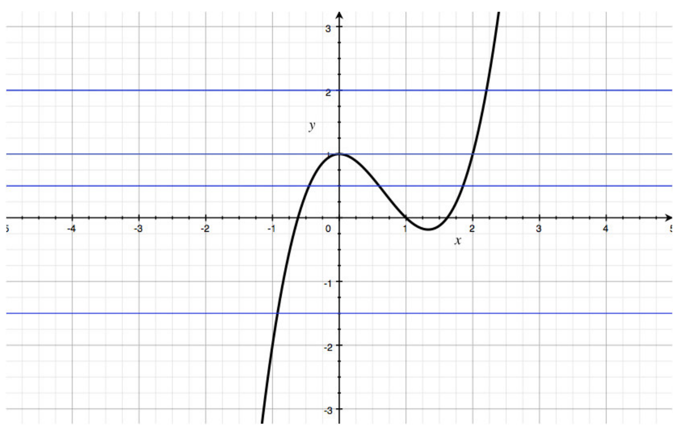
11. $j^{-1}(x) = \frac{x+5}{2}$.

    It is a function.

12. $j(j^{-1}(x)) = 2(\frac{x+5}{2}) - 5 = x + 5 - 5 = x$

    $j^{-1}(j(x)) = \frac{(2x-5)+5}{2} = \frac{2x}{2} = x$

13. The inverse is not a function since the function doesn't pass the horizontal line test.



14. No. The inverse of $g(x)$ is $g^{-1}(x) = e^x - 1$.
15. You could switch the x- and y-coordinates given in the original table to make the table for the inverse.
16. a. F(0) = 9/5(0) + 32 = 32; F(100) = 9/5(100)+32 = 212

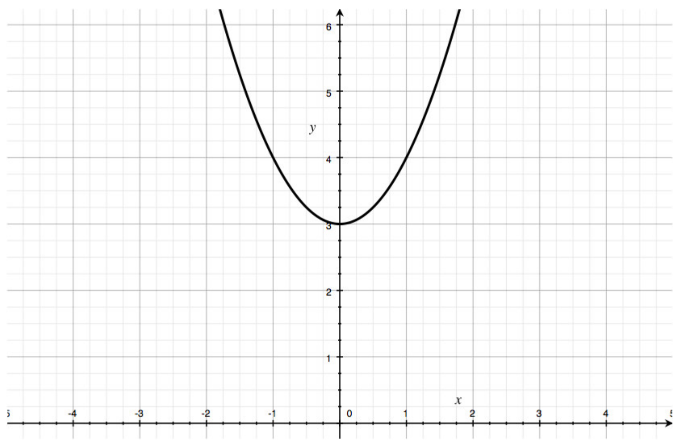
    b. C(F) = (F-32)*5/9

    c. F(C(F) = 9/5((F-32)*5/9) + 32 = F and C(F(C)) = ((9/5C+32)-32)*5/9

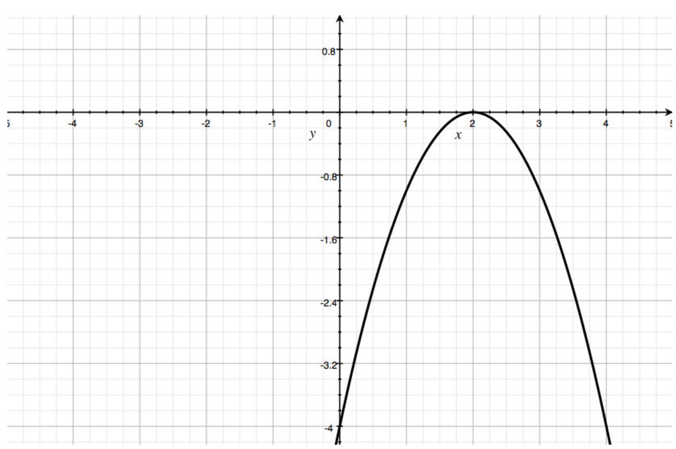
16.3 Answers - Ch 3: Power, Polynomial, and Rational Functions

Section 3.2: Quadratic Functions

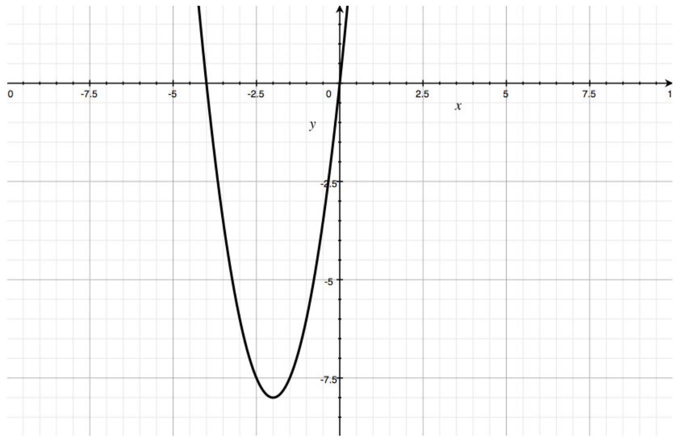
1. The graph of a quadratic is called a *parabola.*
2. A parabola opens up if the leading coefficient is positive.
3. If the coefficient of y is positive, the parabola opens right.
4. The vertex is the extreme (lowest or highest) point of a parabola that opens up or down.
5. The line of symmetry divides a parabola in two symmetrical parts.
6. The graph is a parabola with the vertex at (0, 3).



7. The parabola opens down, with the vertex at (2, 0).



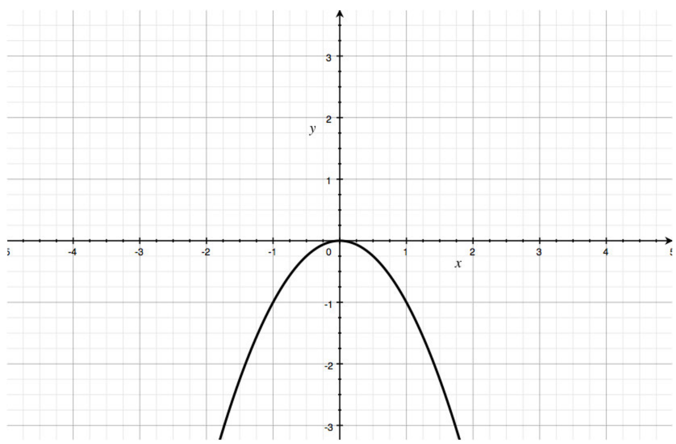
8. The parabola opens up, is narrower than the reference, and has a vertex at (-2, -8).



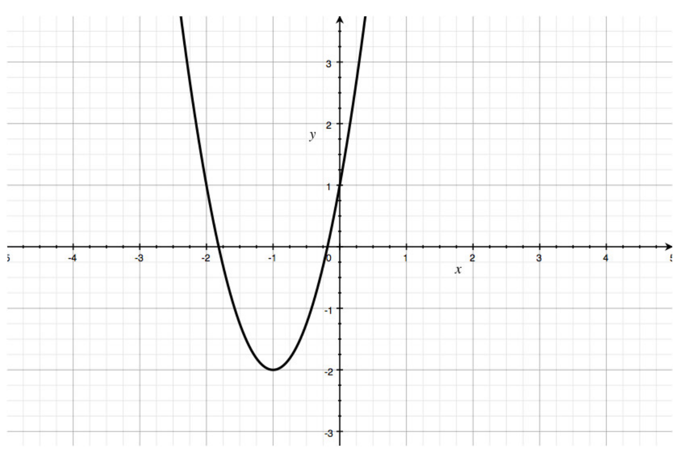
9. a. The parabola opens down.

   b. The vertex is at (-1, 2)

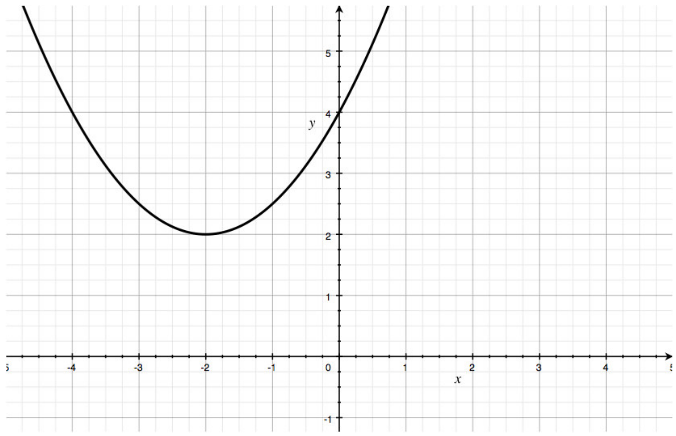
   c. It is not stretched, but reflected across the x-axis and shifted left 1 and up 2.
10. $y = 6x^2$ is the narrowest because it is stretched vertically the most.
11. $y = -x^2$



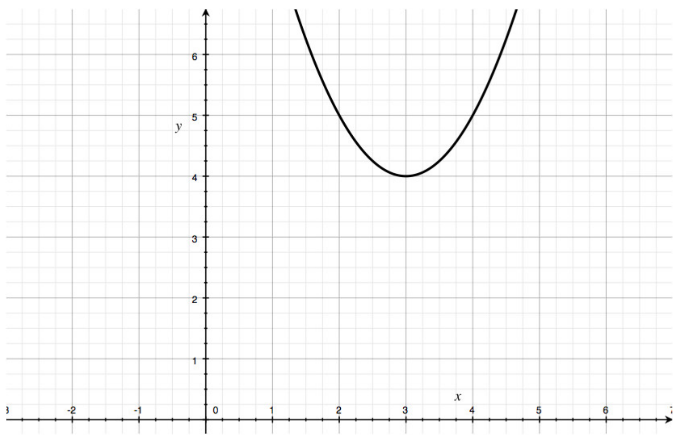
12. $y = 3x^2 + 6x + 1$



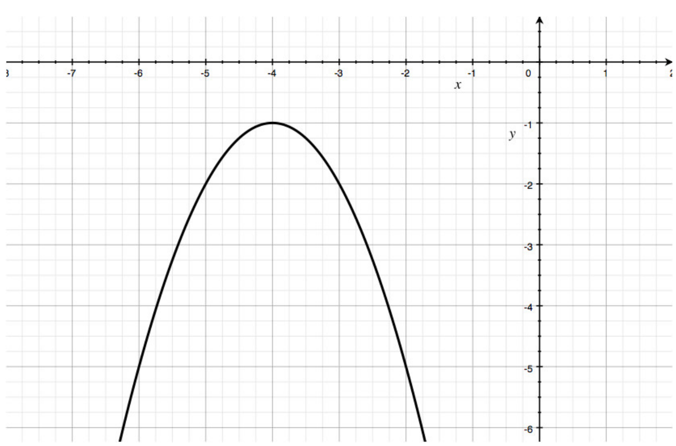
13. $y = \frac{1}{2}x^2 + 2x + 4$



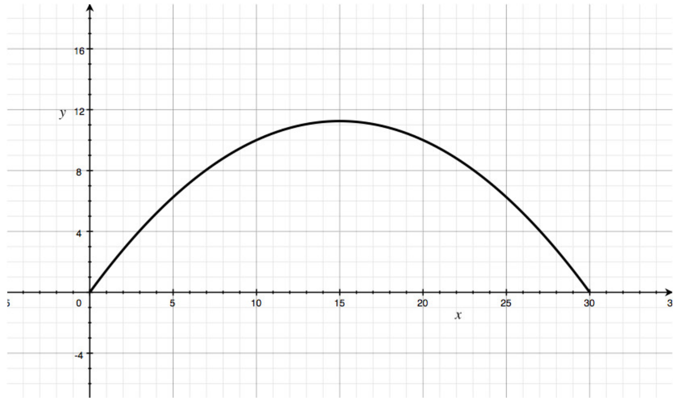
14. $y = (x-3)^2 + 4$



15. $y = -x^2 - 8x - 17$



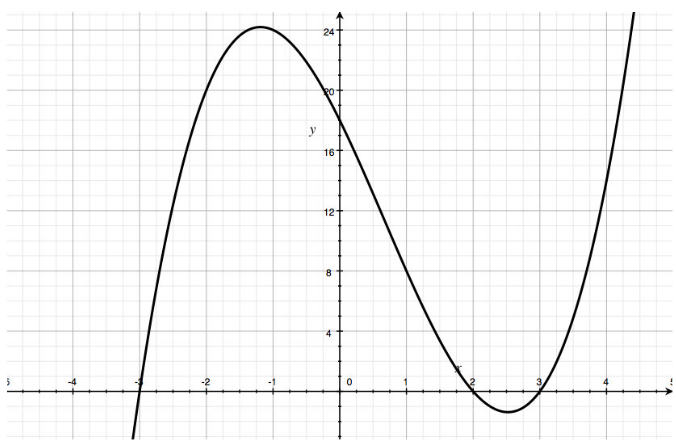
16. Maximum height is 11.25 yds.
17. At max height, the ball is 15 yds down the field.

18. 

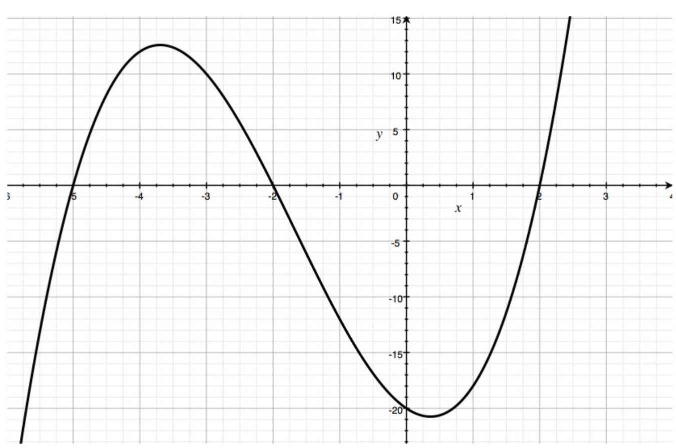
19. You could determine the height and distance by approximating the vertex (the highest point), knowing that the line of symmetry is halfway between 0 and 30. Thus, the vertex is about (15, 11).

Section 3.3: Polynomial Functions

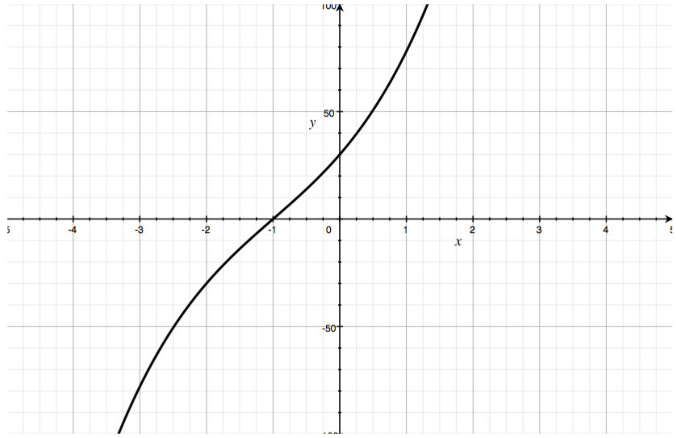
1. The real roots are -3, 2, and 3.



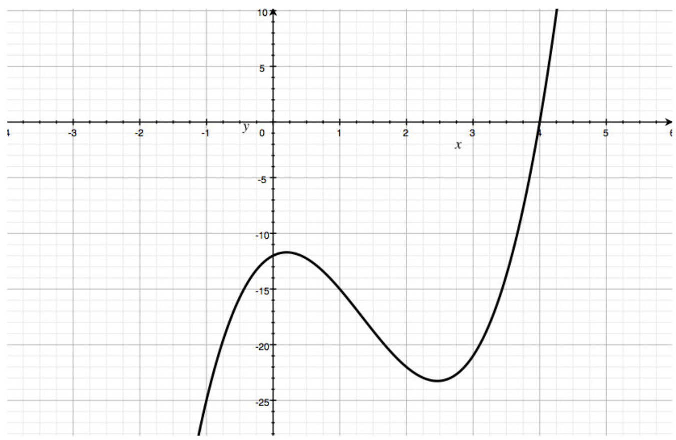
2. The real roots are -5, -2, and 2.



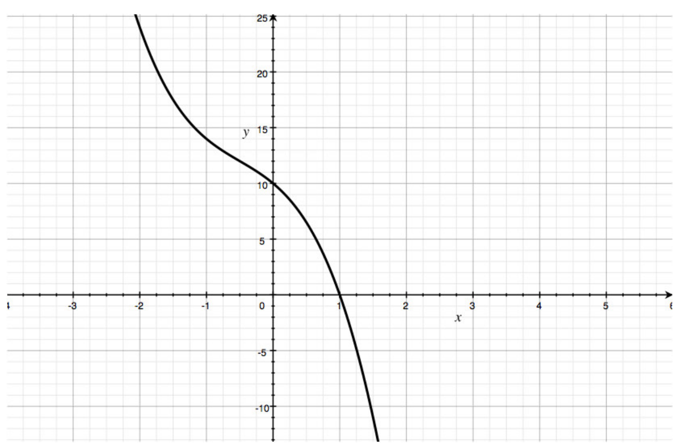
3. The real root is -1.



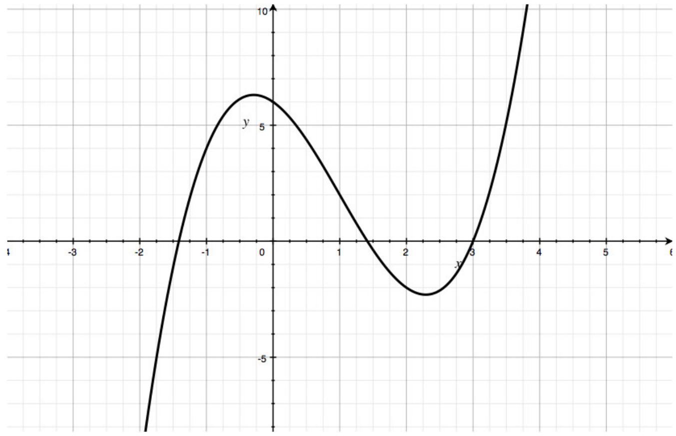
4. The real root is 4.



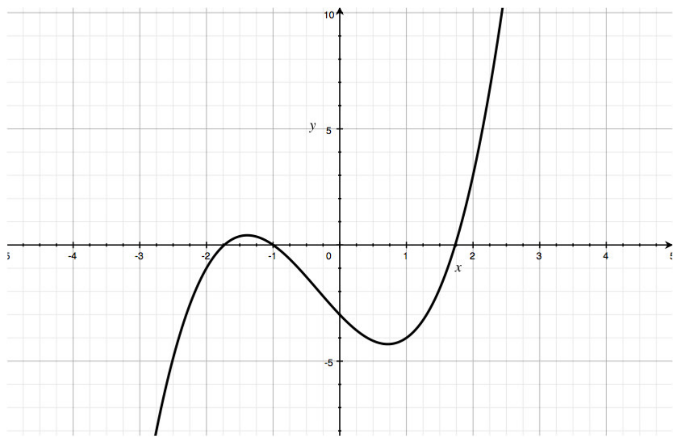
5. The real root is 1.



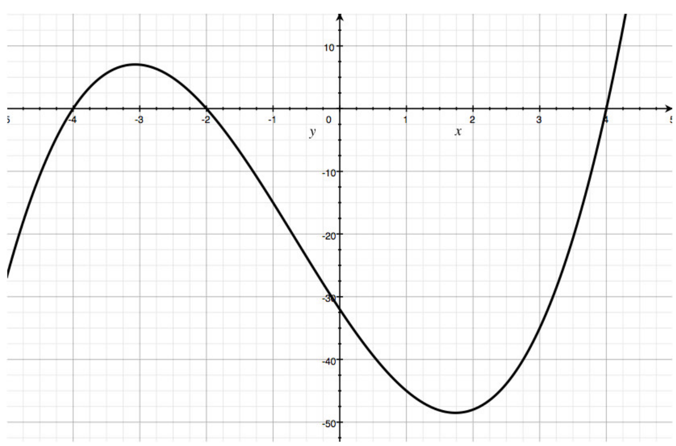
6. One factor is $(x-3)$.



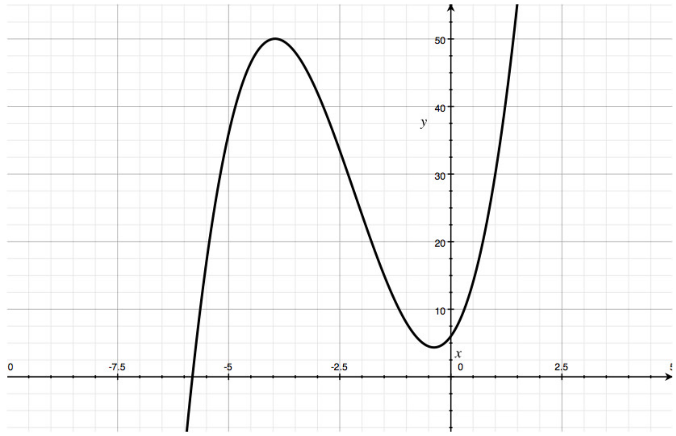
7. One factor is $(x+1)$.



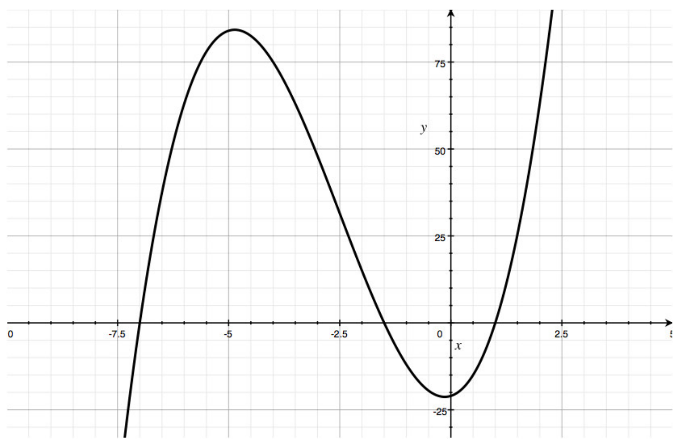
8. Factors are $(x-4)$, $(x+4)$, and $(x+2)$.



9. There are no integer roots.



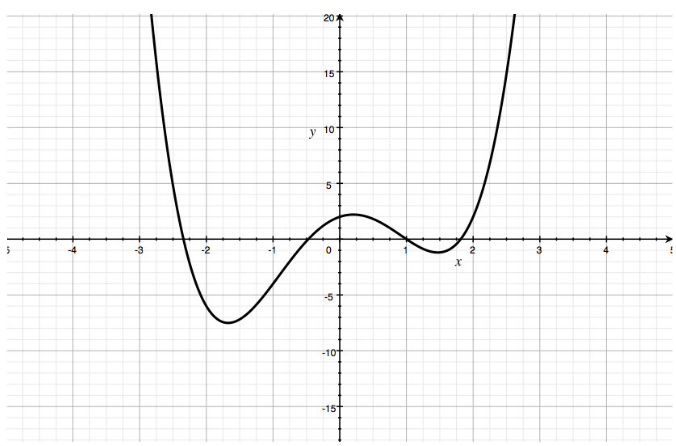
10. Factors are $(x+7)$, $(2x+3)$, and $(x-1)$.



11. Zeros: 1 and approximately -2.343, -0.471, and 1.814

    End behavior: As x approaches $\pm\infty$, y approaches $+\infty$.

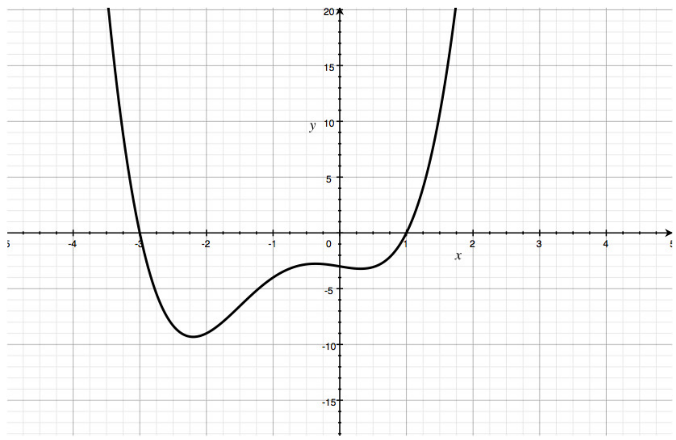
    Sample test points: (-1, -4), (0, 2), and (1.47, -1.195)



12. Zeros: -3 and 1

    End behavior: As x approaches $\pm\infty$, y approaches $+\infty$.

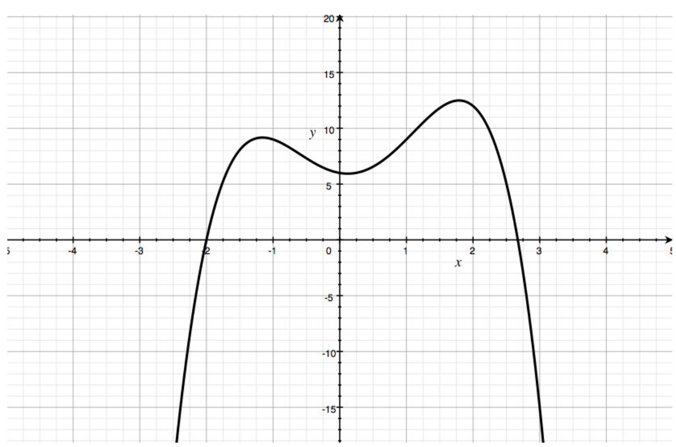
Sample test points: (-4, 65), (0, -3), and (2, 35)



13. Zeros: -2 and approximately 2.672

    End behavior: As x approaches $\pm\infty$, y approaches $-\infty$.

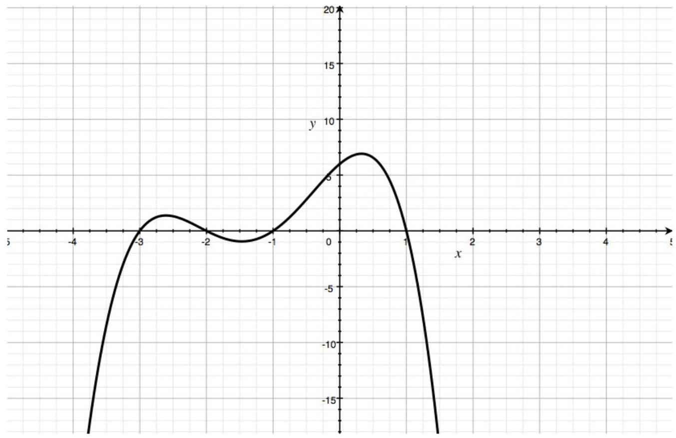
    Sample test points: (-3, -63), (0, 6), and (3, -15)



14. Zeros: -3, -2, -1, and 1

    End behavior: As x approaches $\pm\infty$, y approaches $-\infty$.

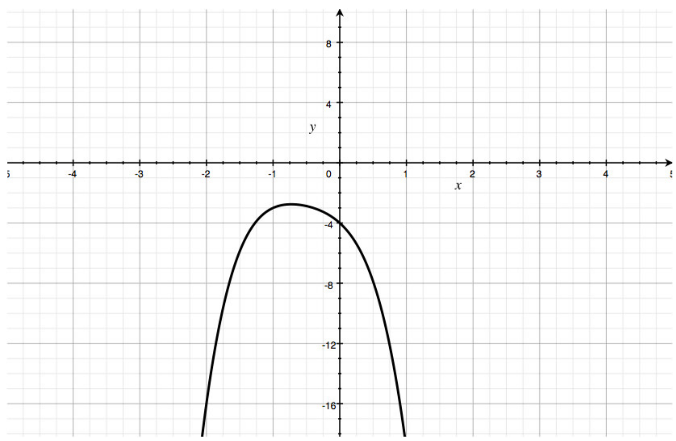
    Sample test points: (-2.607, 1.383), (1.469, -0.941), and (0, 6)



15. Zeros: There are no real zeros.

    End behavior: As x approaches $\pm\infty$, y approaches -∞.

    Sample test points: (-1, -3), (0, -4), and (1, -19)



Section 3.4: Synthetic Division of Polynomials

1. $x+4$
2. $x+1$
3. $a+5$
4. $x+2$
5. $x^2+4x-1+\frac{12}{x+2}$
6. $4x^2+17x+16$
7. $2x-1-\frac{4}{2x+1}$
8. $2x^3-33x^2+267x-2{,}423+\frac{21{,}849}{x+9}$
9. x^2+2x-1
10. $3x^4+3x^3+7x^2+7x+6+\frac{4}{x-1}$
11. Numbers 6 and 9 have no remainder. Having no remainder means the divisor in synthetic division is a root.
12. $(x-k)$ is a factor when k is a zero. $f(k)=0$ if and only if k is a zero.
13. a. $f(\text{-}2) = \text{-}14$

b. The remainder is -14, which is the same as $f(\text{-}2)$.

14. $x = \text{-}4, \frac{1}{6}, \text{-}\frac{5}{2}$
15. $x = 5, \pm\sqrt{2}$
16. $x = 2, \frac{1}{3}, \frac{1}{2}$
17. $x = \text{-}4, \text{-}4, \text{-}1, 2$
18. $x = \text{-}2, 0, \text{-}\frac{3}{2}, \frac{1}{3}$
19. The area of the base is $x^2 - 5x - 12$.
20. $D = \frac{1}{\pi} - \frac{4}{\pi h} + \frac{20}{\pi h^2}$

Section 3.5: Real Zeros of Polynomials

1. $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$
2. $\pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm 1, \pm\frac{5}{4}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm 3, \pm\frac{15}{4}, \pm 5, \pm\frac{15}{2}, \pm 15$
3. $\pm\frac{1}{2}, \pm 1, \pm 2, \pm 4, \pm 8$
4. $\pm 1, \pm 3, \pm 9$
5. $\pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{3}{8}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm 1, \pm\frac{3}{2}, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$
6. $x = \frac{1}{3}, \frac{1}{2}, 2$
7. $x = \text{-}4, \text{-}4, \text{-}1, 2$
8. $x = \text{-}3, \frac{1 \pm \sqrt{2}}{4}$
9. $x = \frac{3}{2}$; other roots are complex
10. $x = 5$; other roots are complex
11. $x = \text{-}\frac{5}{2}, 1, \frac{2 \pm \sqrt{7}}{3}$
12. $x = \text{-}4 \pm \sqrt{3}, \text{-}3, 2, 2$
13. $x = \text{-}4, \text{-}4, \frac{3}{2}, \frac{3}{2}$
14. $x = \text{-}5, \text{-}\frac{1}{3}, \frac{1}{3}, 5$
15. $x = \pm\frac{\sqrt{21}}{3}$. There are two real solutions. The other two solutions are imaginary.

Section 3.6: Fundamental Theorem of Algebra

1. $f(x) = (x-2)^2(x-4)^3(x-1)(x-\sqrt{2}i)(x+\sqrt{2}i)$
$= x^8 - 17x^7 + 118x^6 - 438x^5 + 984x^4 - 1{,}512x^3 + 1{,}760x^2 - 1{,}408x + 512$
2. $f(x) = (x-1)(x+3)^3(x+1)(x-\sqrt{3}i)(x+\sqrt{3}i)$
$= x^7 + 9x^6 + 29x^5 + 45x^4 + 51x^3 + 27x^2 - 81x - 81$
3. $f(x) = (x-5)^2(x+1)^2(x-2i)(x+2i)$
$= x^6 - 8x^5 + 10x^4 + 8x^3 + 49x^2 + 160x + 100$
4. $f(x) = (x-i)(x+i)(x-\sqrt{2}i)(x+\sqrt{2}i)$
$= x^4 + 3x^2 + 2$
5. $f(x) = (x+3)^2(x-2)(x-i)(x+i)$

Roots are -3 (multiplicity 2), $2, i, \text{-}i$
6. $g(x) = (x-1)(x+1)(x-i)(x+i)$

Roots are $1, \text{-}1, i, \text{-}i$
7. $h(x) = (x-4)^2(x-2)^2(x+3i)(x-3i)$

Roots are 4 (multiplicity 2), 2 (multiplicity 2), -3i, 3i

8. $j(x) = (x-1)^2(x-3)^3(x+\sqrt{3}i)(x-\sqrt{3}i)$

   Roots are 1 (multiplicity 2), 3 (multiplicity 3), -$\sqrt{3}i$, $\sqrt{3}i$
9. $k(x) = (x-2)(x+3)(x+4)(x-1)^2$

   Roots are 2, -3, -4, 1 (multiplicity 2)
10. $m(x) = (x-2)(x+2)(x-3)(x+3)(x-i)(x+i)$

    Roots are 2, -2, 3, -3, i, -i
11. $n(x) = (x-6)(x+1)^3(x-\sqrt{5}i)(x+\sqrt{5}i)$

    Roots are 6, -1 (multiplicity 3), $\sqrt{5}i$, -$\sqrt{5}i$
12. $p(x) = (x-2)(x+2)^3(x-\sqrt{7}i)(x+\sqrt{7}i)$

    Roots are 2, -2 (multiplicity 3), $\sqrt{7}i$, -$\sqrt{7}i$
13. The degree of the polynomial is the number of roots with multiplicity.
14. Multiplicity refers to a root that counts more than once, because when the polynomial is in factored form, the degree of its corresponding binomial is greater than 1.
15. -$\sqrt{3}i$

Section 3.7: Approximating Real Zeros of Polynomial Functions

1. a) Leading coefficient: 3; Degree: 5

   b) 1 real zero at approximately -1.4

   c) 4 imaginary zeros
2. a) Leading coefficient: -1; Degree: 3

   b) 1 real zero at approximately 2

   c) 2 imaginary zeros
3. a) Leading coefficient: 1/2; Degree: 4

   b) 2 real zeros at approximately -6.3 and -1

   c) 2 imaginary zeros
4. a) Leading coefficient: 1; Degree: 5

   b) 5 real zeros at approximately -2.6 (multiplicity 2), -1, and 2.6 (multiplicity 2)

   c) 0 imaginary zeros
5. a) Leading coefficient: 1; Degree: 4

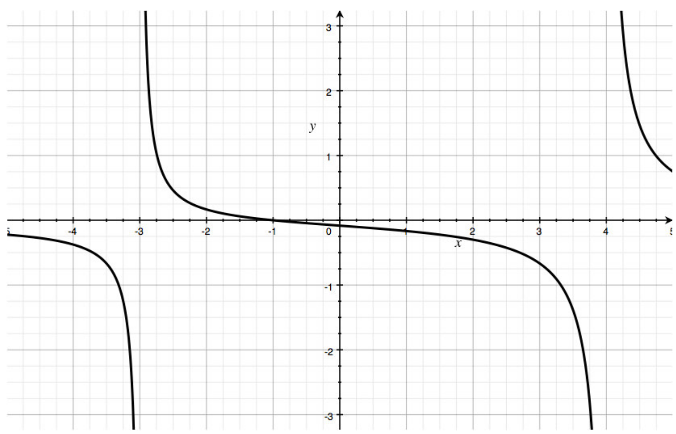
   b) 4 real zeros at approximately -1 and 3 (multiplicity 3)

   c) 0 imaginary zeros
6. [-2, -1]

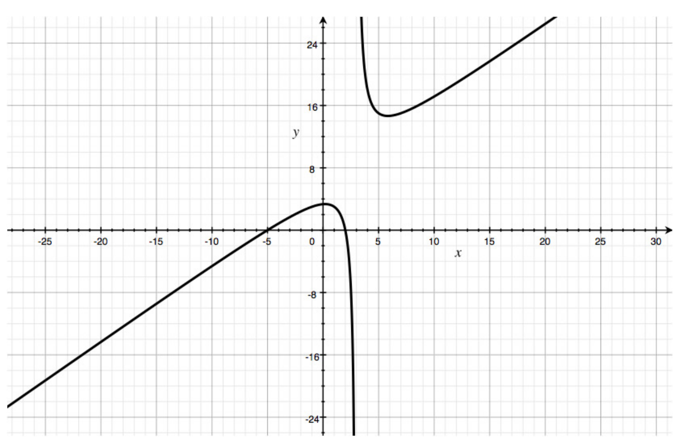
7. [-1, 3]
8. [-2, 4]
9. [-2, 2]
10. The Bounded Roots Theorem is based on continuous functions, and this rational function is discontinuous at x = -3.5.
11. [0.3125, 0.34375], Zero: 0.338
12. [1.875, 1.9375], Zero: 1.893

Section 3.8: Rational Functions

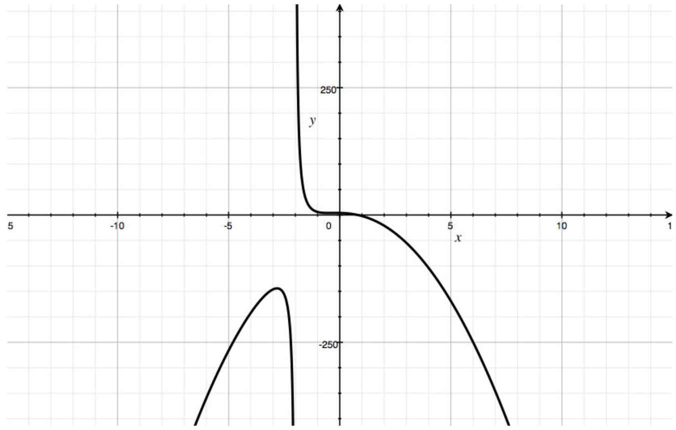
1. VA: x = -4, x= -2; HA: y = 0
2. VA: x = -5; SA: y = x-5
3. VA: x = 3; SA: y = x+3
4. x-intercepts: (-2, 0) and (2, 0)
5. x-intercept: (0, 0)
6. VA: x = -3, x = 4; HA: y = 0; Intercepts: (0, -1/12), (-1, 0)



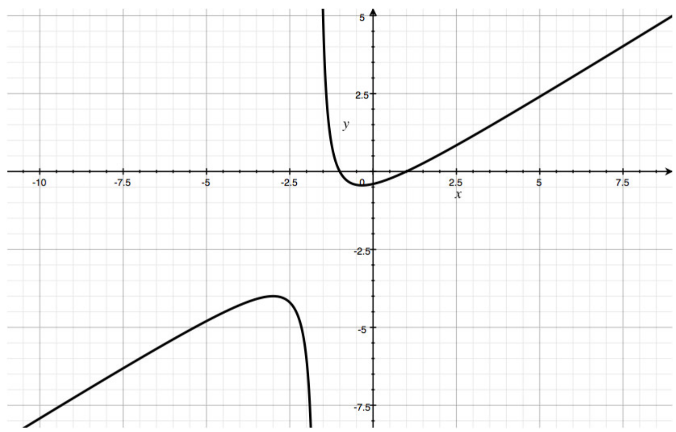
7. VA: x = 3; SA: y = x + 6; Intercepts: (0, 10/3), (2, 0), (-5, 0)



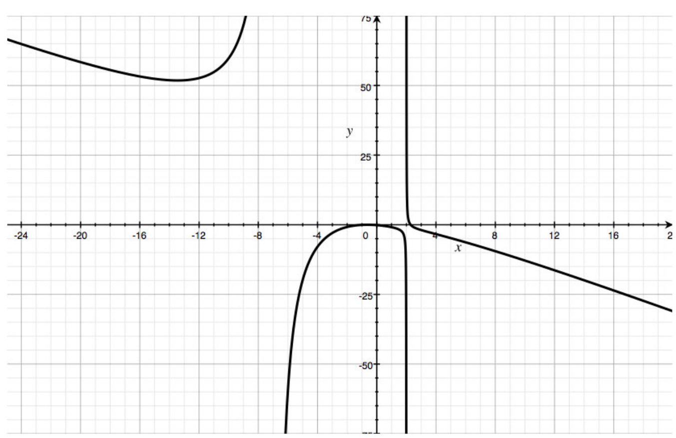
8. VA: x=-2; SA: y = -8x^2+8x-14 Intercepts: approximately (0.81, 0) and (0, 4)



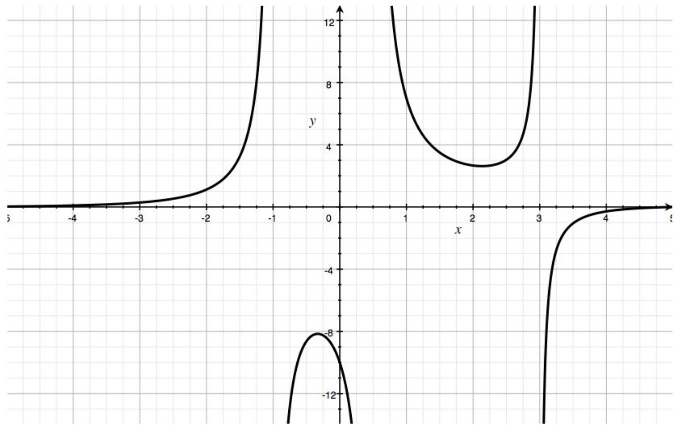
9. VA: x = -5/3; SA: y = 2/3x -10/6; Intercepts: (0, -2/5), (1, 0), (-1, 0)



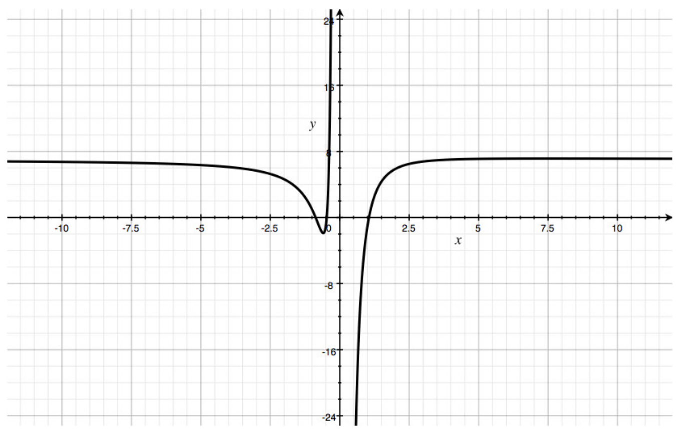
10. VA: x = 2, x = -7; SA: y = -2x + 12 Intercepts: approximately (2.285, 0); (0, -1/7)



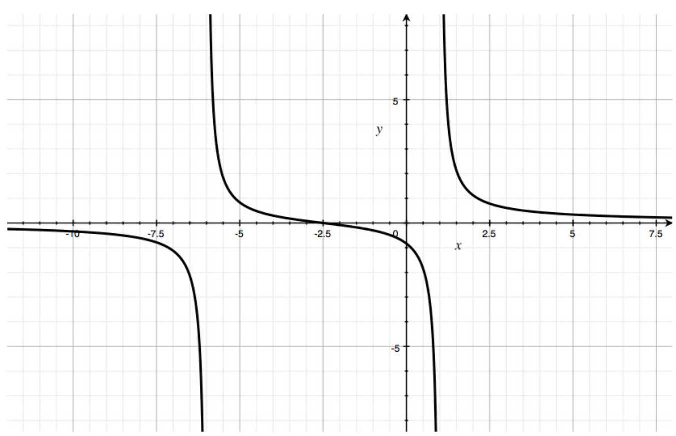
11. VA: x= 1/2, x= -1, x = 3; HA: y = 0; Intercepts: (0, -10), (5, 0), (-6, 0)



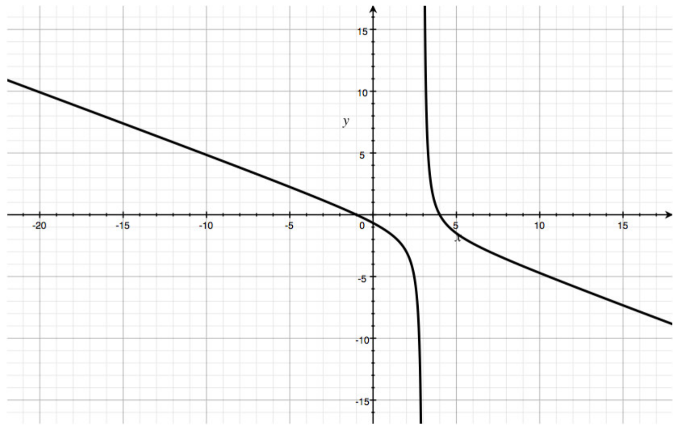
12. VA: x = 0; HA: y = 7 Intercepts: approximately (-0.87, 0), (-0.47, 0), and (1.05, 0); no y-intercept



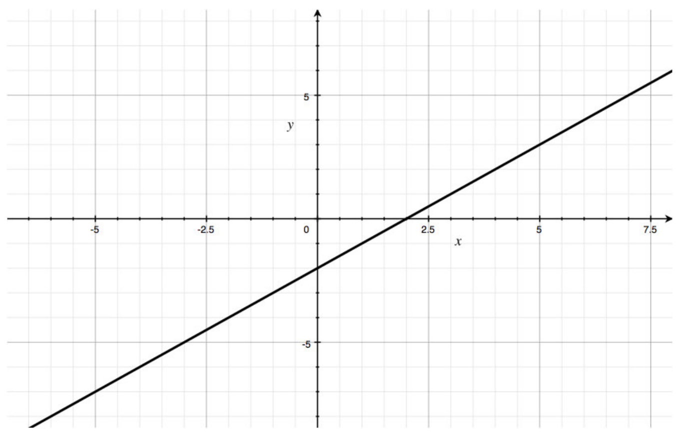
13. VA: x = -6, x = 1; HA: y = 0; Intercepts: (0, -5/6), (-5/2, 0)

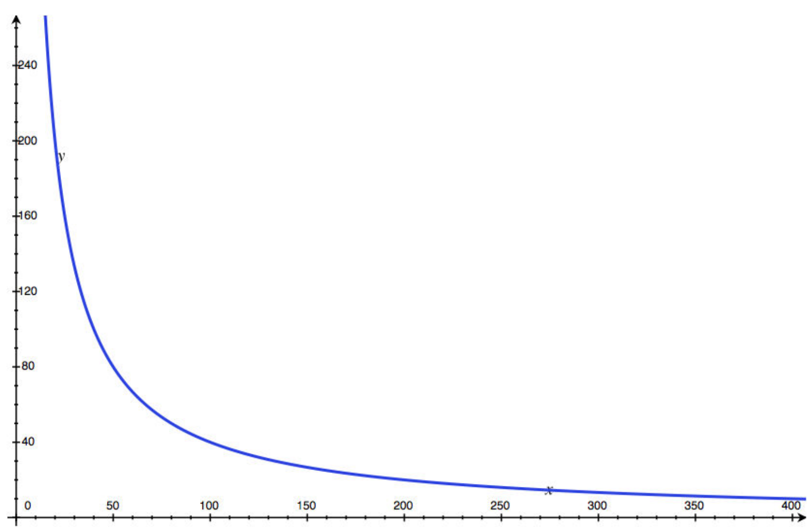


14. VA: x = 3; SA: y = -1/2x; Intercepts: (0, -2/3), (-1, 0), (4, 0)



15. If we divide, $\frac{3x^2-x-10}{3x+5} = x-2$. This can also be found by factoring the numerator and denominator, and canceling the like factor of $3x+5$. This creates a **hole,** not an asymptote.



16. 

Section 3.9: Analysis of Rational Functions

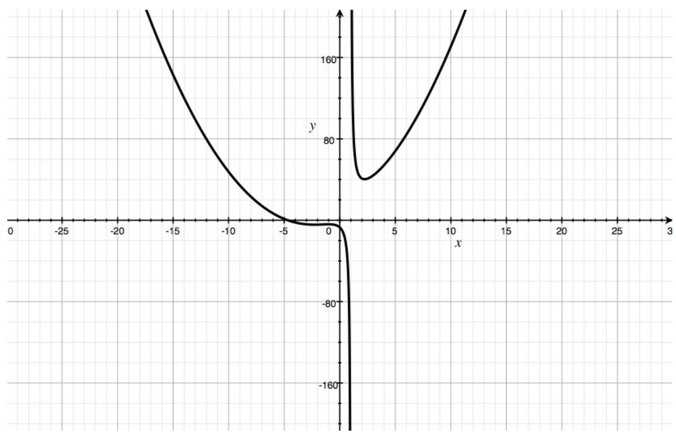
1. $y = \frac{(x-2)(x+5)}{x-2}; x \neq 2$
2. $y = \frac{(x+6)(x-4)}{x-4}; x \neq 4$

3. $f(x) = \frac{(x-8)(x-4)}{x-4}; x \neq 4$
4. $f(x) = \frac{(x+\frac{3}{4})(x+\frac{4}{5})}{x+\frac{4}{5}}; x \neq -\frac{4}{5}$
5. $y = \frac{(x+6)(x+7)}{x+7}; x \neq -7$
6. Asymptotes: VA: x = 1; SA: x^2+6x+9

   Holes: none

   Intercepts: approximately (-4.7, 0), (0, -7)

   Sketch: $\frac{x^3+5x^2+3x+7}{x-1}$

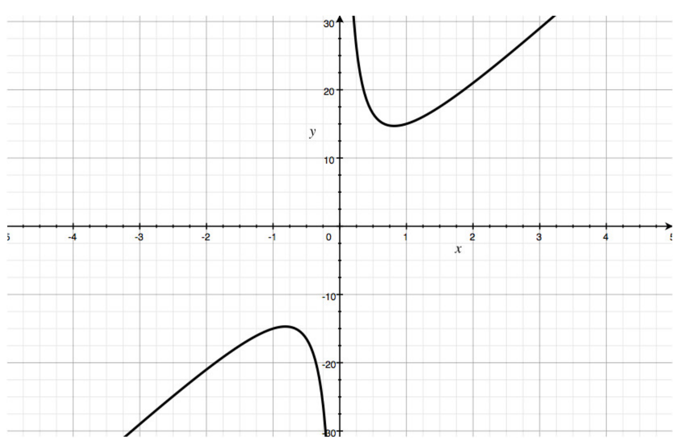


7. Asymptotes: VA: x=0; SA: y=9x

   Holes: none

   Intercepts: none

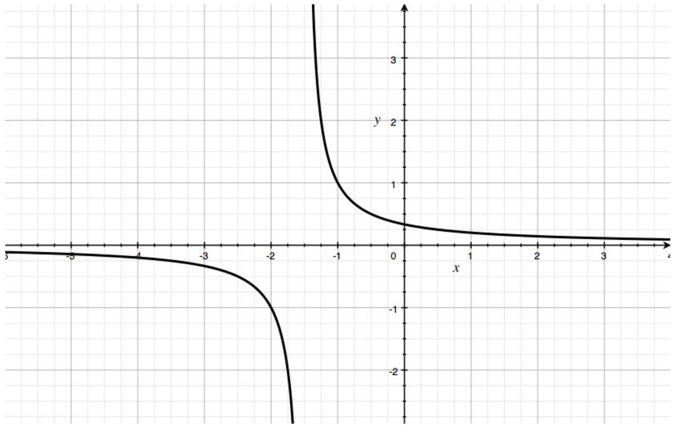
   Sketch: $\frac{(9x^2+6)}{x}$



8. Asymptotes: VA: x = -3/2; HA: y = 0

   Intercepts: (0, 1/3), there is no x-intercept

   Hole: (7, 1/17)

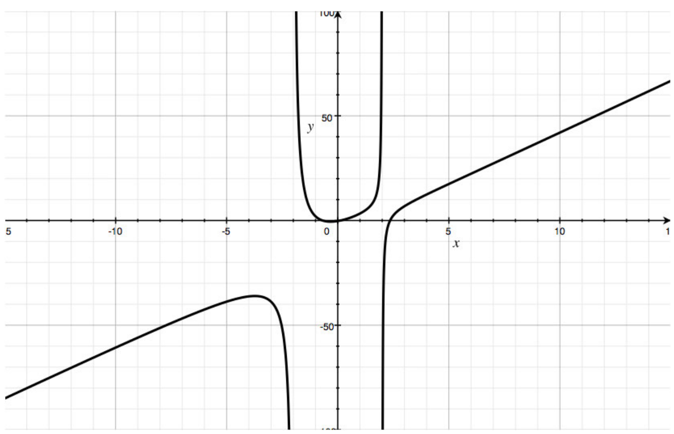


9. Asymptotes: VA: x = -2, x = 2; SA: y = 5x - 9

   Holes: none

   Intercepts: approximately (-0.68, 0), (0.12, 0); (0, -0.25)

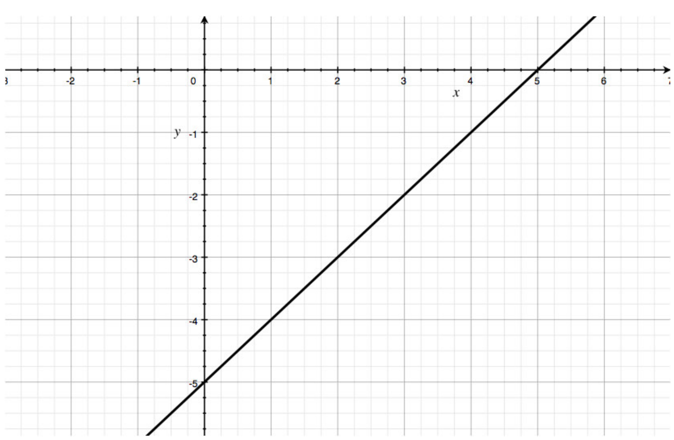
   Sketch: $\frac{5x^3-9x^2-7x+1}{x^2-4}$



10. No asymptotes

    Intercepts: (0, -5), (5, 0)

    Hole (-6, -11)

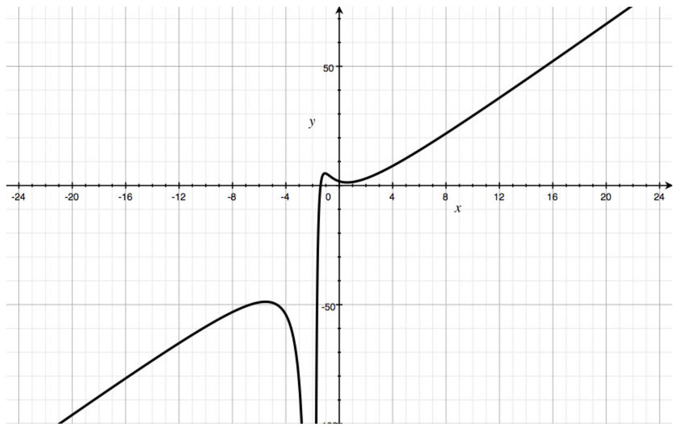


11. Asymptotes: VA: x = -2; SA: y = 4x - 14

    Holes: none

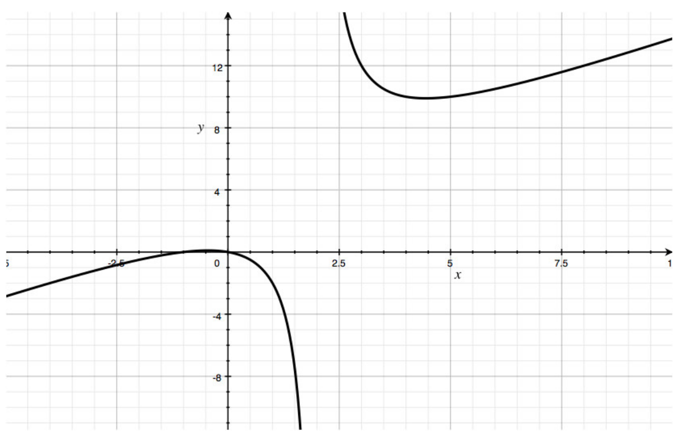
    Intercepts: approximately (-1.4, 0); (0, 7/4)

    Sketch: $\frac{4x^3+2x^2+7}{(x+2)^2}$



12. Asymptotes: VA: x = 2; SA: y = x + 3
    Intercepts (0, 0), (-1, 0)

    Hole: (3, 12)

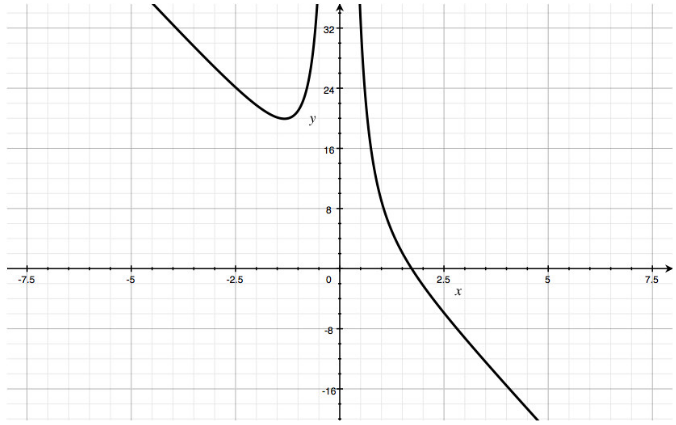


13. Asymptotes: VA: x = 0; SA: y = -6x+8

    Holes: none

    Intercepts: approximately (1.73, 0); no y-intercept

    Sketch: $\frac{-6x^3+8x^2+7}{x^2}$

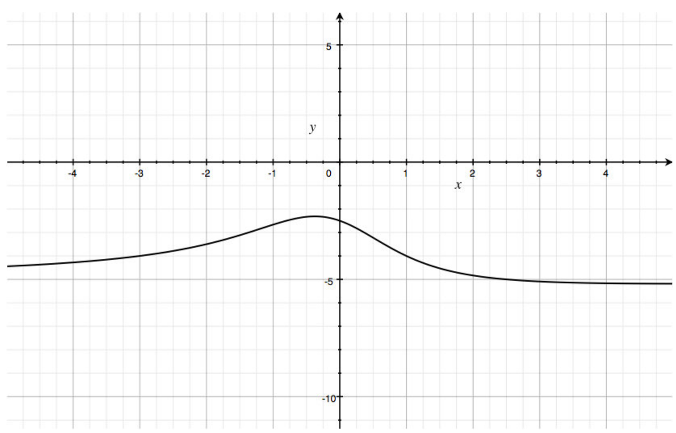


14. Asymptotes: VA: none; HA: y = -5

    Holes: none

    Intercepts: no x-intercept; (0, -5/2)

    Sketch: $\frac{-5x^2-2x-5}{x^2+2}$

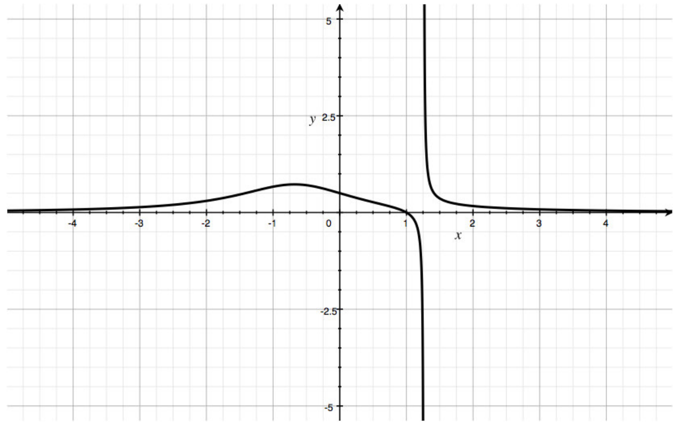


15. Asymptotes: VA: x = cubic root of 2, or approx. 1.26; HA: y = 0

    Holes: none

    Intercepts: (1 , 0); (0, 1/2)

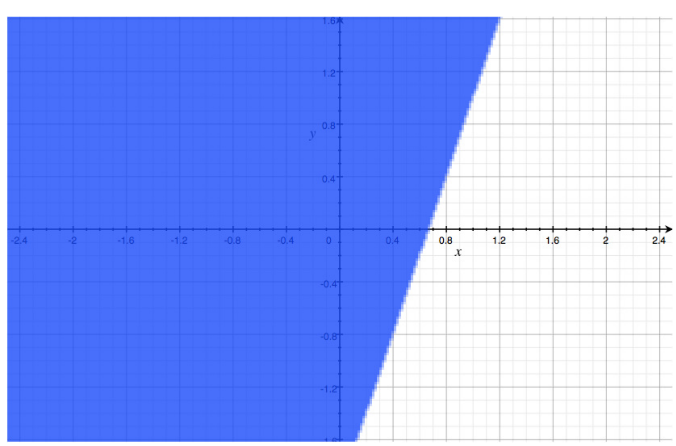
    Sketch: $\frac{x-1}{x^3-2}$



Section 3.10: Polynomial and Rational Inequalities

1. $-3 \leq x \leq 1$
2. $x < \frac{1}{3}$ or $x > 2$
3. $-\frac{5}{2} \leq x \leq \frac{1}{3}$
4. $x < \frac{1}{5}$ or $x > 2$
5. $x < 0$ or $x > \frac{1}{2}$
6. $-\frac{1}{2} < x < 0$ or $x > \frac{3}{2}$
7. $-2 < x < 0$ or $0 < 2$
8. $-\frac{1}{2} \leq x \leq \frac{1}{2}$ or $x \geq 2$
9. $-3 < n < -1$ or $1 < n < 2$
10. $-3 \leq n \leq -2$ or $2 \leq n < 5$
11. $n \leq -3$ or $-\frac{5}{2} \leq n < \frac{1}{3}$ or $n \geq 3$
12. $n < -\frac{3}{2}$ or $-\frac{4}{3} < n < -\frac{1}{2}$ or $n > \frac{1}{2}$
13. $0 \leq t \leq 36.490$
14. $x \leq -4.567$ or $-1.294 \leq x \leq 1.861$
15. $-4.667 < x < 4.044$ or $5.000 < x < 6.623$
16. $y > 3x - 2, x \neq -\frac{2}{3}$

    The graph would be as follows, with missing values anywhere $x = -\frac{2}{3}$:



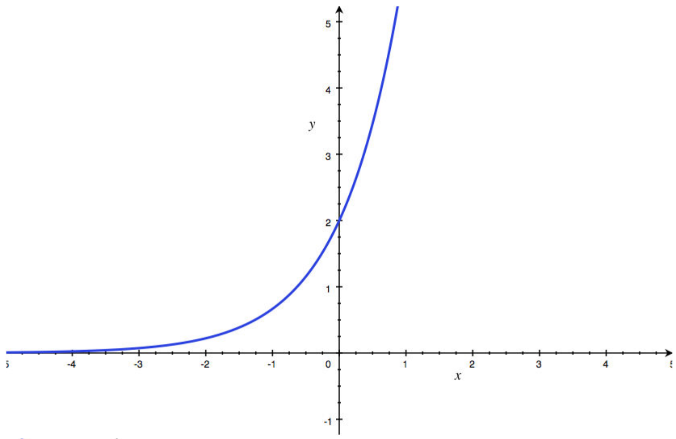
17. a) $R_2 < 60$, so 60 ohms is the max resistance.

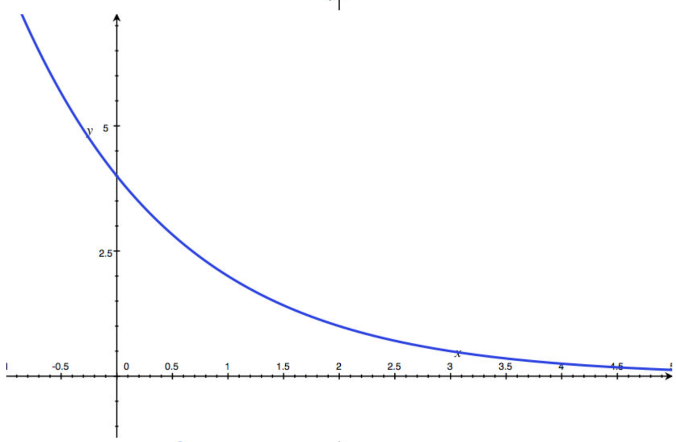
b) 20 ohms, based on how the 2nd resistor would cancel out in this equation.

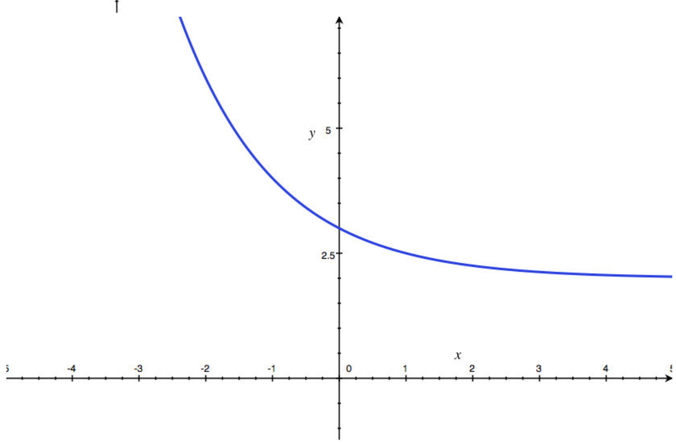
18. Width is greater than 4.

16.4 Answers - Ch 4: Exponential and Logarithmic Functions

Section 4.2: Graphing and Evaluating Exponential Functions

1. $y = 5 \cdot 5^x$
2. $y = 2 \cdot 4^x$
3. $y = 16 \cdot 3^x$
4. 

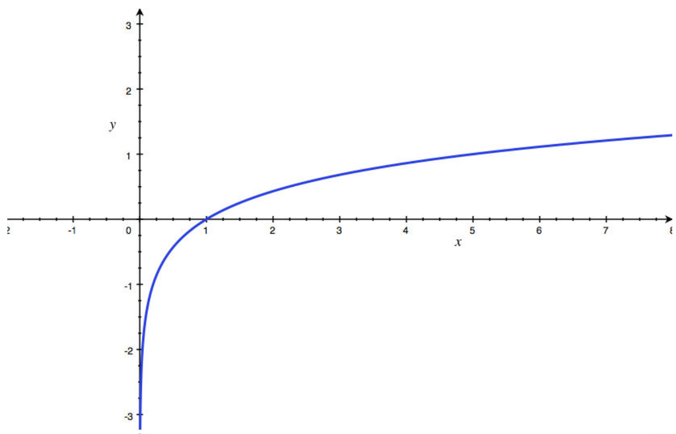
5. 

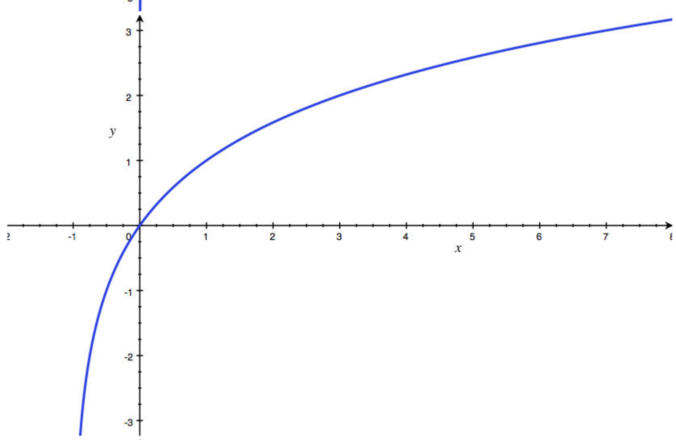
6. 

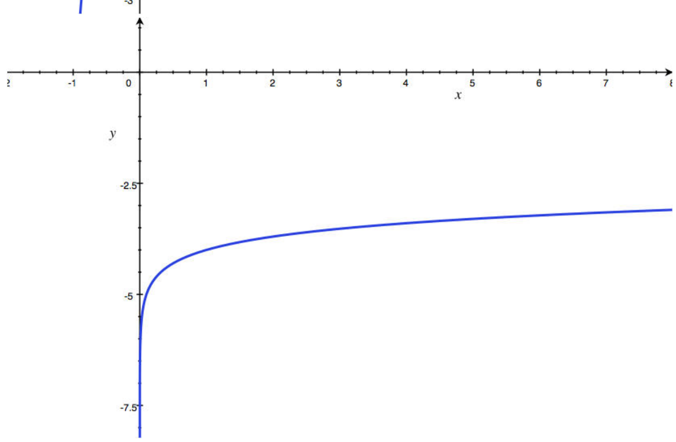
7. $y = 2^x$
8. $y = 2^{-x}$
9. $y = -2^x$
10. $f(x) = 3^{x-4}$ will be moved to the right 4 spaces.

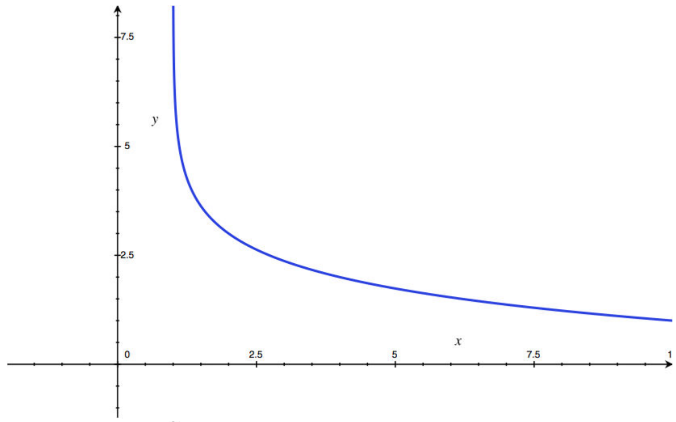
11. $f(x) = \text{-}4^x$ will be reflected across the x axis.
12. $f(x) = 3^x - 2$ will be moved down 2 spaces.
13. $f(x) = \text{-}5^{x+2}$ will move left 2 spaces, and be reflected over the x axis.
14. $f(x) = 5^{x-4} - 3$ will be moved down 3 spaces and right 4 spaces.
15. Because the y-intercept occurs where $x = 0$ and $y = a \cdot b^0 \rightarrow y = a \cdot 1 \rightarrow y = a$.
16. 10,485,760 rabbits
17. Approximately 1.138 grams

Section 4.3: Graphing and Evaluating Logarithmic Functions

1. 

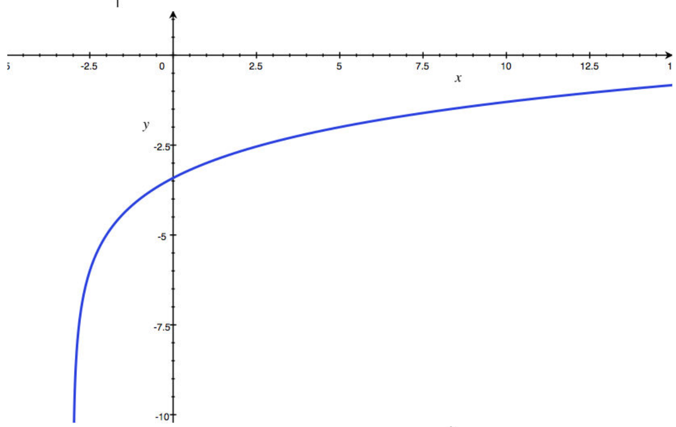
2. 

3. 



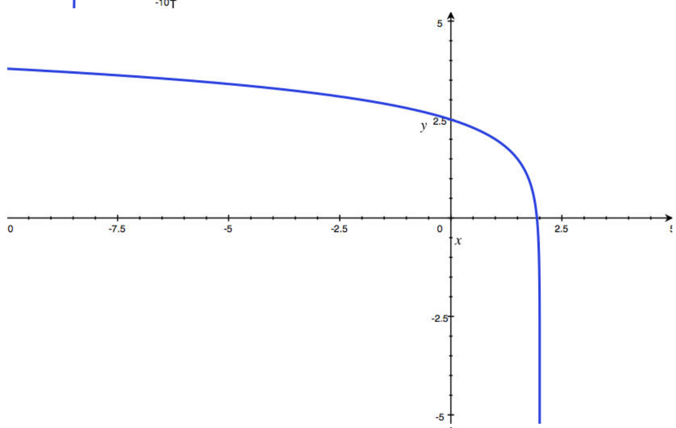
7.5
y
5
2.5
0
2.5
5
x
7.5

4.



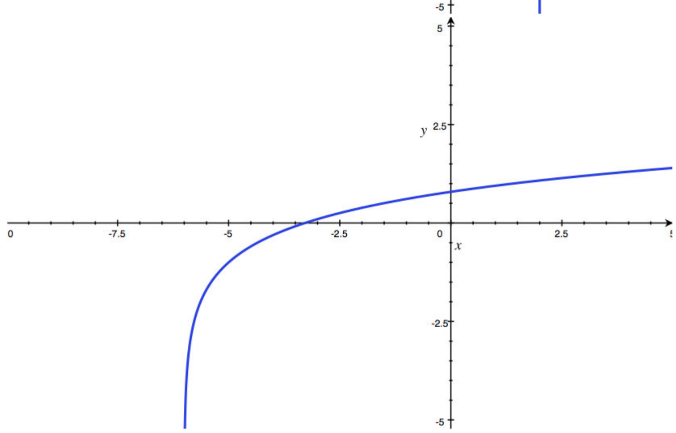
-2.5
0
2.5
5
7.5
10
12.5
x
y
-2.5
-5
-7.5
-10

5.



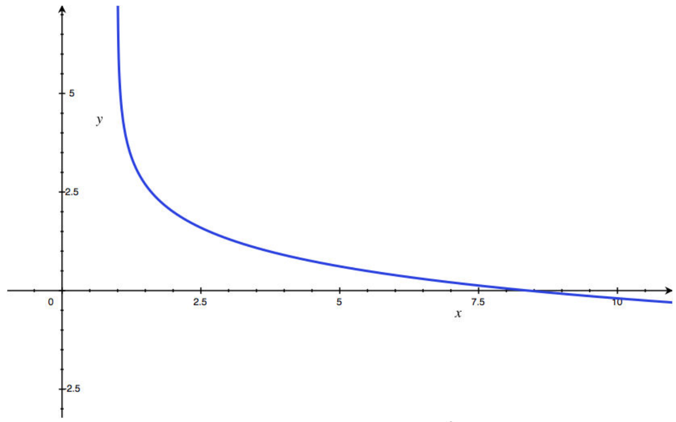
5
y
2.5
-7.5
-5
-2.5
0
x
2.5
-2.5
-5

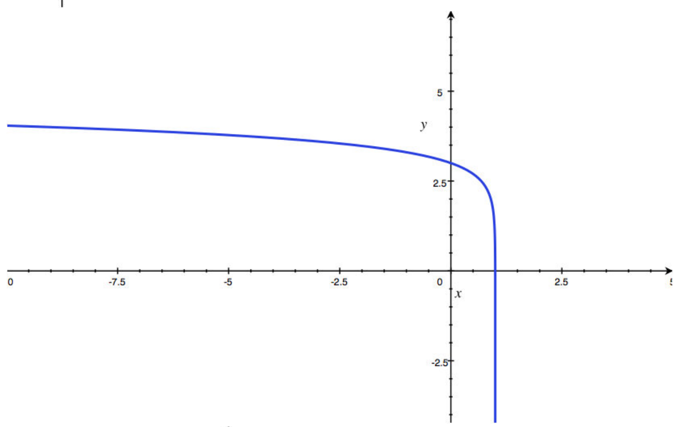
6.

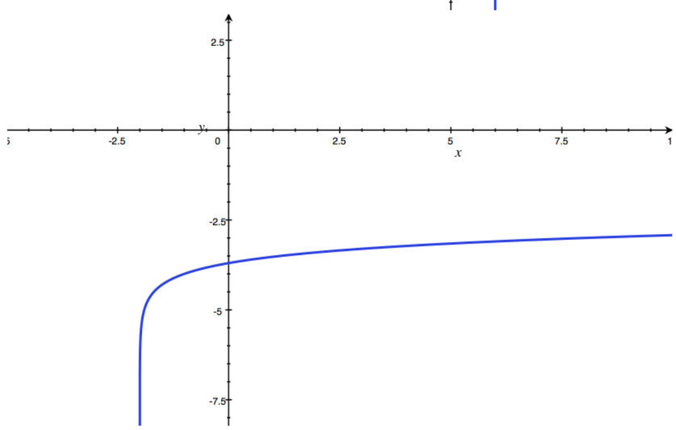


5
y
2.5
-7.5
-5
-2.5
0
x
2.5
-2.5
-5

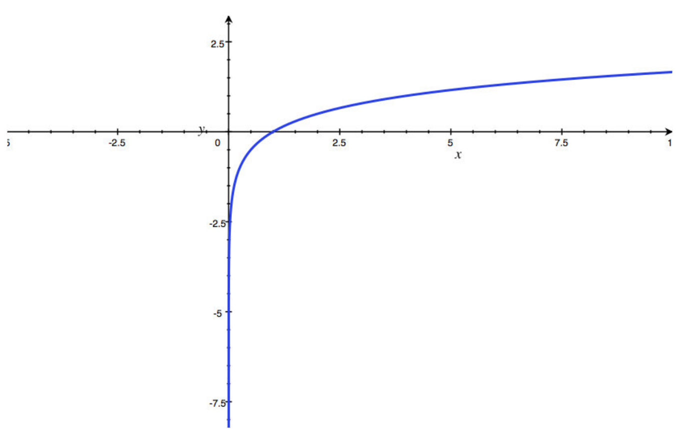
7.

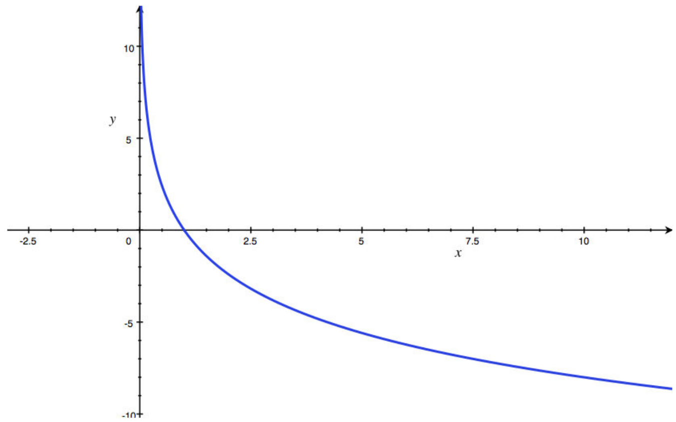
8. 

9. 

10. 

11. You could graph it as $\frac{\log(x)}{\log(4)}$.



12. 

13. Yes
14. No
15. No

4.4 Properties of Logs

1. False
2. False
3. True
4. $\log(8x^2+16x)$
5. $\log x^6$
6. $\log_2\left(\frac{3x^4}{y}\right)$
7. $\log_3 z^4$
8. $\log_4 2+3\log_4 x-\log_4 5$
9. $\ln 4+\ln x+2\ln y-\ln 15$
10. $2\log x+3\log y+3\log z-\log 3$
11. $\log_2 10=x+1$
12. $2^{12}=x-1$
13. Let $y=\log_{b^n} x$. Then $(b^n)^y=x$ and $b^{ny}=x$. Take the log base b of both sides, and you have $(ny)\log_b b=\log_b x$. Simplify to get $ny=\log_b x$. Solve for y, and you have $y=\frac{1}{n}\log_b x$. Since both $\frac{1}{n}\log_b x$ and $\log_{b^n} x$ are equal to y, they are equal to each other.
14. From the previous problem, you know that $\log_{b^n}(x^n)=\frac{1}{n}\log_b x^n$. Use the exponentiation property, and you have $\frac{1}{n}\log_b x^n=n(\frac{1}{n})\log_b x$, which simplifies to $\log_b x$.
15. $y=\log_{\frac{1}{b}}\frac{1}{x}$. Let $y=\log_{\frac{1}{b}}\frac{1}{x}$. Then $\left(\frac{1}{b}\right)^y=\frac{1}{x}$. You can rewrite this equation as $x=b^y$. You can then rewrite in logarithmic form as $\log_b x=y$. Since both $\log_b x$ and $\log_{\frac{1}{b}}\frac{1}{x}$ are equal to y, they are equal to each other.

Section 4.5: Solving Exponential Equations Using Logs

1. $x=1.292$
2. $x=0.431$
3. $x=0.697$
4. $x=1.333$
5. $x=3.754$
6. $x=0.783$

7. $x = -2$
8. $x = 1$
9. $x = 3.0259$
10. $x = 8.9872$
11. $x = \frac{\log(c+a)}{\log b}$
12. $x = -0.057$
13. $x = 1.8$
14. $x = 2$
15. $x = 27$

Section 4.6: Solving Logarithmic Equations

1. $x = 32,768$
2. $x \approx 498.831$
3. $x = 86$
4. $x = 170$
5. $x \approx 1.132$
6. $x = \sqrt[3]{3} \approx 1.442$
7. $x = 2$
8. $x = 1$
9. $x = 8$
10. $x = 27$
11. $x = 2$
12. $x \approx 3.272$
13. $x = \frac{4}{3}$
14. $x = 2 + 2\sqrt{11} \approx 8.633$
15. $x = 6$

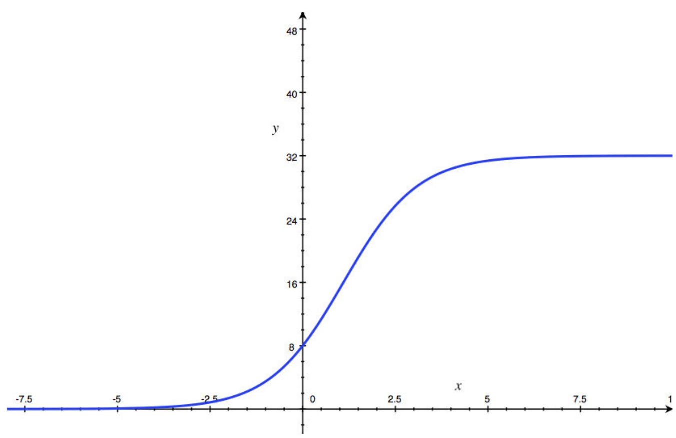
Section 4.7: Compound Interest

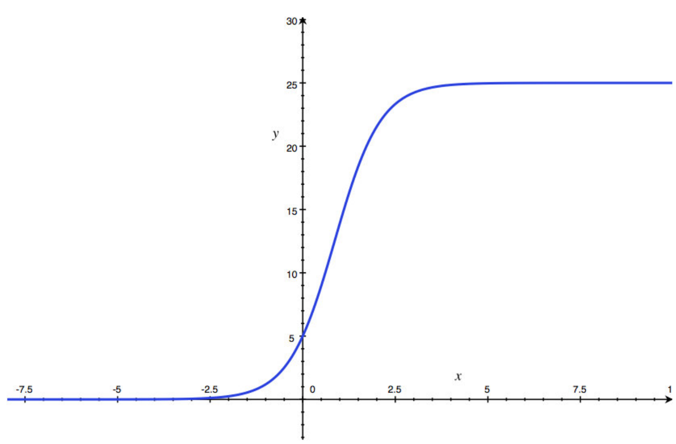
1. The compound interest formula is $A(t) = p(1+r)^t$.
2. They would have earned $373.82.
3. Kyle's balance would be $1,098.81.
4. Yearly simple interest is effectively the same as compound interest compounding yearly. Roberta would have $18,022.10 in the account.
5. The bank has been paying approximately 3.5% annually.
6. She could expect to have $3,122.
7. There is a balance of $885.08.
8. Her original deposit was $650.
9. $1000(1+.05)^7$
10. $f(x) = 3000(1.14^x)$

    $7,500
11. He will owe a total of $318.27.
12. $600
13. $715.56

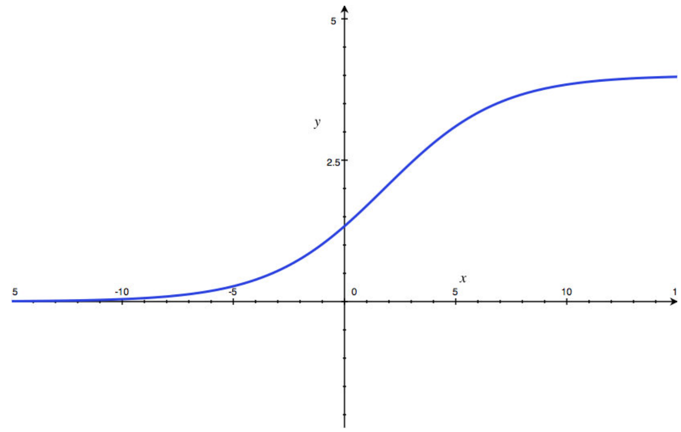
14. \$15,300.14
15. \$13,138.75

Section 4.8: Population Growth Models and Logistic Functions

1. $f(x) = \frac{12}{1+1.4 \cdot 0.51^x}$
2. $f(x) = \frac{200}{1+\frac{1}{3} \cdot 0.8027^x}$
3. $f(x) = \frac{1,500}{1+9 \cdot 0.749^x}$
4. $f(x) = \frac{1,000,000}{1+9 \cdot 0.959^x}$
5. $f(x) = \frac{30,000,000}{1+2.75 \cdot 0.89^x}$
6. 32
7. 8
8. 

9. 25
10. 5
11. 

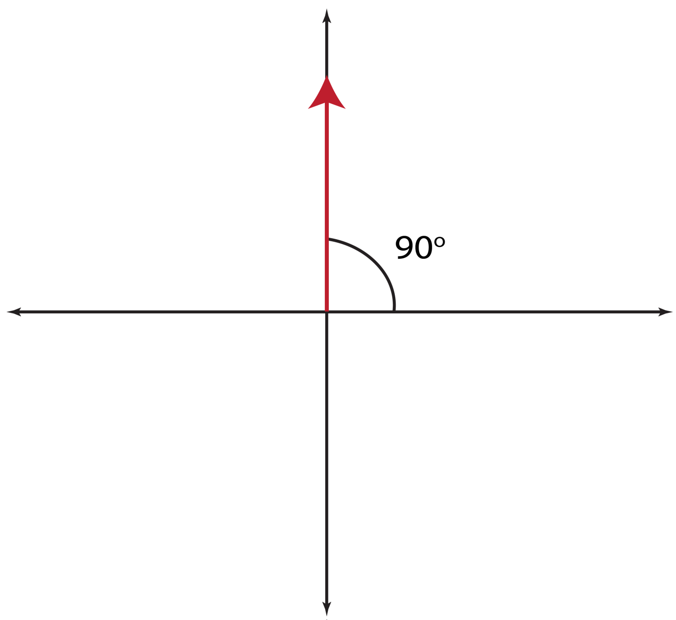
12. 4
13. $\frac{4}{3}$

14. 

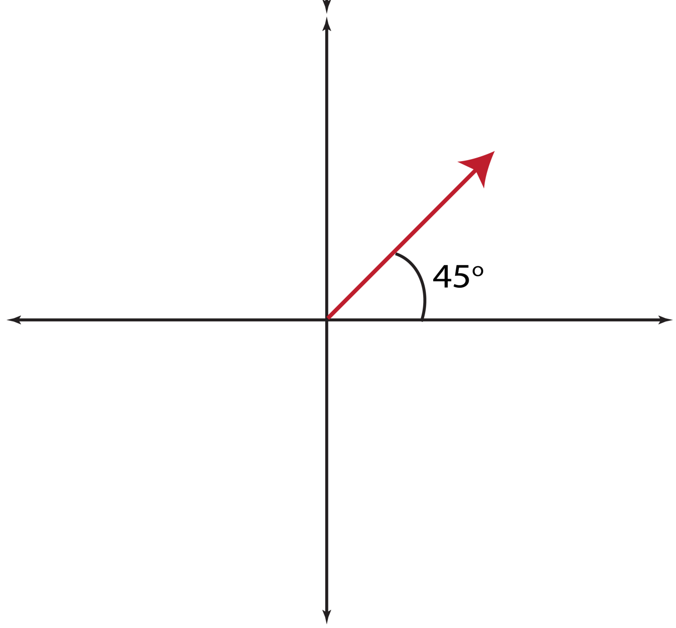
15. Any function in the form $f(x) = \frac{c}{1+a \cdot b^{-x}}$, where $0 < 1$.

16.5 Answers - Ch 5: Angles

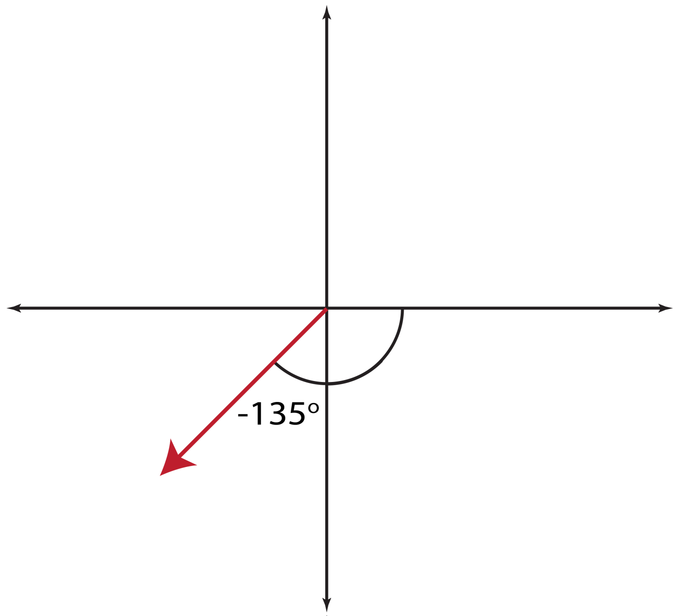
Section 5.2: Angles of Rotation in Standard Positions

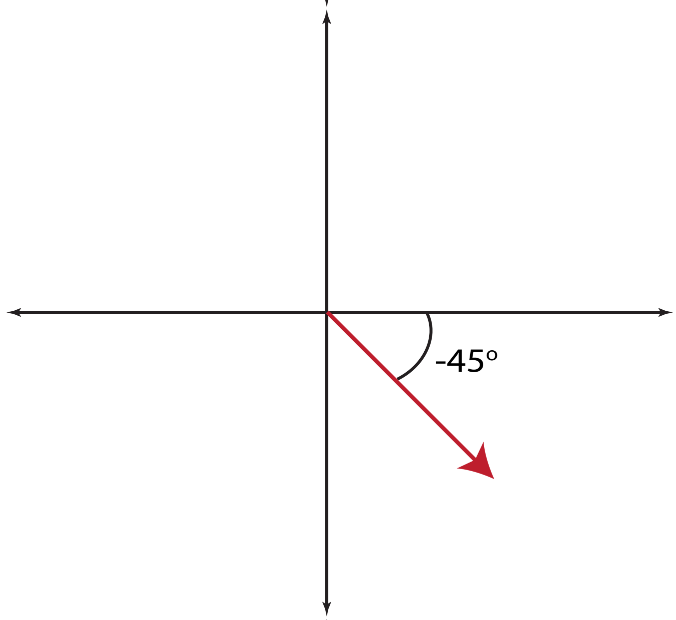


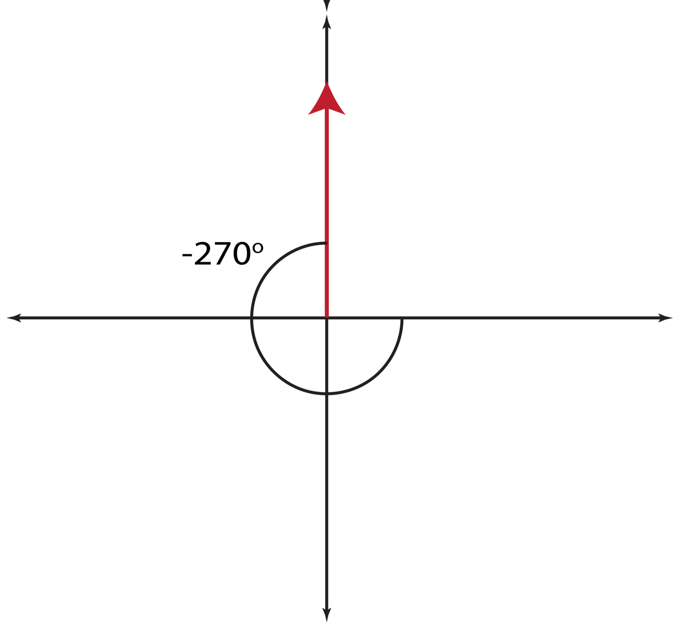
1.

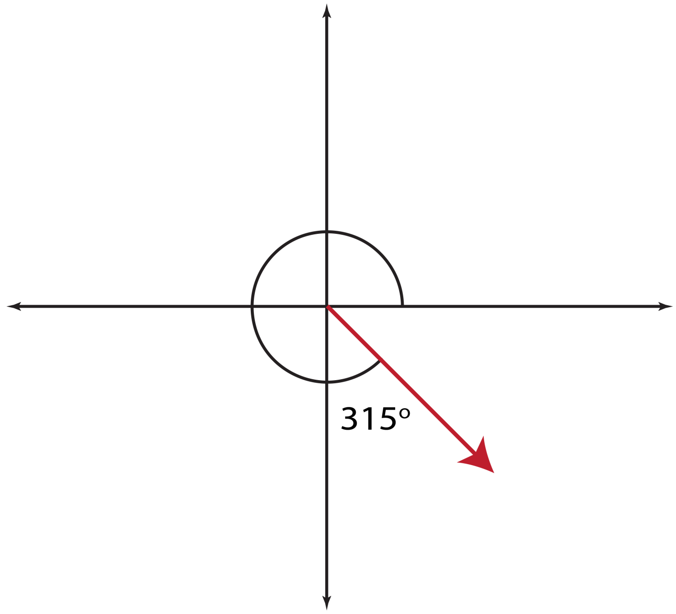


2.

3. 

4. 

5. 

6. 

7. 270°
8. 45°
9. 135°
10. -180°
11. -45°
12. -315°
13. 240°
14. 30°

Section 5.3: Angles in Radians and Degrees

1. $\frac{2\pi}{3}$
2. $\frac{5\pi}{3}$
3. $\frac{\pi}{2}$
4. $\frac{11\pi}{6}$
5. $\frac{3\pi}{2}$
6. $\frac{\pi}{4}$
7. $\frac{\pi^2}{36}$
8. 210°
9. 225°
10. 270°
11. 300°
12. 180°
13. 30°
14. $\frac{540}{\pi}^\circ$
15. There are 360 degrees and 2π radians in a circle. Therefore, each degree is $\frac{\pi}{180}$ radians.
16. Multiply by $\frac{\pi}{180}$ to give you $\frac{5}{12}\pi$, or approximately 0.417π radians.

Section 5.4: Length of an Arc

1. 8π meters

2. 16π meters
3. 32π meters
4. $\frac{3}{10}$ radians
5. 8π inches
6. 12π inches
7. 35π inches
8. 48.02 revolutions
9. 384.8 inches
10. 48π radians
11. $\frac{10\pi}{3}$ inches
12. $\frac{5\pi}{6}$ inches
13. $\frac{55\pi}{12}$ inches
14. $\frac{25\pi}{9}$ inches
15. $\frac{215\pi}{72}$ inches

Section 5.5: Area of a Sector

1. 22.34 in^2
2. 37.7 in^2
3. 43.98 in^2
4. 65.45 in^2
5. 87.27 in^2
6. .56 radians
7. 1.875 radians
8. 2.67 radians
9. 6.18 inches
10. 15.14 inches
11. 7.46 inches
12. 8.74 inches
13. 15 in^2
14. 125.66 in^2
15. 226.19 in^2

Section 5.6: Angular Speed

1. Beth went 14π ft, and Steve went 28π ft.
2. $\frac{\pi}{6}$ radians per second
3. Beth: $\frac{7\pi}{6}$ ft/sec; Steve: $\frac{14\pi}{6}$ ft/sec
4. 6 feet from the center
5. 6 seconds
6. Beth: $\frac{7\pi}{3}$ ft/sec; Steve: $\frac{14\pi}{3}$ ft/sec
7. 16 feet from the center
8. 2.67 feet from the center
9. $\frac{\pi}{30}$ radians/minute
10. $\frac{\pi}{360}$ radians/minute
11. $\frac{\pi}{30}$ feet per minute
12. $\frac{\pi}{360}$ feet per minute
13. 19 inches

14. 2π radians/minute
15. 0.32 ft or 3.82 inches

16.6 Answers - Ch 6: Basic Triangle Trigonometry

Section 6.2: Special Right Triangles

1. The other sides are 3 and $3\sqrt{2}$.
2. The other sides are 7.2 and $7.2\sqrt{2}$.
3. The other sides are each $8\sqrt{2}$.
4. The other sides are $5\sqrt{2}$ and 10.
5. The other sides are $3\sqrt{6}$ and $6\sqrt{2}$.
6. The other sides are $\frac{4\sqrt{3}}{3}$ and $\frac{8\sqrt{3}}{3}$.
7. The other sides are 7.5 and $7.5\sqrt{3}$.
8. The other sides are $6\sqrt{3}$ and 18.
9. The 67.38° angle
10. The side labeled 13
11. The 22.62° angle
12. $3^2+4^2=25=5^2$
13. $5^2+12^2=169=13^2$
14. $7^2+24^2=625=25^2$
15. $8^2+15^2=289=17^2$
16. $9^2+40^2=1{,}681=41^2$
17. $6^2+8^2=100=10^2$
18. Answers vary. One option would be 22, 120, 122.

Section 6.3: Right Triangle Trigonometry

1. $a=8.06,\ b=4.0,\ C=60.3$^○ B=53Â°, c=10.8, b=14.4
2. $A=75$^○, c=8.3, a=30.9 a = 6.0, b=9.2, B=56.9Â°
3. $C=78$^○, b=4.0, c=18.6 a=13.7, b=17.0, C=36.0^○
4. $C=80$^○, a=11.5, c=11.3 C=86Â°, b=4.3, c=4.29
6. $B=0.57$ radian, $a=17.8,\ b=9.6$
7. $b=15.0,\ c=9,\ C=0.6$ radian
8. $a=10.7,\ A=0.87$ radian, $b=9.0$
9. $C=\frac{\pi}{2}$ radian, $a=5, c=5\sqrt{2}$
10. $B=\frac{\pi}{6},\ C=\frac{\pi}{3},\ c=13\sqrt{3}$
11. $a=10.2,\ A=0.6$ radian, $c=16.0$
12. $A=\frac{\pi}{4}$ radian, $B=\frac{\pi}{4}$ radian, $b=10$

Section 6.4: Inverse Trigonometric Functions

1. Approximately 57.03°

2. Approximately 79.05°
3. Approximately 12.02°
4. Approximately 39.99°
5. Approximately 61.01°
6. Approximately 23.00°
7. $x \approx 45^\circ$ and $y \approx 45^\circ$
8. $x \approx 71^\circ$ and $y \approx 19^\circ$
9. $x \approx 27^\circ$ and $y \approx 63^\circ$
10. $x \approx 45^\circ$ and $y \approx 45^\circ$
11. $x \approx 50^\circ$ and $y \approx 40^\circ$
12. $b \approx 39.7$, $A \approx 39^\circ$, and $B \approx 51^\circ$
13. $c \approx 10.8$, $A \approx 34^\circ$, and $B \approx 56^\circ$
14. $a \approx 10.7$, $A \approx 50^\circ$, and $B \approx 40^\circ$
15. Approximately 34.99°
16. Approximately 21.8°
17. Approximately 88.26°

Section 6.5: The Law of Cosines

1. 36.7°
2. 48.3°
3. 95°
4. 118.7°
5. 27.1°
6. 34.2°
7. 16.85
8. 80.47°
9. 38.53°
10. 4.4 miles
11. Approximately 47.1 feet
12. 560 feet
13. Yes
14. No
15. Yes

Section 6.6: The Law of Sines

1. 35.2° or 144.8°
2. 22.5°
3. 22.6° or 157.4°
4. 58.9°
5. 65° or 115°
6. 71.8° or 108.2°
7. 63.9°
8. No solution
9. No solution

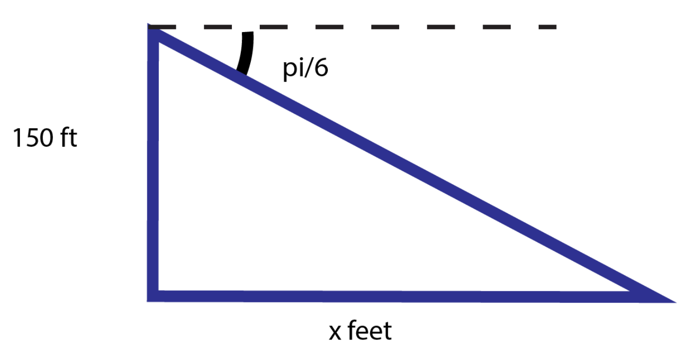
10. 51.5°
11. $b > 6.3$
12. $b < 13.8$
13. $b = 20.74$ or $b \geq 21$
14. Bill is 2.52 miles from school, and Connie is 3.1 miles from school.
15. About 0.62 miles

Section 6.7: Area of a Triangle

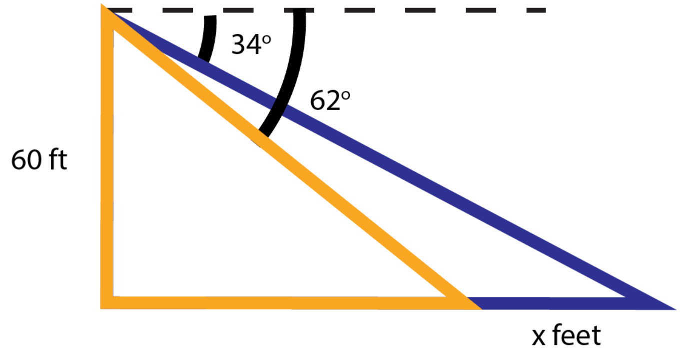
1. 91.6 square units
2. 13.6 square units
3. 403.8 square units
4. 10.4 square units
5. 7.2 square units
6. 11.2 square units
7. 13.6 square units
8. 4.3 square units
9. 17.3 square units
10. 18.8 square units
11. 72.3 square units
12. 48.6°
13. 41.1°
14. 45.3°
15. 22.9°

Section 6.8: Applications of Basic Triangle Trigonometry

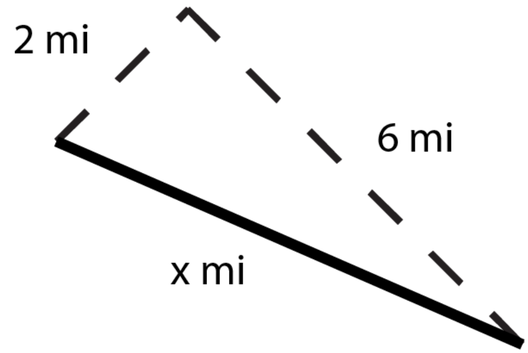
1.



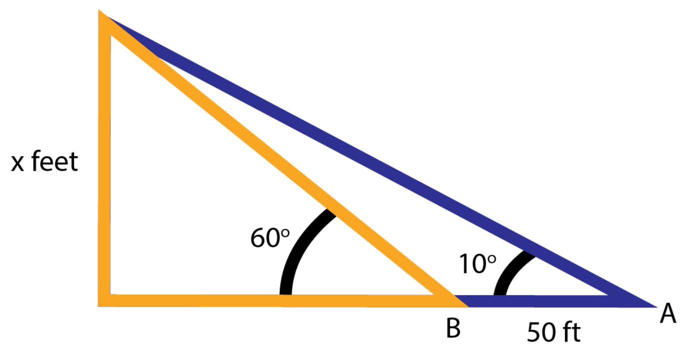
2. Find the angle in the triangle (complementary angle), and then use tangent.
3. 259.8 feet

4. 

5. Tangent
6. 57.05 feet

7. 

8. This is a right triangle, so you can use the Pythagorean Theorem.
9. 6.32 miles

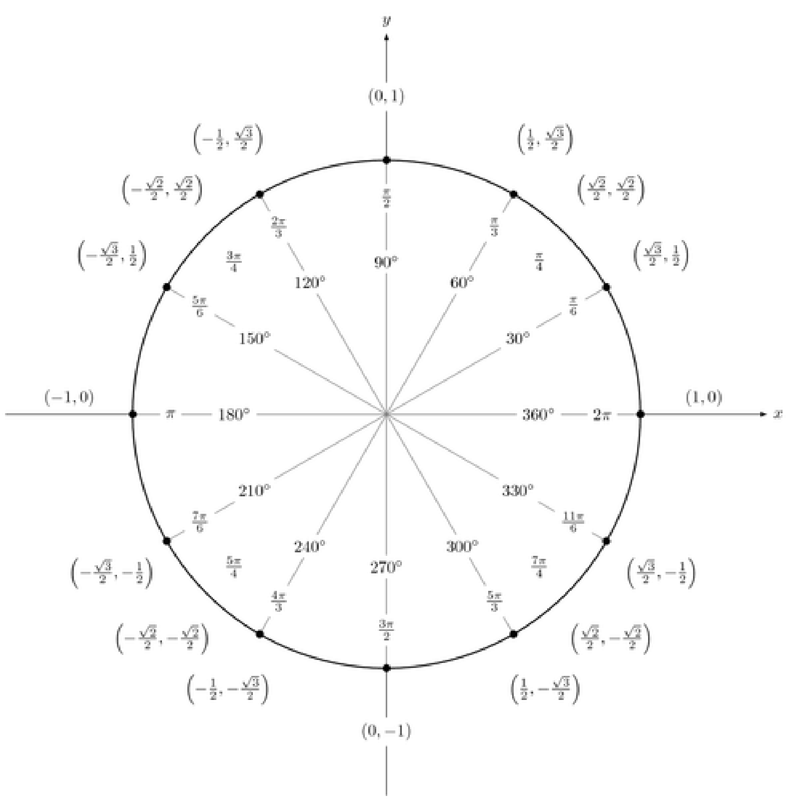
10. 

11. Two tangent equations to solve for the distance from B to the building, then substitute to solve for the height.
12. 9.8 feet
13. 5.32 inches
14. 10.48 inches
15. 10.35 inches

16.7 Answers - Ch 7: The Unit Circle and Trigonometric Functions

Section 7.2: The Unit Circle

1. 340°
2. 115°
3. 140°
4. 330°
5. 315°
6. Negative
7. Positive
8. Positive
9. Positive
10. -$\frac{1}{2}$
11. 2
12. 1
13. -$\frac{1}{\sqrt{3}}$ or -$\frac{\sqrt{3}}{3}$
14. -$\frac{1}{2}$
15. -$\frac{2\sqrt{3}}{3}$



16.

Section 7.3: Graphing Sine and Cosine

1. A.$(\frac{\pi}{2}, 1)$

   B. $(\pi, -1)$

   C. $(\frac{3\pi}{2}, 0)$

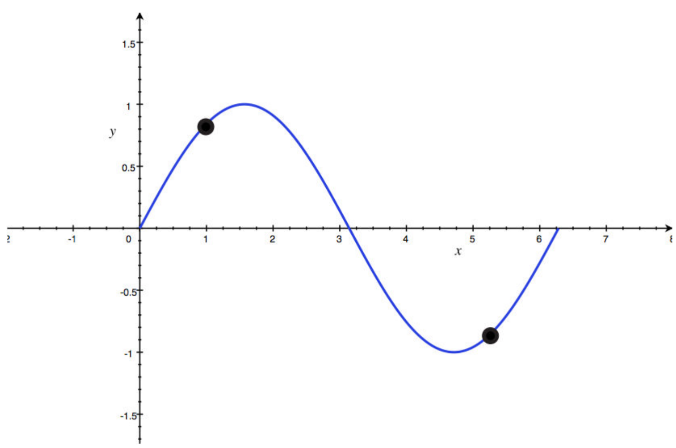
   D. $(\frac{11\pi}{6}, -\frac{1}{2})$

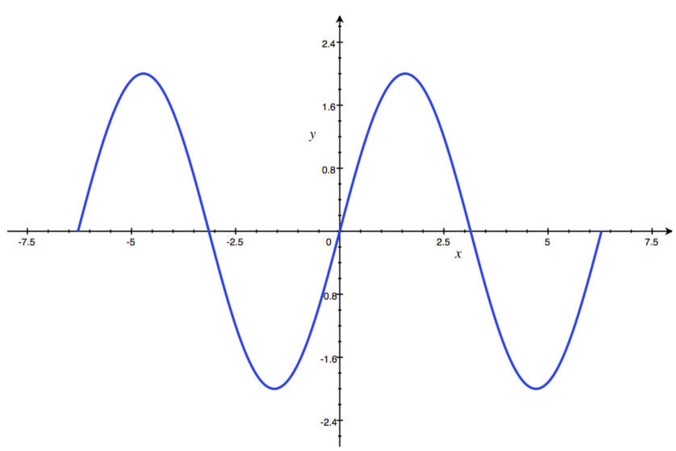
   E. $(2\pi, 1)$

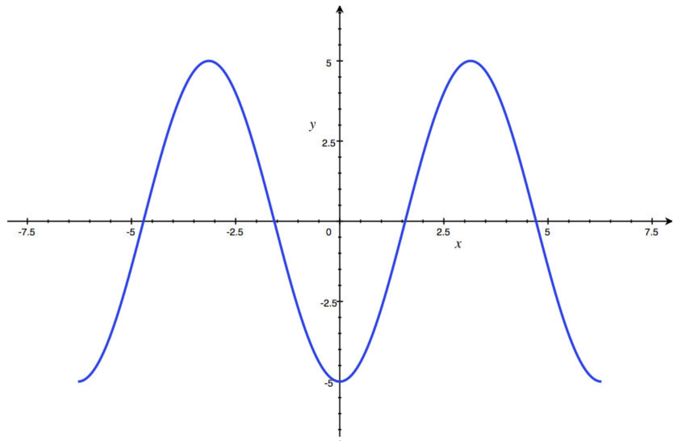
   F. $(\frac{11\pi}{4}, \frac{\sqrt{2}}{2})$

   G. $(\frac{7\pi}{2}, -1)$

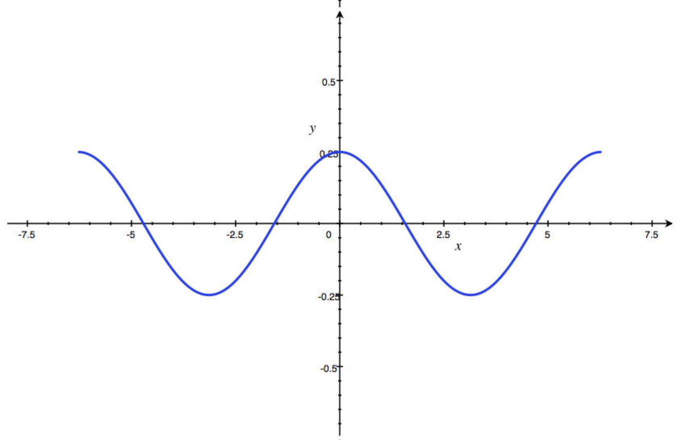
   H. $(\frac{11\pi}{3}, -\frac{\sqrt{2}}{2})$
2. $(\frac{\pi}{4}, \frac{\sqrt{2}}{2}), (\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}), (\frac{9\pi}{4}, \frac{\sqrt{2}}{2}), (\frac{13\pi}{4}, -\frac{\sqrt{2}}{2})$
3. $(\frac{\pi}{3}, \frac{\sqrt{3}}{2}), (\frac{5\pi}{3}, -\frac{\sqrt{3}}{2})$



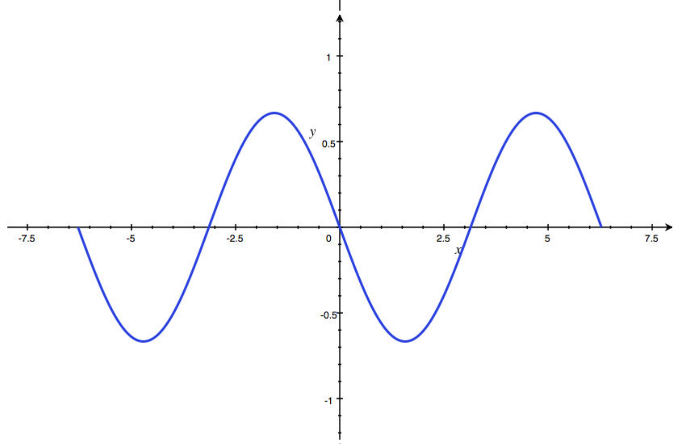
4. 


5
2.5
-2.5
-5
y
-7.5
-5
-2.5
0
2.5
5
7.5
x

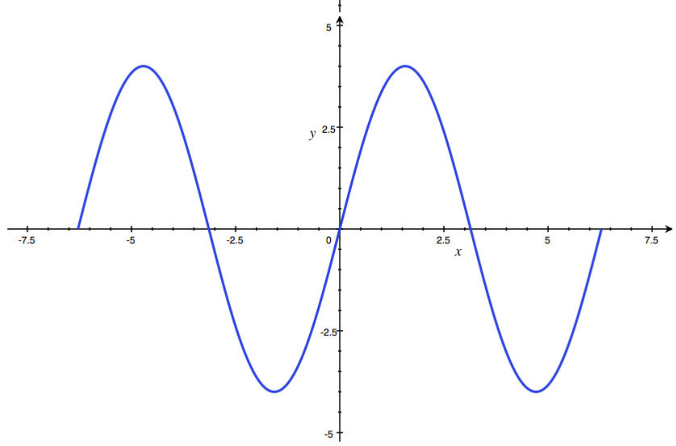
5.


0.5
0.25
-0.25
-0.5
y
-7.5
-5
-2.5
0
2.5
5
7.5
x

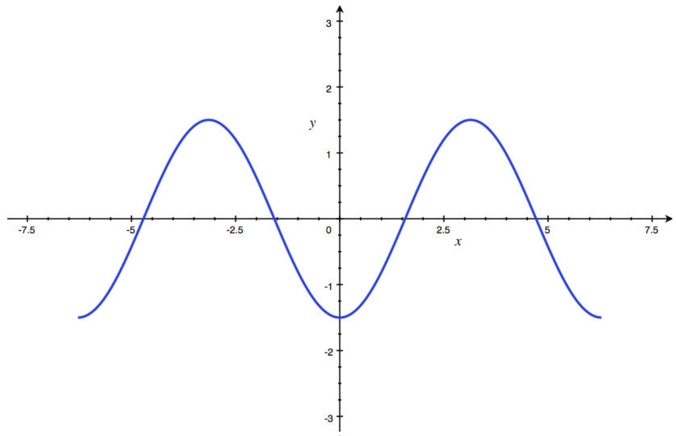
6.

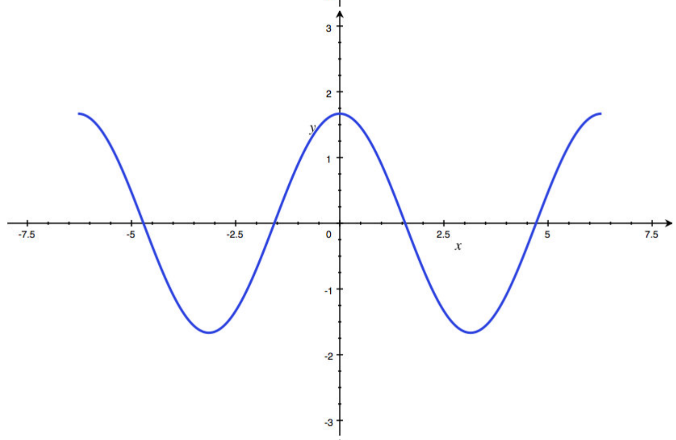

1
0.5
-0.5
-1
y
-7.5
-5
-2.5
0
2.5
5
7.5
x

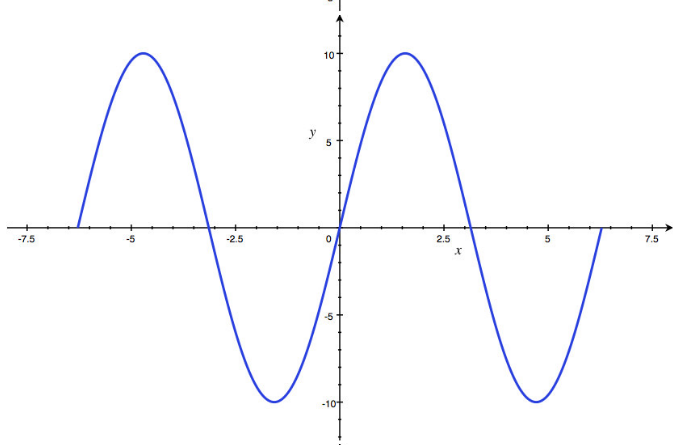
7.

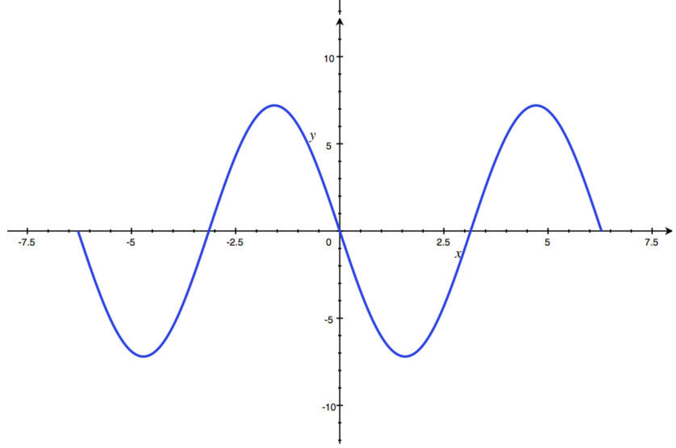

5
2.5
-2.5
-5
y
-7.5
-5
-2.5
0
2.5
5
7.5
x

8.

9. 

10. 

11. 

12. 

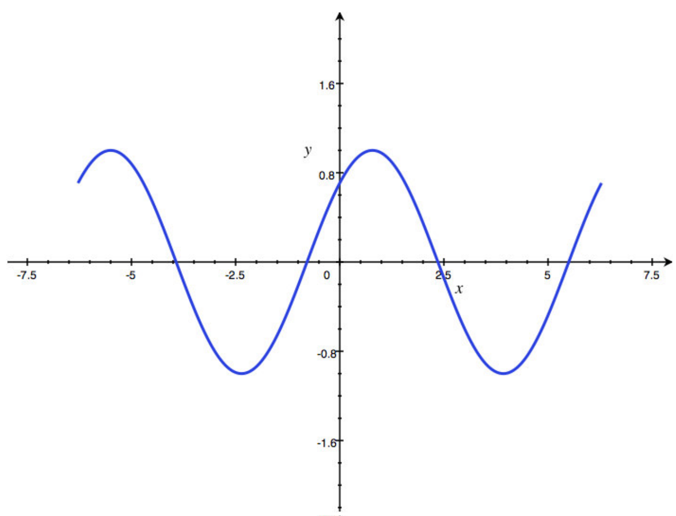
13. $\frac{\pi}{2}$ units
14. $\frac{\pi}{2}$ units
15. $y = 2.5 \sin x$

16. $y = 1.75 \cos x$

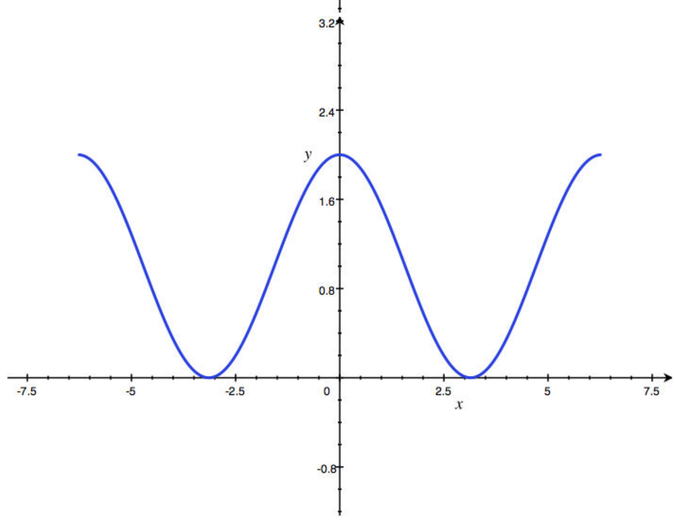
Section 7.4: Translating Sine and Cosine Functions

1. C
2. A
3. D
4. B
5. C
6. D
7. $y = \sin\left(x - \frac{5\pi}{4}\right) + 3$
8. Infinitely many. Explanations will vary.
9. a) $\frac{\pi}{2}$

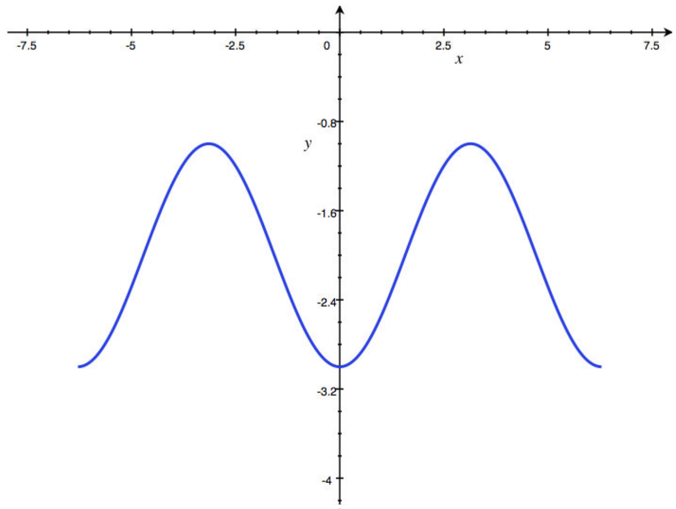
   b) $\frac{\pi}{2}$

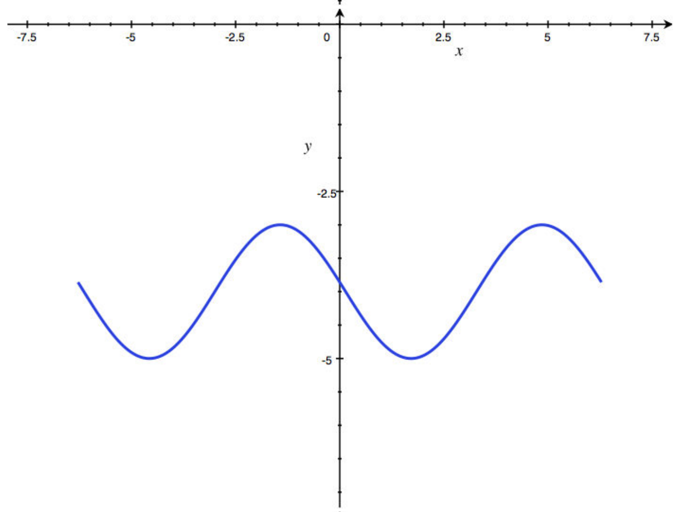


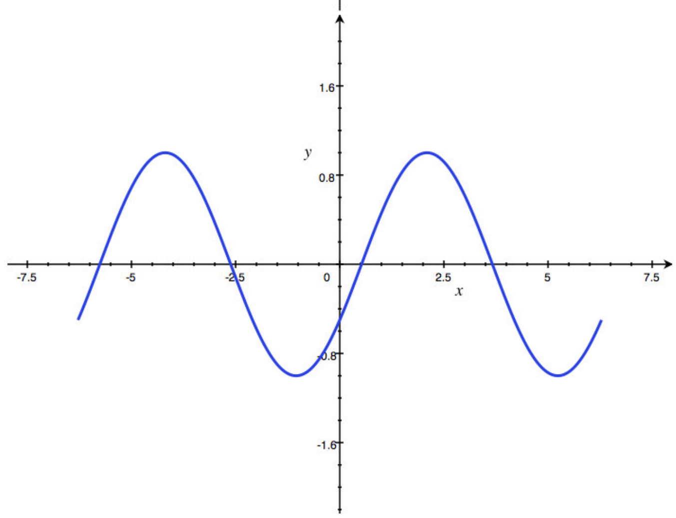
10.

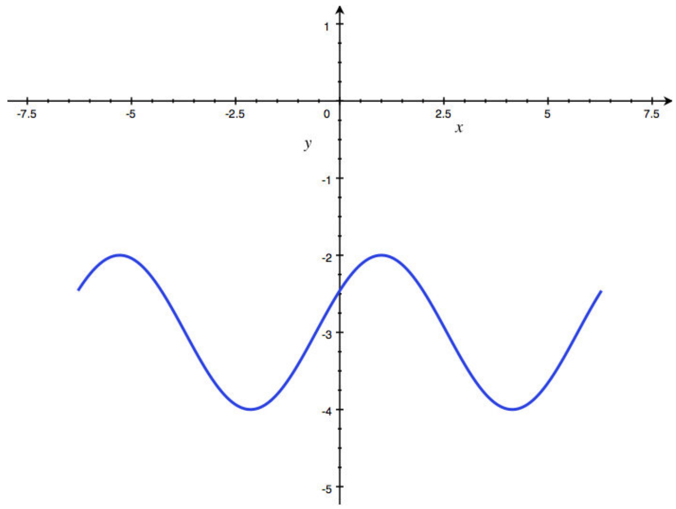


11.

12. 
-7.5
-5
-2.5
0
2.5
5
7.5
x
y
-0.8
-1.6
-2.4
-3.2
-4

13. 
-7.5
-5
-2.5
0
2.5
5
7.5
x
y
-2.5
-5

14. 
1.6
y
0.8
-7.5
-5
-2.5
0
2.5
5
7.5
x
-0.8
-1.6

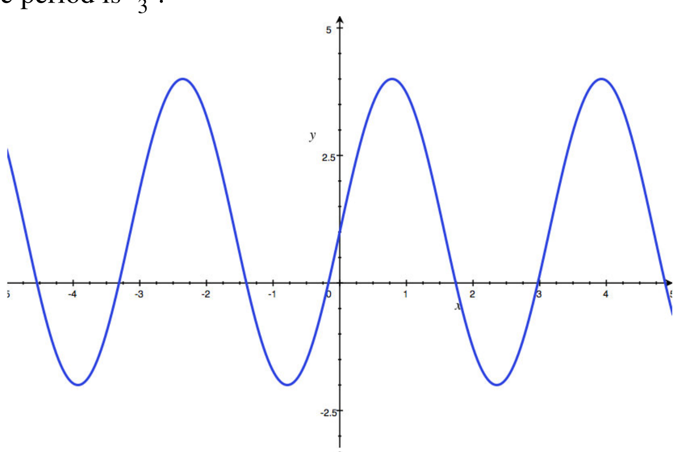


15.

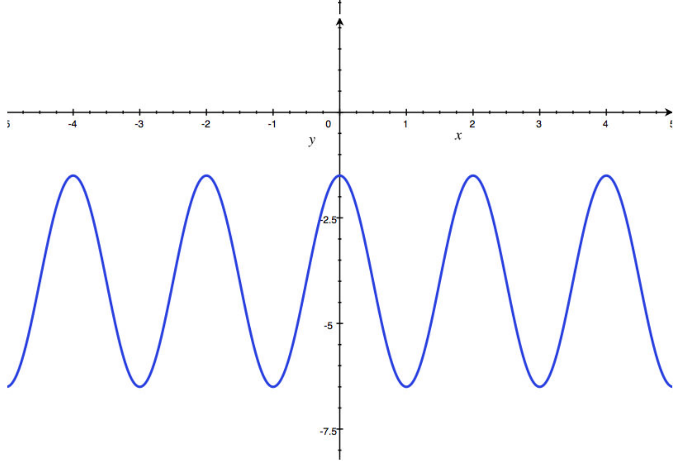
16. Yes; explanations will vary.

Section 7.5: Frequency and Period of Sinusoidal Functions

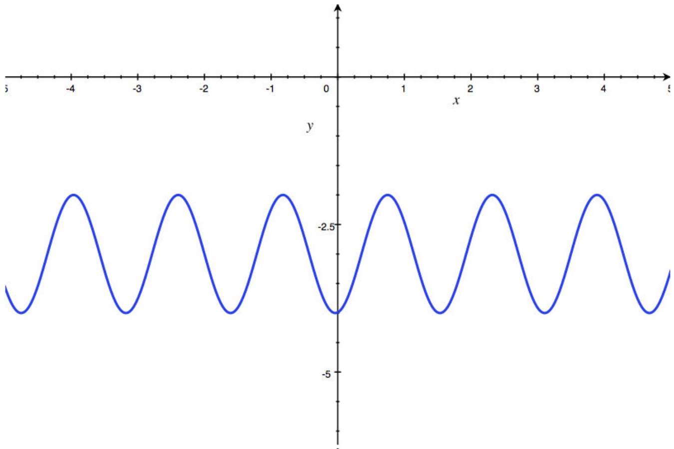
1. The frequency is 4 and the period is $\frac{\pi}{2}$.
2. The frequency is 2 and the period is π.
3. The frequency is $\frac{1}{2}$ and the period is 4π.
4. The frequency is $\frac{3}{4}$ and the period is $\frac{8\pi}{3}$.
5. The frequency is 3 and the period is $\frac{2\pi}{3}$.



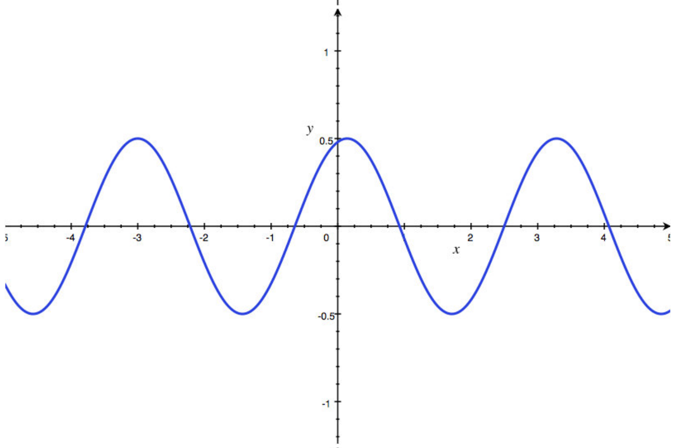
6.



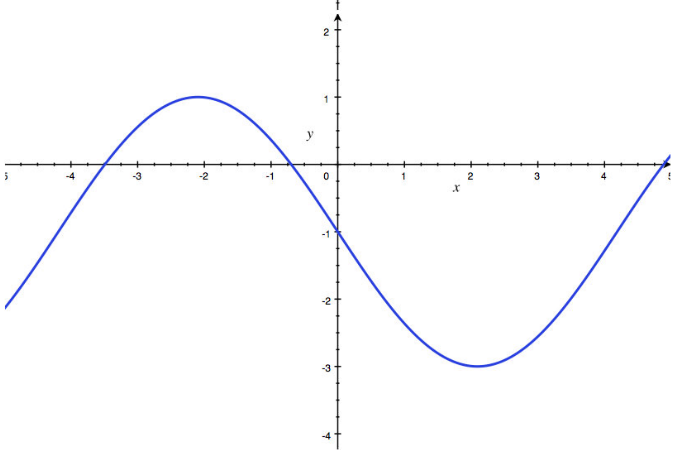
7.



8.

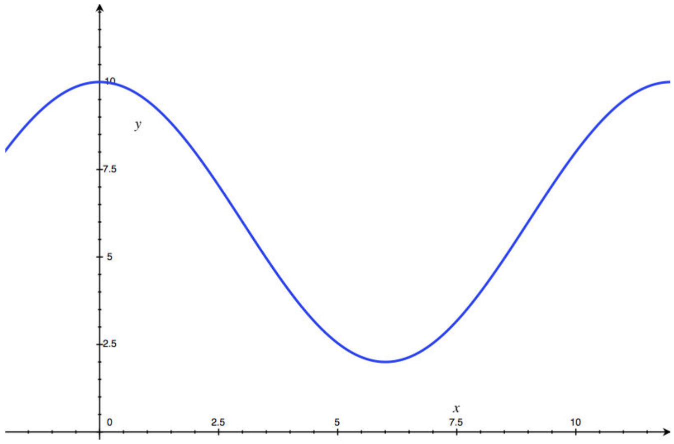


9.



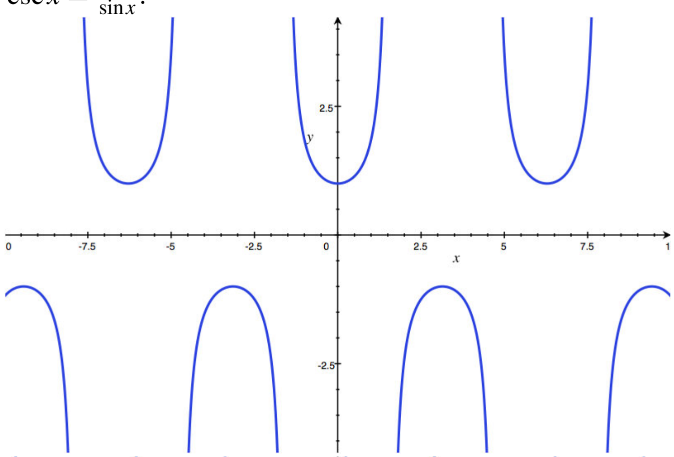
10.

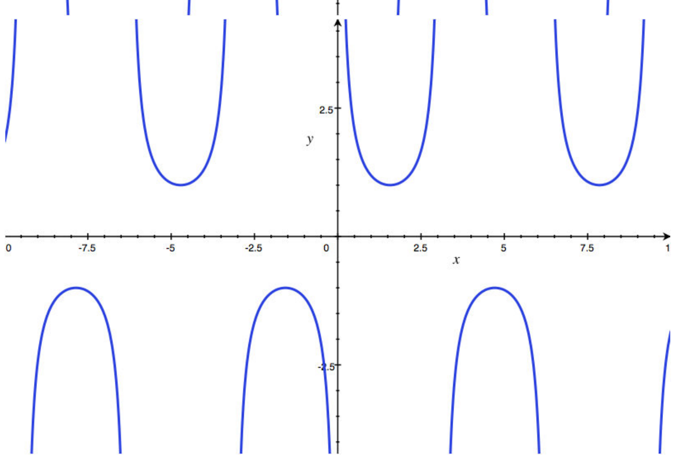
11. $f(x) = -3\sin(3x) + 2$
12. $f(x) = 2\cos(2x)$
13. $f(x) = \cos\left(\frac{x}{2}\right) - 1$

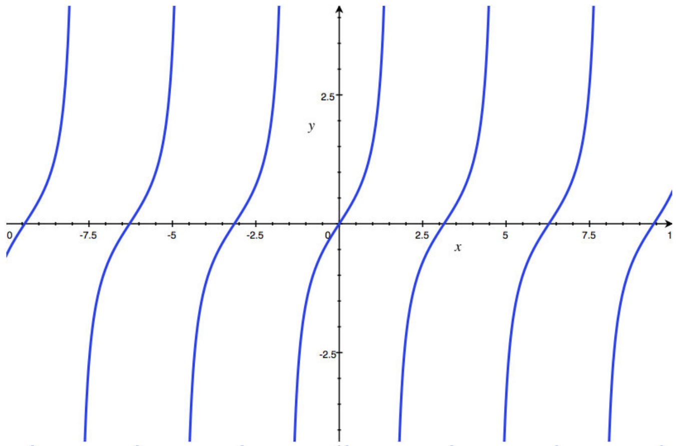
14. 

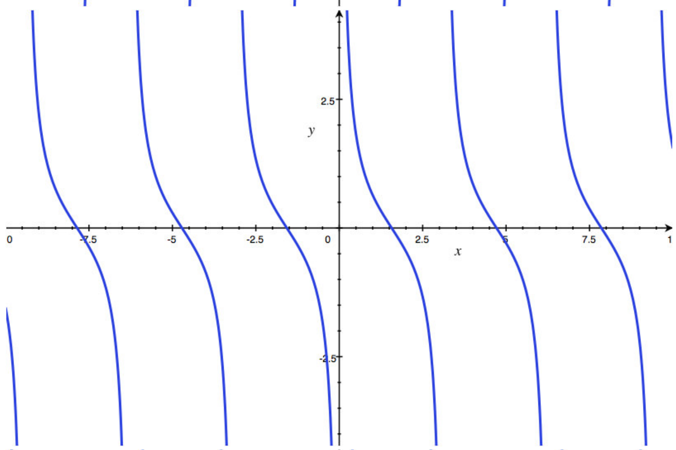
15. $f(x) = 4 \cdot \cos\left(\frac{\pi}{6} \cdot x\right) + 6$

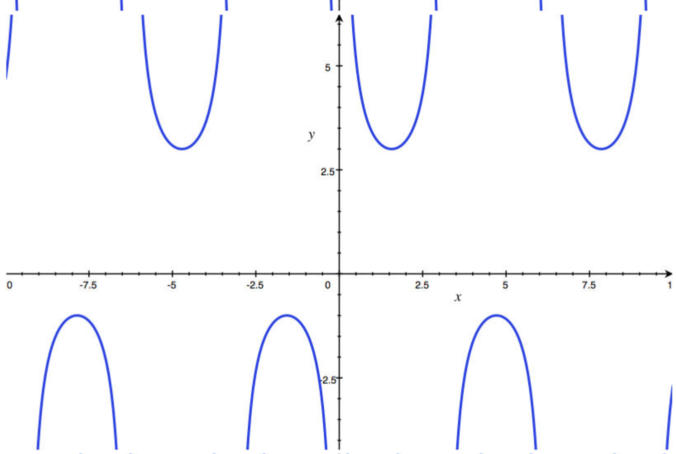
Section 7.6: Graphs of Other Trigonometric Functions

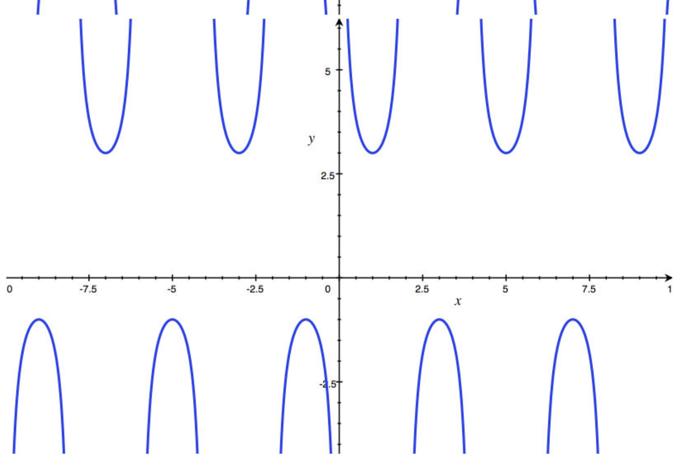
1. You can use $\cos x$ because $\sec x = \frac{1}{\cos x}$.
2. You can use $\sin x$ because $\csc x = \frac{1}{\sin x}$.
3. 

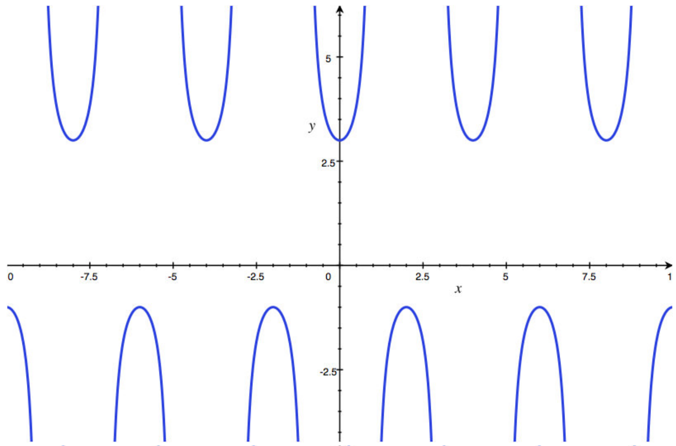
4. 

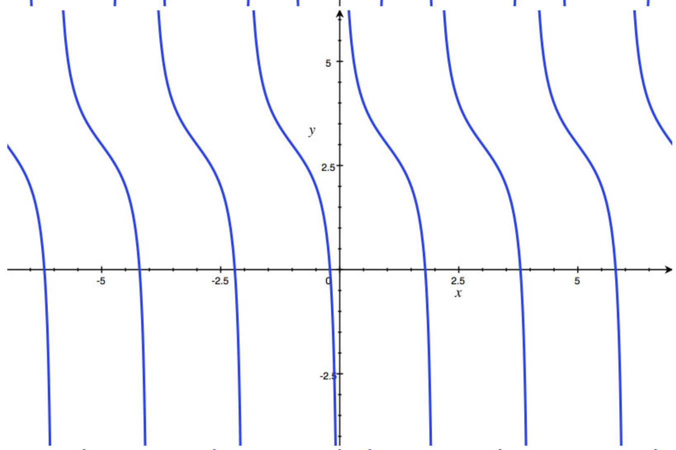
5. 
2.5
y
-7.5
-5
-2.5
0
2.5
x
5
7.5
-2.5

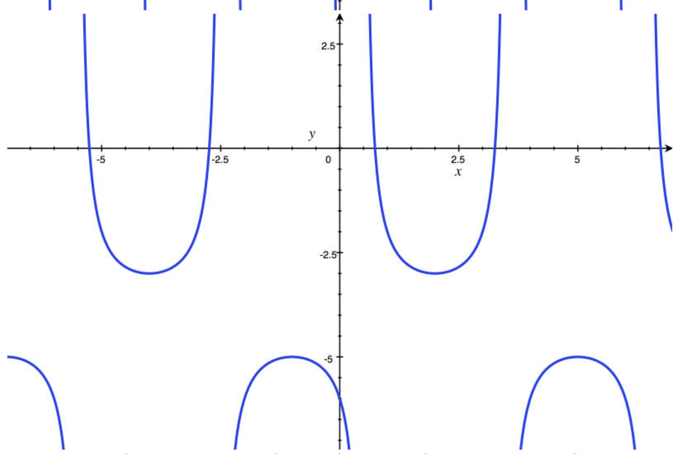
6. 
2.5
y
-7.5
-5
-2.5
0
2.5
x
5
7.5
-2.5

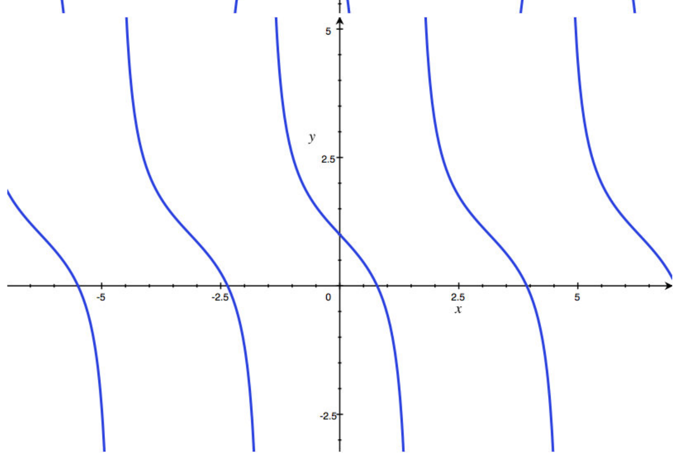
7. 
5
y
2.5
-7.5
-5
-2.5
0
2.5
x
5
7.5
-2.5

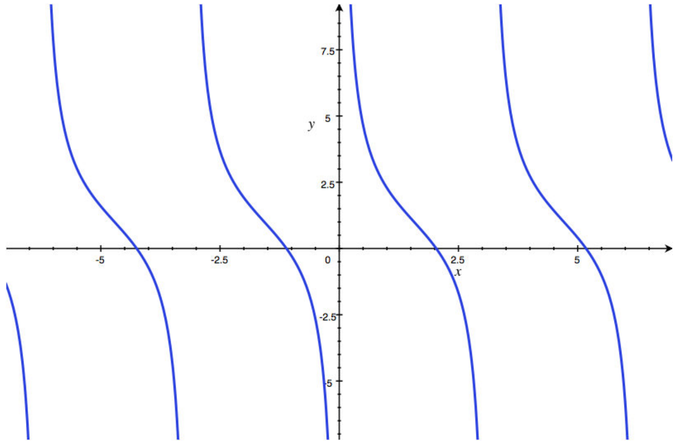
8. 
5
y
2.5
-7.5
-5
-2.5
0
2.5
x
5
7.5
-2.5

9. 

10. 

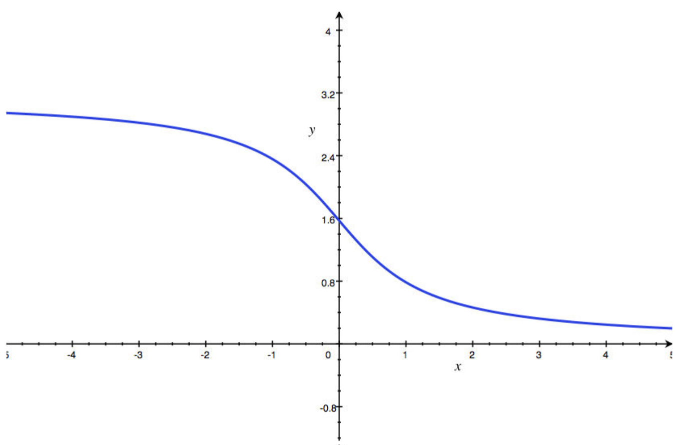
11. 

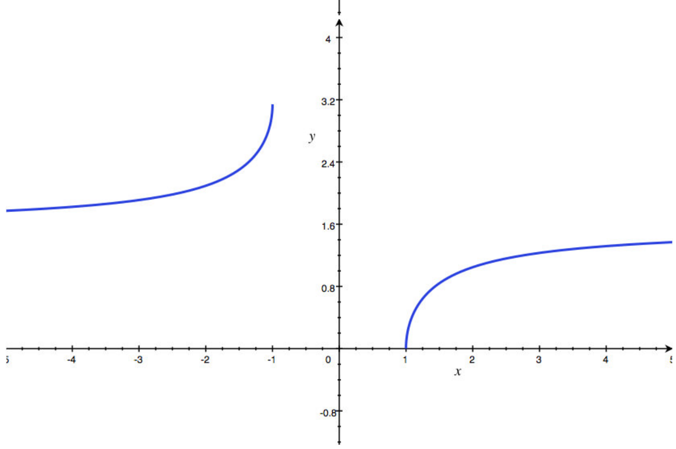
12. 

13. 

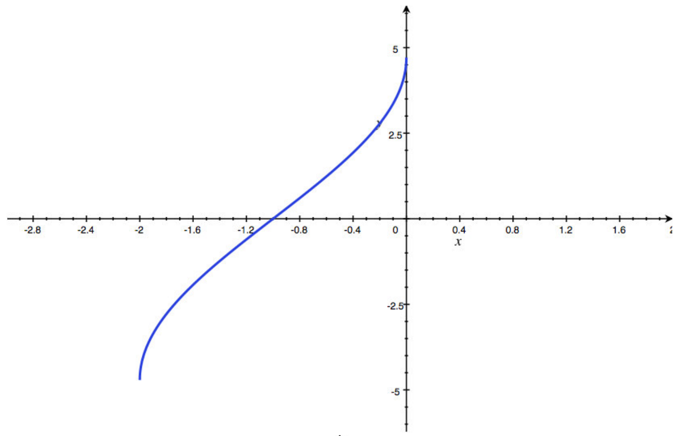
14. Possible answers: $secx = \frac{1}{cosx}; secx = \csc\left(x+\frac{\pi}{2}\right)$
15. Possible answers: $\csc x = \frac{1}{\sin x}; \csc x = \sec\left(x-\frac{\pi}{2}\right)$

Section 7.7: Graphs of Inverse Trigonometric Functions

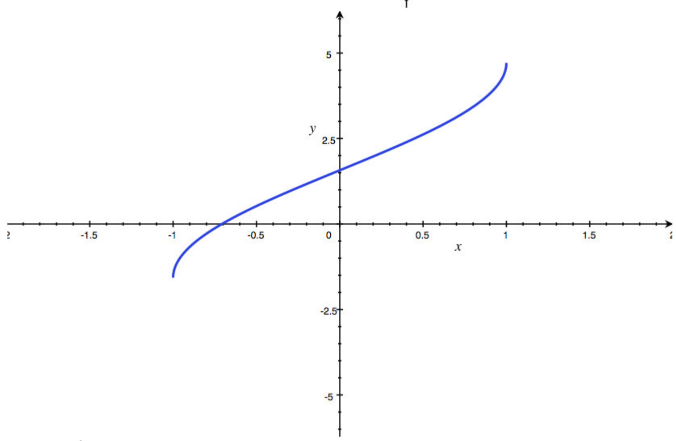
1. 

2. 

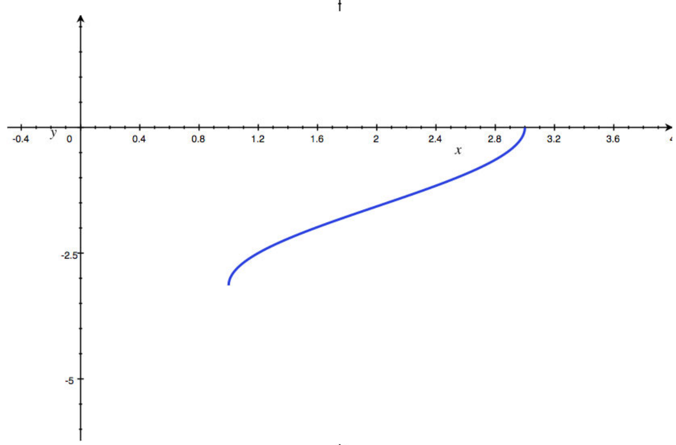
3. $y = \arcsin x$
4. $y = \tan x$
5. $y = \csc x$
6. $y = \frac{1}{\arccos x}$
7. $y = \frac{1}{\arctan x}$


5
2.5
y
-2.8 -2.4 -2 -1.6 -1.2 -0.8 -0.4 0 0.4 0.8 1.2 1.6
x
-2.5
-5

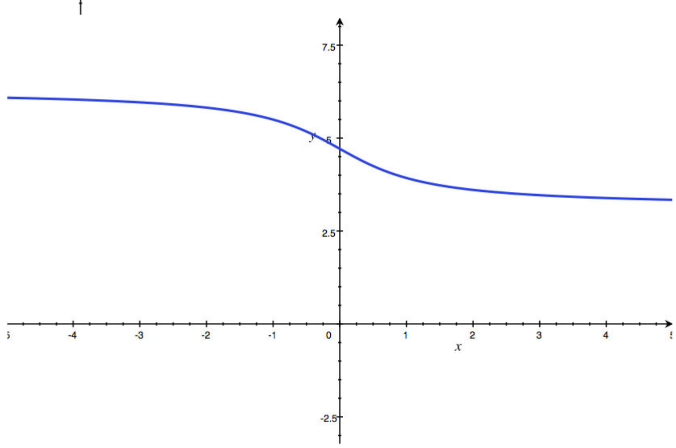
8.


5
y
2.5
-1.5 -1 -0.5 0 0.5 1 1.5
x
-2.5
-5

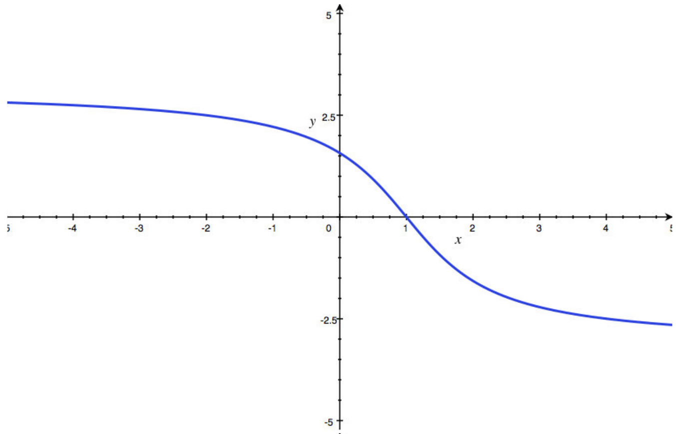
9.

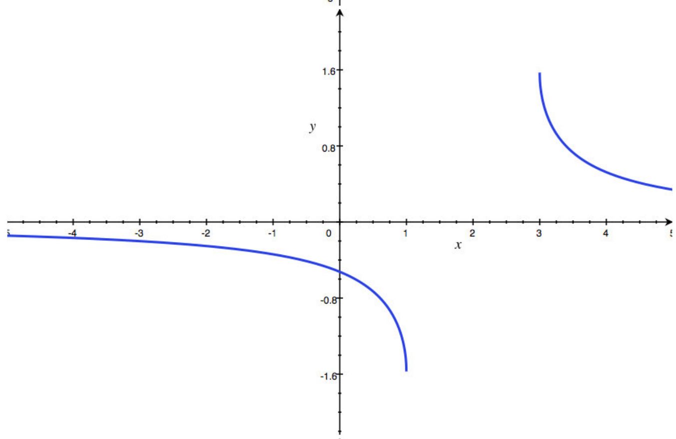

-0.4 y 0 0.4 0.8 1.2 1.6 2 2.4 2.8 3.2 3.6
x
-2.5
-5

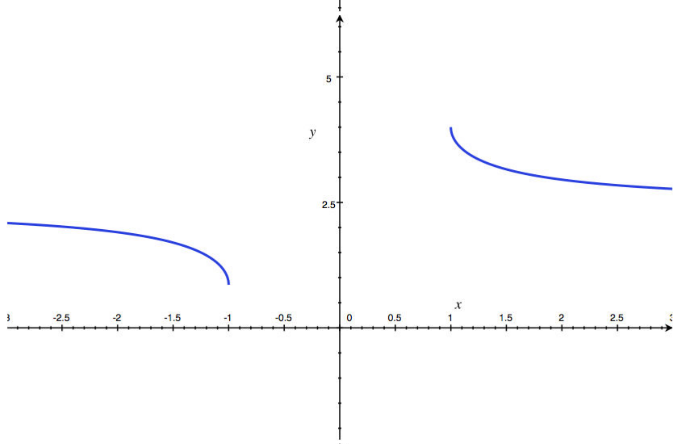
10.

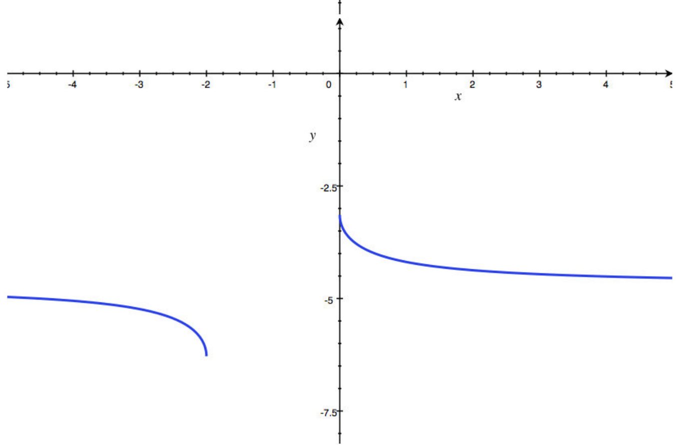

7.5
y 5
2.5
-4 -3 -2 -1 0 1 2 3 4
x
-2.5

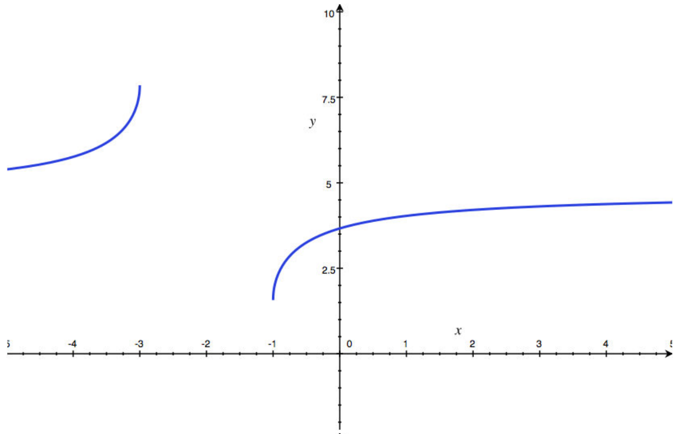
11.

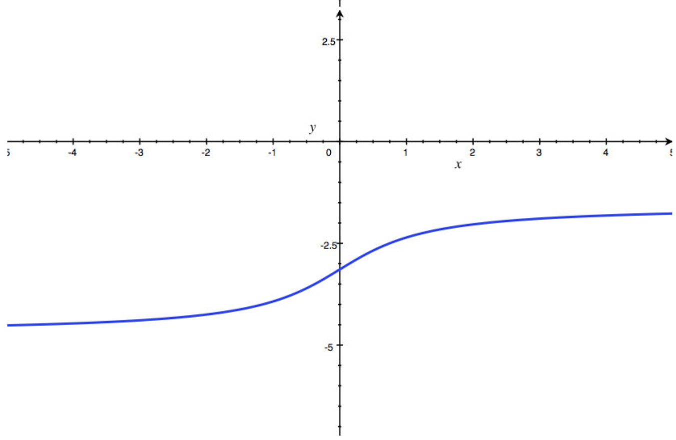
12. 
5
y 2.5
-4 -3 -2 -1 0 1 2 3 4
x
-2.5
-5

13. 
1.6
y
0.8
-4 -3 -2 -1 0 1 2 3 4
x
-0.8
-1.6

14. 
5
y
2.5
x
-2.5 -2 -1.5 -1 -0.5 0 0.5 1 1.5 2 2.5

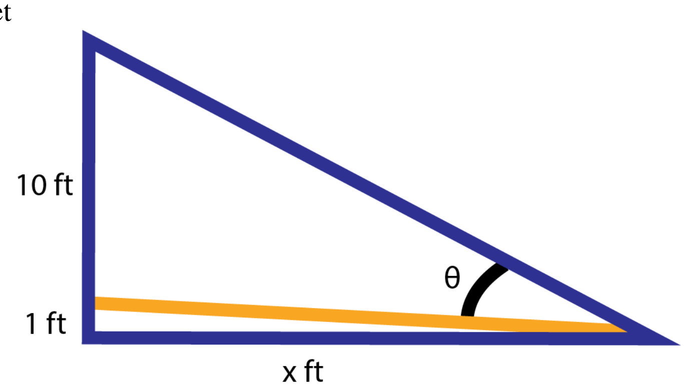
15. 
-4 -3 -2 -1 0 1 2 3 4
x
y
-2.5
-5
-7.5

16. 

17. 

18. $\frac{5}{4}$
19. $\frac{5}{12}$
20. $\frac{5}{4}$

Section 7.8: Applications of Trigonometric Functions

1. Approximately 50.19°
2. Approximately 32.21°
3. Approximately 173.2 feet
4. 

5. $\arctan\left(\frac{11}{x}\right) - \arctan\left(\frac{1}{x}\right)$
6. Approximately 42°
7. $x \approx 17\ ft$
8. Approximately 80.9°
9. Approximately 68.2°
10. $y = 10\sin x + 7$

11. Approximately 13.43 m
12. Approximately 68.2°
13. Approximately 54.2°
14. $y = 6.5 \sin\left(\frac{\pi}{6}x\right)$
15. Approximately -6.28 m or 6.28 m below average sea level

16.8 Answers - Ch 8: Analytic Trigonometry

Section 8.2: Basic Trigonometric Identities

1. $\cot\theta = \frac{adj}{opp} = \frac{\left(\frac{adj}{hyp}\right)}{\left(\frac{opp}{hyp}\right)} = \frac{\cos\theta}{\sin\theta}$
2. Start with the graph of $\cos\theta$. This is the same as the graph of $\cos(\text{-}\theta)$. Then, $\cos\left(\text{-}\left(\theta - \frac{\pi}{2}\right)\right)$ shifts horizontally to the right $\frac{\pi}{2}$, creating the graph of $\sin\theta$.
3. $\sec\theta = \frac{hyp}{adj} = \frac{1}{\left(\frac{adj}{hyp}\right)} = \frac{1}{\cos\theta}$
4. $\tan\theta \cdot \cot\theta = \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} = 1$
5. $\sin\theta \cdot \csc\theta = \sin\theta \cdot \frac{1}{\sin\theta} = 1$
6. $\sin\theta \cdot \sec\theta = \sin\theta \cdot \frac{1}{\cos\theta} = \tan\theta$
7. $\cos\theta \cdot \csc\theta = \cos\theta \cdot \frac{1}{\sin\theta} = \cot\theta$
8. -0.81
9. 0.5
10. 4
11. $\text{-}\frac{1}{0.7} \approx \text{-}1.43$
12. If a function is even, then its graph is symmetric with respect to the y-axis. If a function is odd, then it has 180° rotation symmetry about the origin.
13. $\frac{\tan x \sec x}{\csc x} \cdot \cot x = \frac{\tan x \sin x \cos x}{\cos x \sin x} = \tan x$
14. $\frac{\sin^2 x \sec x}{\tan x} \cdot \csc x = \frac{\sin x \sin x \cos x}{\sin x \sin x \cos x} = 1$
15. $\cos x \cdot \tan x = \cos x \cdot \frac{\sin x}{\cos x} = \sin x$

Section 8.3: Pythagorean Trigonometric Identities

1. $\left(1-\cos^2 x\right)\left(1+\cot^2 x\right) = \sin^2 x \cdot \csc^2 x = \sin^2 x \cdot \frac{1}{\sin^2 x} = 1$
2. $\cos x\left(1-\sin^2 x\right) = \cos x(\cos^2 x) = \cos^3 x$
3. $\sin^2 x = \left(1-\cos^2 x\right) = (1-\cos x)(1+\cos x)$
4. $\frac{\sin^2 x+\cos^2 x}{\csc x} = \frac{1}{\csc x} = \sin x$
5. $\sin^4 x - \cos^4 x = (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) = (\sin^2 x - \cos^2 x) \cdot 1 = \sin^2 x - \cos^2 x$
6. $(\sin^2 x - \sin^4 x)(\cos x) = \sin^2 x(1-\sin^2 x)(\cos x) = \sin^2 x(\cos^2 x)(\cos x) = \sin^2 x(\cos^3 x)$
7. $\sec^3 x$
8. $\sin^2 x$
9. $1-\sin x$
10. $\sin x$
11. $\sin^2 x$
12. $\sec^4 x$
13. $\sec x$
14. $\tan^2 x$
15. $\cos x$

Section 8.4: Sum and Difference Identities

1. $\frac{\sqrt{3}-1}{2\sqrt{2}}$
2. -$\frac{\sqrt{3}-1}{2\sqrt{2}}$
3. -$\frac{1+\sqrt{3}}{2\sqrt{2}}$
4. $\frac{1+\sqrt{3}}{2\sqrt{2}}$
5. -$\sqrt{2}(1+\sqrt{3})$
6. $2+\sqrt{3}$
7. $\sin(\alpha+\beta)=\sin(\alpha-(\text{-}\beta))$
   $=\sin\alpha\cos(\text{-}\beta)-\cos\alpha\sin(\text{-}\beta)$
   $=\sin\alpha\cos\beta-\cos(\alpha)(\text{-}1)(\sin\beta)$
   $=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
8. $\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$
   $=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$
   $=\frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$
   $=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
9. $\tan(\alpha-\beta)=\tan(\alpha+(\text{-}\beta))$
   $=\frac{\tan\alpha+\tan(\text{-}\beta)}{1-\tan\alpha\tan(\text{-}\beta)}$
   $=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$
10. $\frac{1}{2}$
11. $\frac{1}{2}$
12. $\frac{\sqrt{3}}{2}$
13. $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta=2\cos\alpha\cos\beta$
14. $\tan\left(x+\frac{\pi}{4}\right)=\frac{\tan x+\tan\frac{\pi}{4}}{1-\tan x\tan\frac{\pi}{4}}$
    $=\frac{\tan x+1}{1-\tan x}$
    $=\frac{1+\tan x}{1-\tan x}$
15. $\sin(x+\pi)=\sin x\cos\pi+\cos x\sin\pi$
    $=\sin x(\text{-}1)+\cos x(0)$
    $=\text{-}\sin x$

Section 8.5: Double, Half, and Power Reducing Identities

1. $\cos 2x=\cos(x+x)=\cos x\cos x-\sin x\sin x=\cos^2 x-\sin^2 x$
2. $\cos 2x=\cos^2 x-\sin^2 x=(1-\sin^2 x)-\sin^2 x=1-2\sin^2 x$
3. $\cos 2x=\cos^2 x-\sin^2 x=\cos^2 x-(1-\cos^2 x)=\cos^2 x-1+\cos^2 x=2\cos^2 x-1$
4. $\frac{1+\cos 2x}{2}=\frac{1+\cos^2 x-\sin^2 x}{2}=\frac{1+\cos^2 x-(1-\cos^2 x)}{2}=\frac{2\cos^2 x}{2}=\cos^2 x$
5. $\tan^2 x=\frac{\sin^2 x}{\cos^2 x}=\frac{\frac{1-\cos 2x}{2}}{\frac{1+\cos 2x}{2}}=\frac{1-\cos 2x}{1+\cos 2x}$
6. $\tan\frac{x}{2}=\pm\sqrt{\tan^2\frac{x}{2}}=\pm\sqrt{\frac{1-\cos x}{1+\cos x}}$
7. $\csc 2x=\frac{1}{\sin 2x}=\frac{1}{2\sin x\cos x}=\frac{1}{2}\csc x\sec x$

8. $\cot 2x = \frac{\cos 2x}{\sin 2x} = \frac{\cos^2 x - \sin^2 x}{2\sin x \cos x} = \frac{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}}{\frac{2\sin x \cos x}{\sin^2 x}} = \frac{\cot^2 x - 1}{2\cot x}$
9. $\tan \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} \cdot \sqrt{\frac{1-\cos x}{1-\cos x}} = \frac{1-\cos x}{\sqrt{1-\cos^2 x}} = \frac{1-\cos x}{\sin x}$
10. $\tan \frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{1-\cos x}{\sin x} \cdot \frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{\sin x(1+\cos x)} = \frac{\sin^2 x}{\sin x(1+\cos x)} = \frac{\sin x}{1+\cos x}$
11. $\frac{1}{8}(4\cos(2x) + \cos(4x) + 3)$
12. $\sqrt{\frac{2}{1-\frac{1}{\sqrt{2}}}}$
13. $2-\sqrt{3}$
14. $\sqrt{2}-1$
15. $\sqrt{\frac{2}{1+\frac{1}{\sqrt{2}}}}$

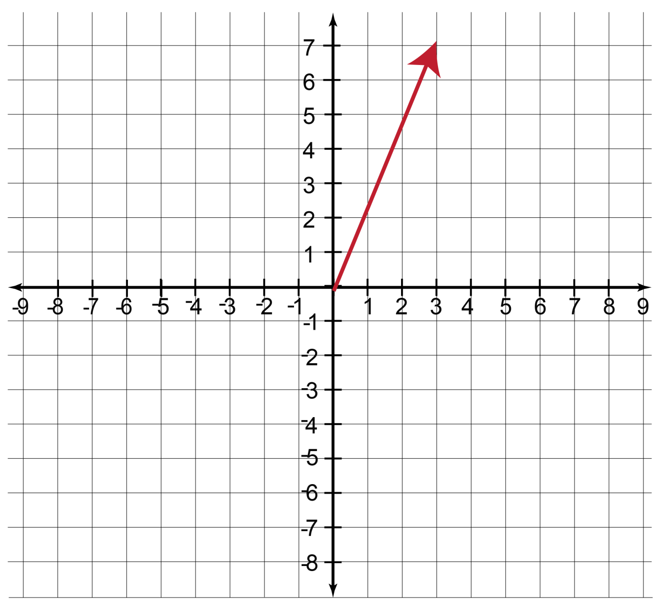
Section 8.6: Trigonometric Equations

1. $x = 0$
2. $x = 0, \frac{\pi}{2}, \frac{3\pi}{2}$
3. $x = 1.786, 4.497$
4. No solution
5. $x = 0.916, 1.98, 4.058, 5.12$
6. No solution
7. Identity
8. $x = 120^\circ$ or 240°
9. $x = 180^\circ$
10. $x = 3\pi$
11. $x = \frac{13\pi}{6}, \frac{17\pi}{6}$
12. $x = 2\pi, 3\pi$
13. $x = \frac{5\pi}{2}$
14. $x = \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}$
15. Identity

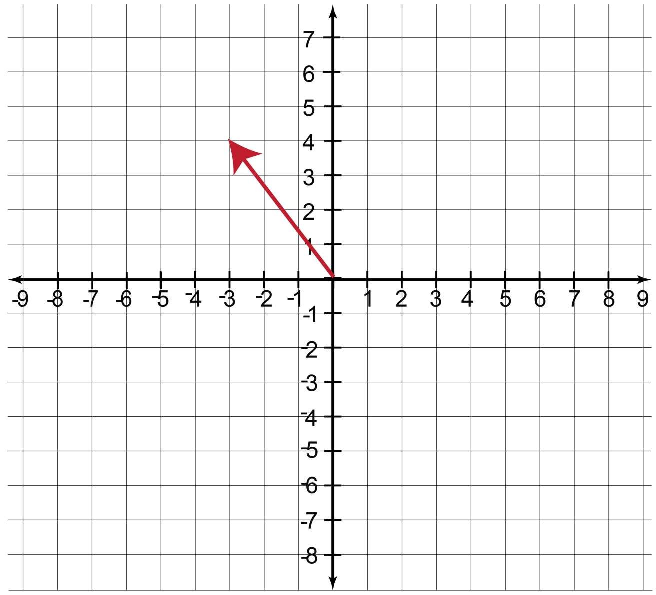
16.9 Answers - Ch 9: Vectors

Section 9.2: Two-Dimensional Vectors

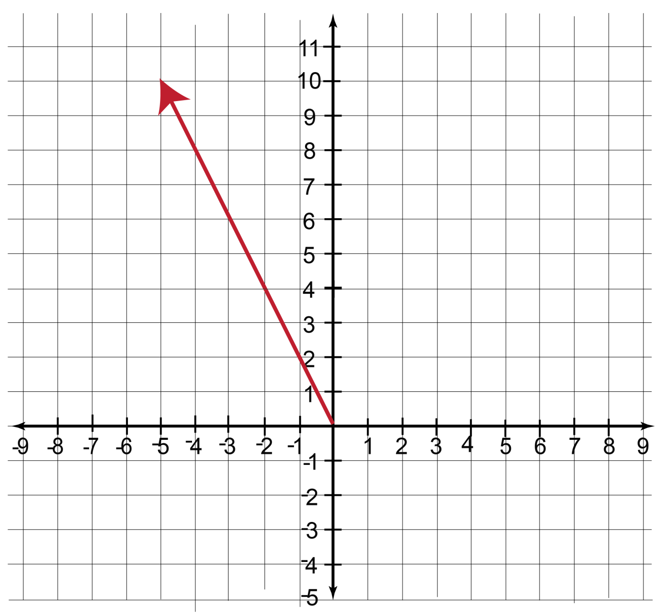
1. Answers vary.
2. <-5, -9>
3. <5, 9>
4. <-2, -17>
5. <-3, 8>
6. <-8, 2>
7. <8, -2>
8. Magnitude: 7.62



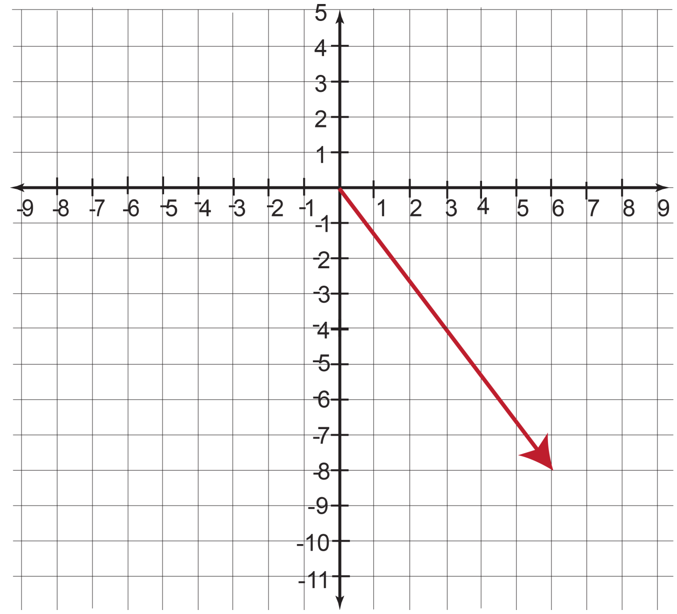
9. Magnitude: 5



10. Magnitude: 11.18



11. Magnitude: 10



12. No
13. No
14. <-4.97, -12.01>
15. NE at 10 knots
16. Pick a direction for the boat's movement (along one axis), then set the direction of the stream along the other. If, for example, you set the boat to move in the positive *x* direction and the stream to move in the negative *y* direction, then the vector would be <35, -25>.

Section 9.3: Operations with Vectors

1. $< 9,7 >$
2. $< -10,10 >$
3. $< 4,22 >$
4. $< 15,-15 >$

5. 11.4

6. 14.14

7. 22.36

8. 21.21
9. 53.13°
10. 116.57°
11. 306.87°
12. 17 miles
13. The bird is actually moving in a direction of 28° south of east (62° east of south).
14. 50.6 miles per hour
15. 9° east of north

Section 9.4: Unit Vectors

1. $y = 18$
2. $< \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} >$
3. $< \frac{9}{\sqrt{82}}, \text{-}\frac{1}{\sqrt{82}} >$
4. $< \frac{1}{\sqrt{10}}, \text{-}\frac{3}{\sqrt{10}} >$
5. $< \frac{1}{\sqrt{82}}, \frac{9}{\sqrt{82}} >$
6. (-0.32, 0.66)
7. (-0.37, 0.68)
8. (1.47, -0.24)
9. $< 37.1, 29.8 >$
10. $< 200, 200\sqrt{3} >$
11. $< 10.26, 28.19 >$
12. $< \text{-}153.9, 422.86 >$
13. $< \text{-}423.5, \text{-}79.9 >$
14. 374.2 mph at 263.14°
15. 16.9 mph north and 36.25 mph east

Section 9.5: Dot Products

1. 24
2. 16
3. -14
4. 21
5. 34
6. 49.4°
7. 56.3°
8. 171.87°
9. 8.1°
10. 26.6°
11. ${v_1}^2 + {v_2}^2$
12. The magnitude of a vector is the square root of the dot product of the vector with itself.
13. 0
14. $(cu) \cdot v = < cu_1 > \cdot < v_1, v_2 >$
    $= < cu_1v_1 + cu_2v_2 >$
    $= < u_1 \cdot cv_1 + u_2 \cdot cv_2 >$
    $= < u_1, u_2 > \cdot$
    $= u \cdot (cv)$
15. To prove that these two vectors are perpendicular to each other, show that their dot products are equal to zero.
16. 53.46 J
17. 142.6 J
18. 246.2 J

Section 9.6: Scalar and Vector Projections

1. $< 6, 114 >$
2. $<$ -56, -64 $>$
3. $<$ -105, 45 $>$
4. $<$ 99, -220 $>$
5. $< 144, 160 >$
6. $proj_A \to B = < \frac{27}{17}, \frac{45}{17} >$
7. $proj \to = <\frac{16}{5}, \frac{48}{5}>$
8. $proj_c \to D = <\frac{23}{122}, \frac{253}{122}>$
9. $proj_E \to F = <-\frac{59}{325}, \frac{1062}{325}>$
10. $proj \ \to = <-\frac{80}{61}, \frac{96}{61}>$
11. $proj_H \to I = <-\frac{298}{229}, \frac{2235}{229}>$
12. $proj \ \to = <\frac{11}{5}, \frac{22}{5}>$
13. $proj \to = <\frac{399}{113}, \frac{456}{113}>$
14. $proj_J \to K = <-\frac{224}{41}, \frac{280}{41}>$
15. $proj_L \to M = <\frac{161}{338}, \frac{391}{338}>$
16. 34.64
17. 62.5
18. 1,719.35
19. 1,871.13
20. 29.34

Section 9.7: Vector Equation of a Line

1. $\vec{r} = <2, -2, 5> +k <-1, 7, -1>$
2. $\vec{r} = <2, -9, 5> +k <6, 13, -11>$
3. $\vec{r} = <15, 3, -3> +k <-11, -6, 12>$
4. $\vec{r} = <-1, -1, 7> +k <4, 12, 1>$
5. $\vec{r} = <1, -3, 2> +k <-6, 6, -3>$
6. $\vec{r} = <25, 17, 42> +k <41, 5, 19>$
7. Skew
8. Skew
9. Skew
10. Skew
11. (1.5, 1.5, 4.5)
12. (5, -2.5, -0.5)
13. (9.5, 0. 3)
14. (1, 5, 7.5)
15. $t = 3: <9, 16, -8>; t = 5: <13, 30, -14>$
16. $t = 3: <28, -25, 1>; t = 5: <46, -39, 7>$
17. $t = 3: <8, -1, -9>; t = 5: <14, -3, -13>$

16.10 Answers - Ch 10: Systems and Matrices

Section 10.2: Systems of Two Equations and Two Unknowns

1. (-7, 3)
2. (3, 8)
3. $\left(\frac{143}{18}, \frac{-28}{9}\right)$
4. (6, 10)
5. (3, 9)
6. (8, 2)
7. (10, -10)
8. (8, -5)
9. (0, 5)
10. $\left(\frac{33}{19}, \frac{3}{19}\right)$
11. A system of equations has no solution if, when you are solving, you end up with a false statement like 0 = 2.
12. Answers will vary. Possible answers: If a system of equations has no solution, it means that the graphs do not intersect. For a system of two variables, the lines would be parallel.
13. A system where the two equations are equivalent (one is a multiple of the other) will produce an infinite number of solutions.
14. (2, 9)
15. (7, 1)
16. 99 and 51
17. $68,750 in Company A, and $31,250 in Company B
18. Clownfish cost $1.95, and goldfish cost $1.25.
19. 26 and 9
20. 5 cappuccinos and 4 lattes

Section 10.3: Solving Linear Systems in Three Variables

1. Yes
2. No
3. Yes
4. Yes
5. (2, 7, 8)
6. (-4, 1, 6)
7. (5, 9, -2)
8. (2, 1, 2)
9. (-$\frac{8}{3}$, 0, 0)
10. (-7, 5, 4)
11. (1, -4, 0)
12. No solution
13. (-1, 0, -1)
14. (4, -2, 1)
15. $y = 3 - x$; $z = 0$

16.

$$A+O+B=9$$
$$2A+2O=10$$
$$2O+B=10$$

Apples cost \$2; onions cost \$3; a basket of blueberries costs \$4.

17. 735
18. \$24,000 in the savings account; \$11,000 in the time deposit; \$7,000 in the bond
19. 7 nickels, 6 dimes, and 12 quarters
20. 126 adult, 92 children, and 42 student tickets

Section 10.4: Matrices to Represent Data

1. 2 x 4
2. 2 x 2
3. 4 x 2
4. 4 x 3
5. 1 x 2
6. Answers will vary.
7. Answers will vary.
8. A diagonal matrix
9. $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$
10. The rows represent the days of the week, and the columns represent the weeks of working.
11. \$124
12. \$83
13. Fridays
14. Thursdays
15. No. The entries of matrices must be numbers, not words.

Section 10.5: Matrix Algebra

1. $\begin{bmatrix} 35 & 26 \\ 50 & 34 \end{bmatrix}$
2. Not possible—you can't multiply a 2 x 3 matrix by a 2 x 2 matrix.
3. $\begin{bmatrix} 46 & 146 \\ 8 & 23 \end{bmatrix}$
4. $\begin{bmatrix} 0 & 12 \\ 20 & 12 \\ 4 & 24 \end{bmatrix}$
5. $\begin{bmatrix} 16 & 13 \\ 4 & 10 \end{bmatrix}$
6. $\begin{bmatrix} 3 & -7 \\ -2 & -6 \end{bmatrix}$

7. $\begin{bmatrix} 22 & 26 \\ 6 & 16 \end{bmatrix}$
8. $\begin{bmatrix} 39 & 132 & 94 \\ 30 & 60 & 64 \end{bmatrix}$
9. Not possible—you can't multiply a 2 x 3 matrix by a 2 x 2 matrix.
10. They both equal $\begin{bmatrix} 52 & 40 \\ 73 & 50 \end{bmatrix}$.
11. They both equal $\begin{bmatrix} 18 & 12 \\ 27 & 18 \end{bmatrix}$.
12. $\begin{bmatrix} 356 & 866 & 122 \\ 528 & 653 & 285 \\ 939 & 132 & 205 \end{bmatrix}$
13. $\begin{bmatrix} 624 & 118 & 68 \\ 684 & 312 & 378 \\ 1,566 & 46 & 266 \end{bmatrix}$
14. $\begin{bmatrix} 132 & 288 & 84 \\ 372 & 164 & 376 \\ 248 & 300 & 288 \end{bmatrix}$
15. $\begin{bmatrix} 21,148 & 315,181 & 23,377 \\ 49,708 & 419,179 & 29,539 \\ 37,679 & 672,733 & 56,811 \end{bmatrix}$
16. Answers will vary.
17. Friday: $1,019.25; Saturday: $1,295.25; Sunday: $857.75. Total: $3,172.25

Section 10.6: Row Operations and Row Echelon Forms

1. Answers will vary.
2. Answers will vary.
3. Add a multiple of one row to another row; scale a row by multiplying through by a non-zero constant; swap two rows.
4. The rows are linearly independent.
5. Answers will vary.
6. The rows are linearly independent because reduced row echelon form is the identity matrix.
7. Answers will vary.
8. The rows are linearly independent because reduced row echelon form is the identity matrix.
9. Answers will vary.
10. Reduced row echelon form is $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Because there are no zero-only rows, the rows of the original matrix were linearly independent.
11. Answers will vary.
12. Reduced row echelon form is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. The rows of matrix D were not linearly independent.
13. Answers will vary.
14. Reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. The rows of matrix E were not linearly independent.
15. Answers will vary.

16. The rows are linearly independent because reduced row echelon form is the identity matrix.

Section 10.7: Augmented Matrices

1. (-3, 4)
2. (1, 6)
3. Infinite number of solutions
4. (3, 2)
5. No solution. The lines are parallel and do not intersect.
6. (1, 4, 6)
7. (-3, 1, 5)
8. (2, 5, 3)
9. Infinite number of solutions
10. (3, 2, 8)
11. (5, -1, 2)
12. No solution. If you multiply R_1 by 2 and add it to R_3, you end up with 0 = 13. Therefore, no solution exists.
13. (-4, 5, 3)
14. (1, 6, 8)
15. (-3, 2, 5)

Section 10.8: Determinants

1. 2
2. -27
3. -4
4. 1
5. -22
6. -9
7. -89
8. 294
9. 8
10. -186
11. -56
12. 88
13. 124
14. 176
15. Only square matrices have determinants.
16. If the determinant is zero, then the rows are not linearly independent.
17. No, they are not collinear. Area = 21.5
18. No, they are not collinear. Area = 112.5
19. No, they are not collinear. Area = 35

Section 10.9: Cramer's Rule

1. (-3, 4)
2. (1, 6)
3. There is not one solution because the determinant of the coefficient system is 0. The rows of the coefficient matrix are not linearly independent.
4. (3, 2)
5. There is not one solution because the determinant of the coefficient system is 0. The rows of the coefficient matrix are not linearly independent.
6. $x = 1$
7. $y = 1$
8. $z = 3$
9. $x = 2$
10. $y = 2$
11. $z = 2$
12. There is not one solution because the determinant of the coefficient system is 0. The rows of the coefficient matrix are not linearly independent.
13. (-4, 5, 3)
14. (1, 6, 8)
15. (-3, 2, 5)
16. Look at the other relevant determinants for Cramer's Rule. If they are also zero, then the system has infinite solutions. If they are non-zero, then the system has no solution.
17. Paperbacks: \$8; Hardcover books: \$15
18. Corndogs: \$2.75; Cotton candies: \$1.75

Section 10.10: Inverse Matrices

1. $$\begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}$$
2. $\begin{bmatrix} -\frac{5}{27} & \frac{2}{9} \\ \frac{2}{27} & \frac{1}{9} \end{bmatrix}$
3. $\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$
4. $\begin{bmatrix} 1 & -6 \\ 0 & 1 \end{bmatrix}$
5. $\begin{bmatrix} \frac{1}{11} & \frac{5}{22} \\ \frac{2}{22} & -\frac{3}{11} \end{bmatrix}$
6. No inverse
7. $\begin{bmatrix} -\frac{8}{41} & \frac{7}{41} & \frac{5}{41} \\ \frac{19}{41} & \frac{9}{41} & -\frac{17}{41} \\ -\frac{6}{41} & -\frac{5}{41} & \frac{14}{41} \end{bmatrix}$
8. $\begin{bmatrix} -\frac{3}{294} & \frac{34}{294} & \frac{5}{294} \\ \frac{18}{294} & -\frac{8}{294} & \frac{68}{294} \\ \frac{27}{294} & -\frac{12}{294} & -\frac{45}{294} \end{bmatrix}$
9. $\begin{bmatrix} -1 & -1 & \frac{1}{2} \\ \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{17}{4} & \frac{21}{4} & -\frac{7}{4} \end{bmatrix}$

10. $\begin{bmatrix} \frac{40}{186} & \frac{24}{186} & -\frac{22}{186} \\ -\frac{5}{186} & -\frac{3}{186} & \frac{26}{186} \\ -\frac{12}{186} & \frac{30}{186} & -\frac{12}{186} \end{bmatrix}$
11. $\begin{bmatrix} \frac{14}{56} & \frac{12}{56} & \frac{58}{56} \\ 0 & -\frac{16}{56} & -\frac{8}{56} \\ -\frac{7}{56} & \frac{6}{56} & -\frac{4}{56} \end{bmatrix}$
12. $\begin{bmatrix} \frac{1}{22} & 0 & -\frac{3}{22} \\ -\frac{1}{44} & \frac{1}{4} & \frac{3}{44} \\ \frac{9}{22} & -\frac{1}{2} & -\frac{5}{22} \end{bmatrix}$
13. Students should show that the matrix times itself equals the identity matrix.
14. Students should show that the matrix times itself equals the identity matrix.
15. A non-square matrix cannot be multiplied on both sides by the same matrix because the order of the matrices would not work. Therefore, for a non-square matrix there cannot exist just one inverse matrix.

Section 10.11: Partial Fraction Decomposition

1. $-\frac{\frac{1}{5}}{x-1} + \frac{\frac{16}{5}}{x+4}$
2. $-\frac{\frac{7}{9}}{x} - \frac{\frac{1}{3}}{x^2} + \frac{\frac{7}{9}}{x-3}$
3. $-\frac{\frac{1}{5}}{x} + \frac{\frac{6}{5}}{x-5}$
4. $\frac{-\frac{1}{18}}{x} + \frac{\frac{19}{54}}{x+6} + \frac{\frac{19}{27}}{x-3}$
5. $-\frac{\frac{1}{2}}{x^2} + \frac{\frac{7}{4}}{x+2} + \frac{\frac{5}{4}}{x}$
6. $-\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1}$
7. $\frac{\frac{9}{4}}{x^2} + \frac{\frac{9}{16}}{x} + \frac{\frac{55}{16}}{x-4}$
8. $\frac{\frac{9}{5}}{x+7} + \frac{\frac{1}{5}}{x-3}$
9. $\frac{4-3x}{x^2+1} - \frac{4}{x^2} + \frac{3}{x}$
10. $\frac{\frac{-11x-7}{13}}{x^2+4} + \frac{\frac{11}{13}}{x-3}$
11. $\frac{\frac{-x-13}{5}}{x^2+1} + \frac{\frac{5}{3}}{x^2} + \frac{\frac{14}{45}}{x-3} - \frac{\frac{1}{9}}{x}$
12. Students should verify that the partial fractions sum to the original function.
13. Students should verify that the partial fractions sum to the original function.
14. Students should verify that the partial fractions sum to the original function.
15. Students should verify that the partial fractions sum to the original function.

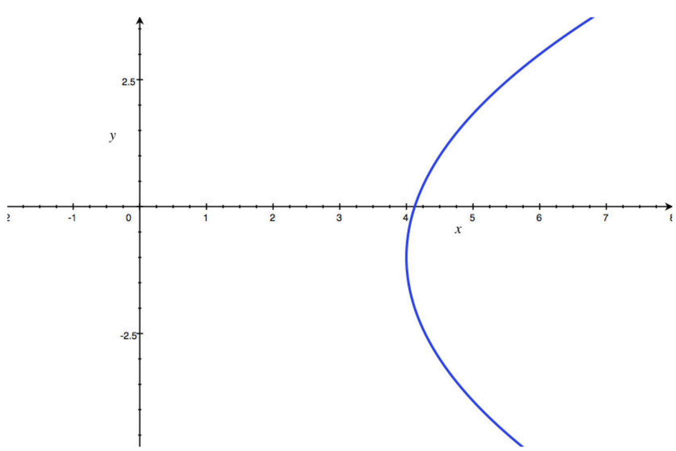
16.11 Answers - Ch 11: Conics

Section 11.2: General Form of a Conic

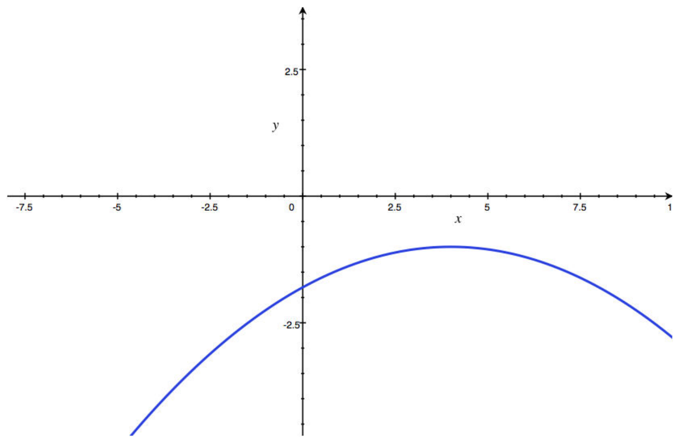
1. Ellipse
2. Circle
3. Ellipse
4. Sideways parabola
5. Hyperbola
6. Parabola
7. $(x+2)^2-4$
8. $(y-4)^2-16$
9. $3(x+1)^2+1$
10. $3(y+\frac{3}{2})^2+\frac{33}{4}$
11. $2(x-3)^2-17$
12. $4(x-2)^2+(y+1)^2=16$
13. $9(x-3)^2+(y-1)^2=1$
14. $3(x-1)^2+4y^2=12$
15. $y=(x+2)^2-3$

Section 11.3: Parabolas

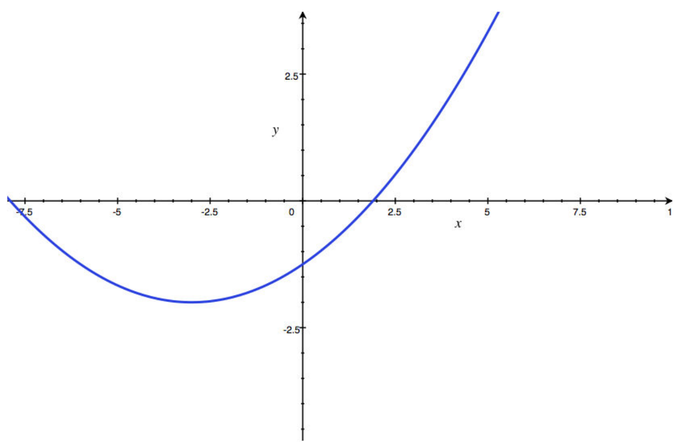
1. $(x-1)^2=4\cdot 3(y-1)$
2. $(y+1)^2=\text{-}4\cdot 3(x-1)$
3. $(y-4)^2=4\cdot 3(x-5)$
4. $(x-1)^2=\text{-}4\cdot 4(y-8)$
5. $(y-4)^2=\text{-}4\cdot 3(x-1)$
6. $(x-6)^2=\text{-}4\cdot 5(y-4)$
7. $(x-1)^2=4(y-11)$
8. $(y+1)^2=4\cdot 2(x-4)$



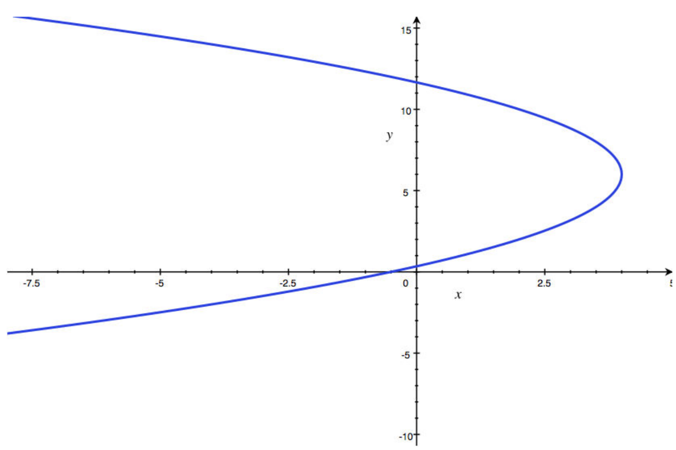
9. $(x-4)^2 = -4 \cdot 5(y+1)$



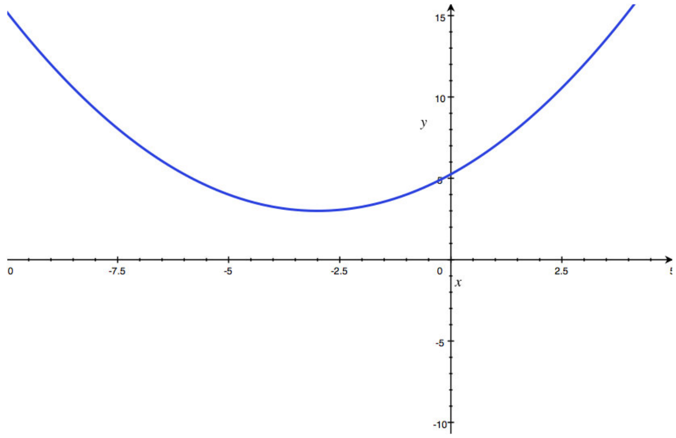
10. $(x+3)^2 = 4 \cdot 3(y+2)$



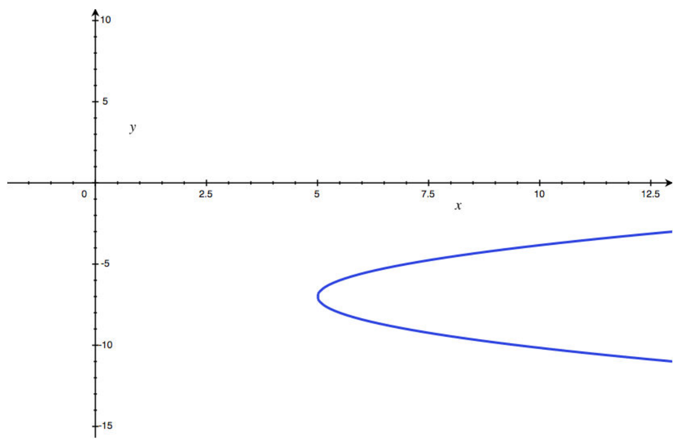
11. $(y-6)^2 = -4 \cdot 2(x-4)$



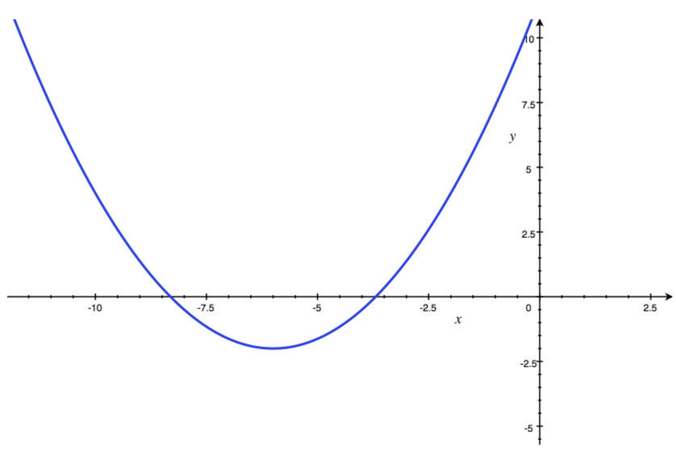
12. $(x+3)^2 = 4(y-3)$



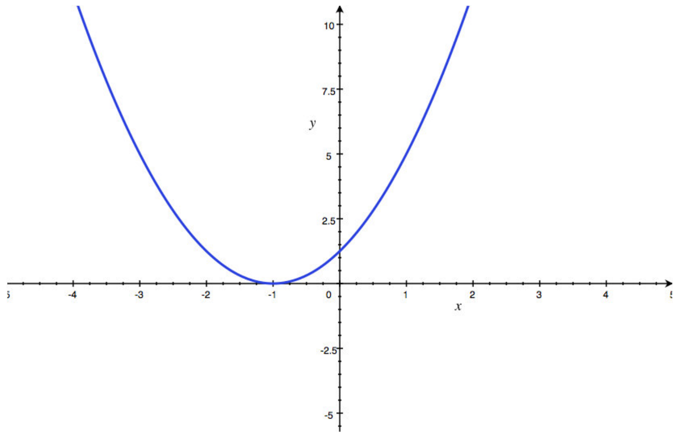
13. $(y+7)^2 = 4 \cdot \frac{1}{2}(x-5)$



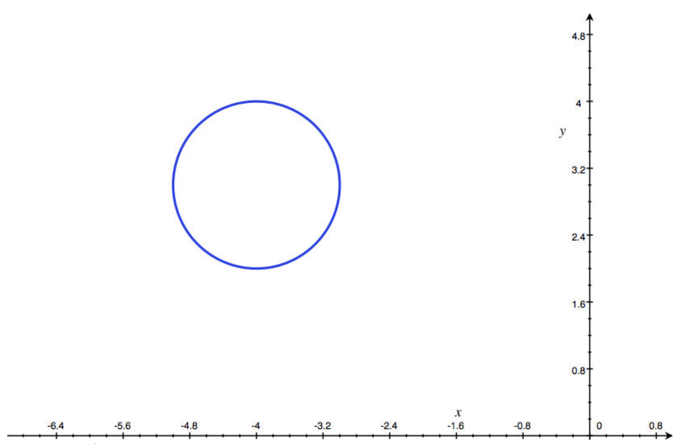
14. $(x+6)^2 = 4 \cdot \frac{2}{3}(y+2)$

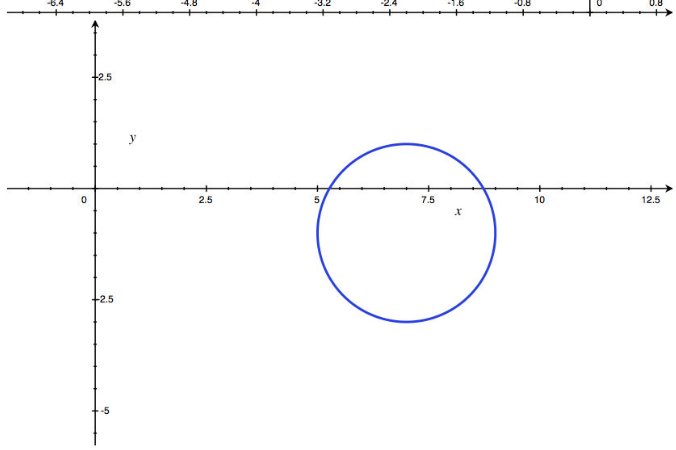


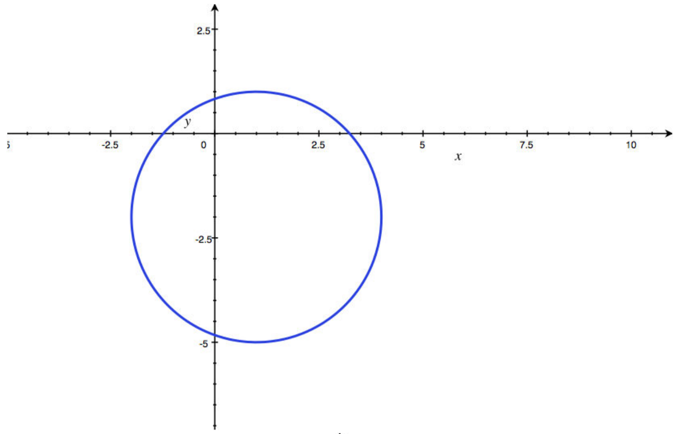
15. $(x-1)^2 = 4 \cdot \frac{1}{5}(y)$



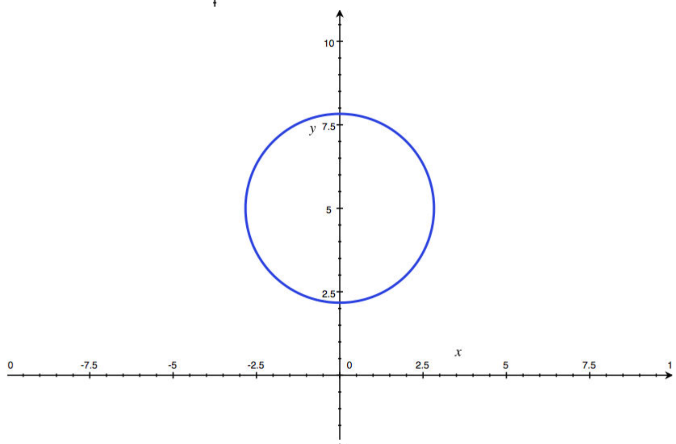
Section 11.4: Circles

1. 

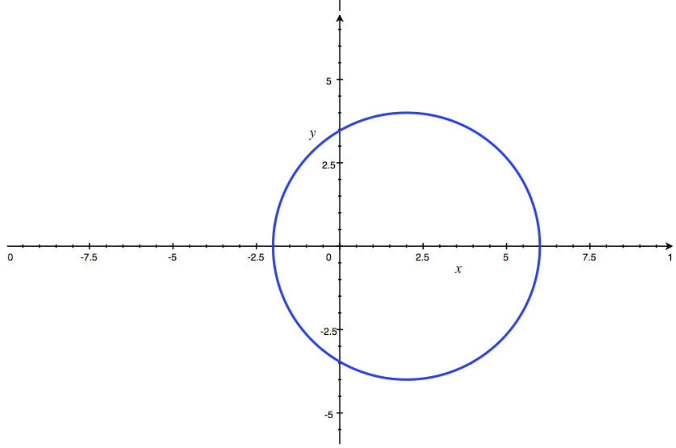
2. 



3.



4.



5.

6. $(x-2)^2+(y+5)^2=11$
7. $(x+1)^2+(y-4)^2=16$
8. $(x-3)^2+(y-2)^2=1$
9. $(x+1)^2+(y+7)^2=25$
10. $(x-1)^2+(y-1)^2=2$
11. $(x+2)^2+(y-1)^2=9$
12. $(x+7)^2+(y-1)^2=16$
13. $x^2+(y+5)^2=25$
14. $x^2+y^2=4$
15. $(x+2)^2+y^2=1$

Section 11.5: Ellipses

1. Vertices: (1, -1) and (1, -9)

   Foci: (1, -5$+\sqrt{12}$), (1, -5$-\sqrt{12}$)

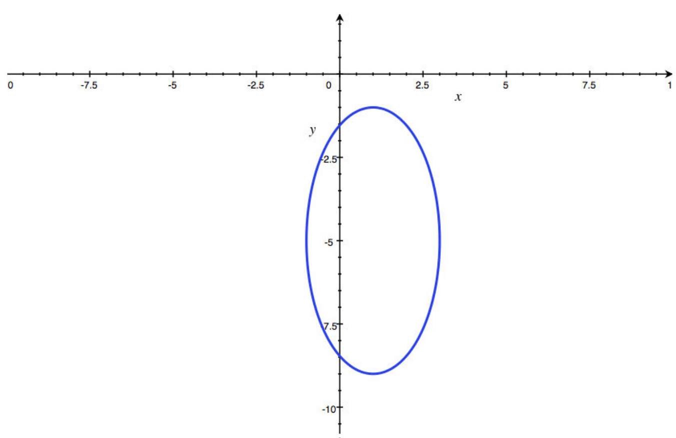
   Eccentricity: $\frac{\sqrt{12}}{4}$
2. Vertices: (2, -2) and (-4, 2)

   Foci: (-1$+\sqrt{5}$, -2), (-1$-\sqrt{5}$, -2)

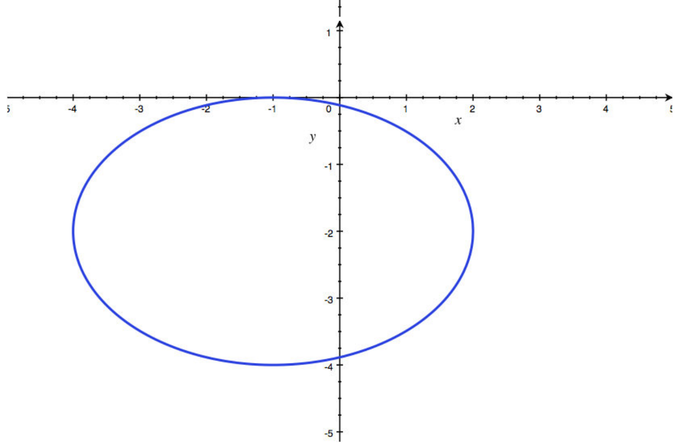
   Eccentricity: $\frac{\sqrt{5}}{3}$
3. Vertices: (2, 3) and (2, -1)

   Foci: (2, 1$+\sqrt{3}$), (2, 1$-\sqrt{3}$)

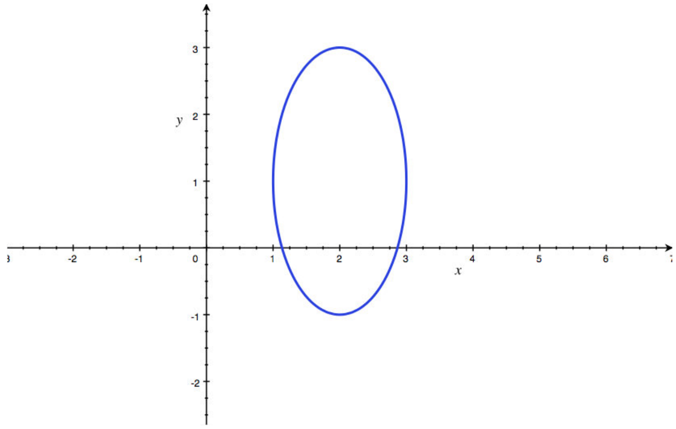
   Eccentricity: $\frac{\sqrt{3}}{2}$



4.

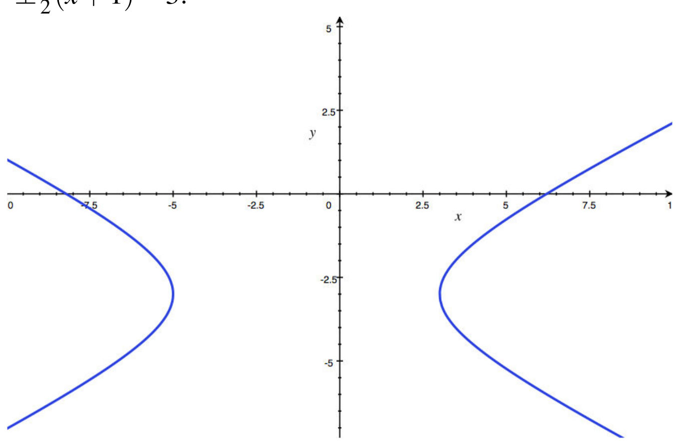


5.

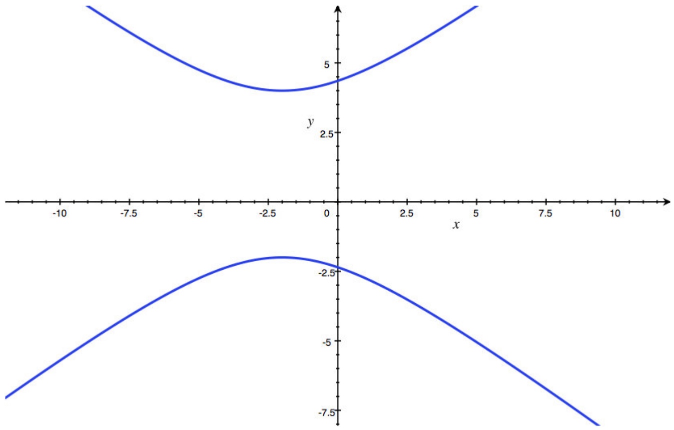
6. 

7. $\frac{(x+1)^2}{16}+\frac{(y+7)^2}{4}=1$
8. $\frac{(x-4)^2}{9}+(y+1)^2=1$
9. $\frac{(x-2)^2}{4}+\frac{(y+2)^2}{9}=1$
10. $\frac{(x-4)^2}{9}+\frac{(y-3)^2}{25}=1$
11. $\frac{(x-4)^2}{36}+\frac{(y-1)^2}{4}=1$
12. $\frac{(x+2)^2}{25}+\frac{(y-4)^2}{9}=1$
13. $\frac{(x-6)^2}{25}+\frac{(y+2)^2}{169}=1$
14. Possible answer: $\frac{x^2}{50^2}+\frac{y^2}{25^2}=1$
15. 17.86 feet

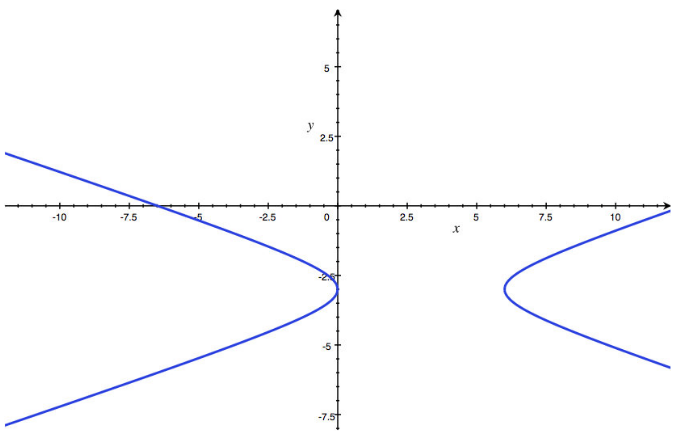
Section 11.6: Hyperbolas

1. $\frac{(x+1)^2}{16}-\frac{(y+3)^2}{4}=1$. This is a hyperbola because the coefficients of x^2 and y^2 have opposite signs.
2. This hyperbola opens side to side.
3. The vertices are at (-5, -3) and (-3, -3).
4. The asymptotes are at $y=\pm\frac{1}{2}(x+1)-3$.
5. 

6. $-\frac{(x+2)^2}{16}+\frac{(y-1)^2}{9}=1$. This is a hyperbola because the coefficients of x^2 and y^2 have opposite signs.
7. Opens up and down.
8. (-2, 4) and (-2, -2)
9. $y=\pm\frac{3}{4}(x+2)+1$

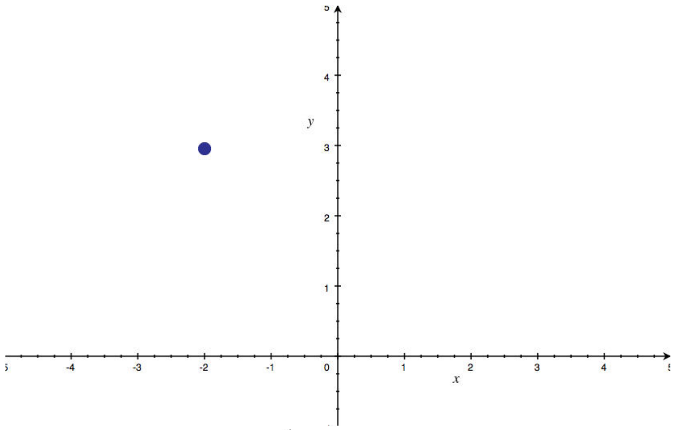
10. 

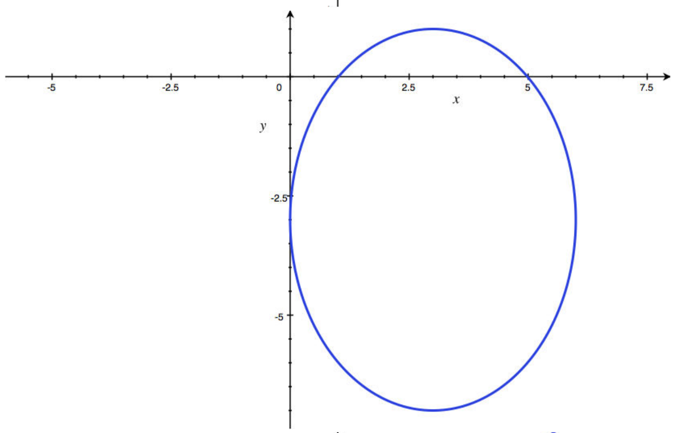
11. $\frac{(x-3)^2}{9} - \frac{(y+3)^2}{1} = 1$. This is a hyperbola because the coefficients of x^2 and y^2 have opposite signs.
12. Opens side to side
13. Vertices are at (6, -3) and (0, -3).
14. $y = \pm\frac{1}{3}(x-3) - 3$

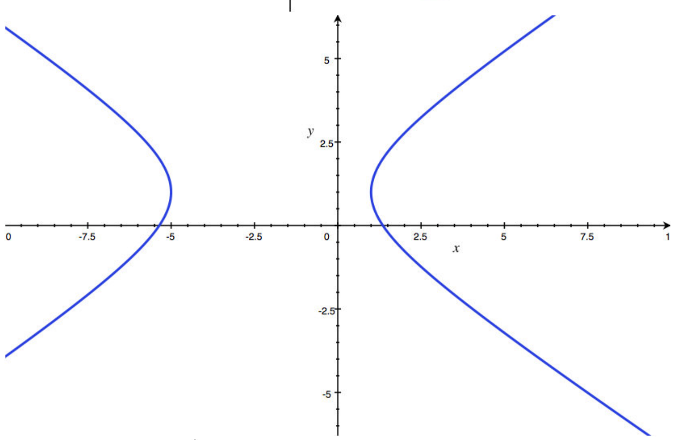
15. 

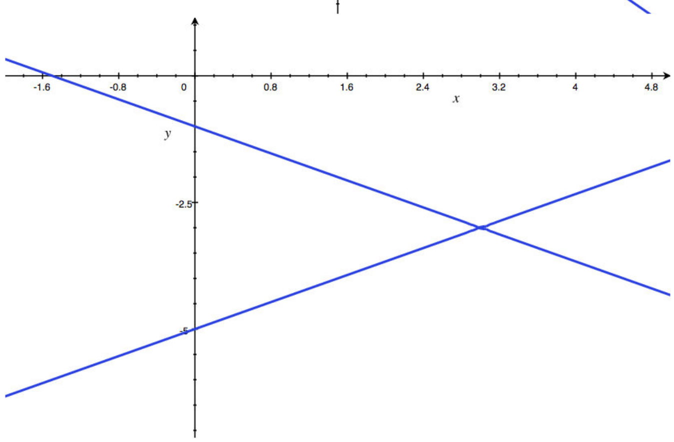
Section 11.7: Degenerate Conics

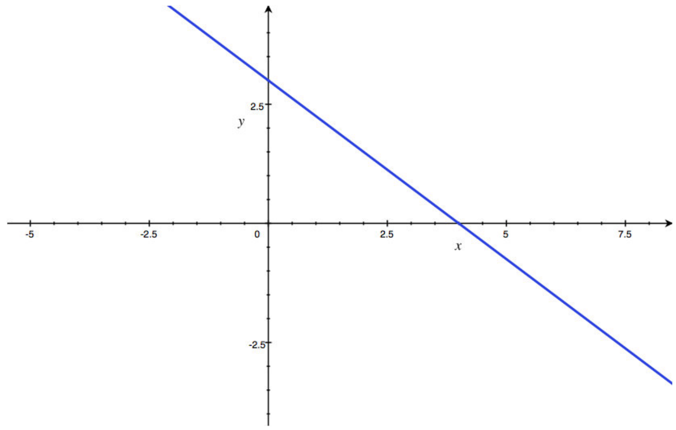
1. The degenerate form of a hyperbola is two intersecting lines, of a circle is a point, and of a parabola is a line or two parallel lines.
2. $\frac{(x-3)^2}{9} - \frac{(y+3)^2}{1} = 0$; degenerate hyperbola
3. $\frac{(x+2)^2}{9} - \frac{(y-1)^2}{4} = 1$; hyperbola
4. $\frac{(x+2)^2}{4} + \frac{(y-3)^2}{9} = 0$; point
5. $\frac{(x+2)^2}{4} + \frac{(y-3)^2}{9} = 1$; ellipse
6. $\frac{5}{2}x + \frac{1}{2} = y$; line
7. $y = (x+2)^2 + 4$; parabola
8. $(x-1)^2 + (y-3)^2 = 4$; circle
9. $\frac{(x-1)^2}{4} - (y-3)^2 = 0$; degenerate hyperbola
10. $\frac{(x-1)^2}{4} + (y-3)^2 = 1$; ellipse

11. 
y
x
-4
-3
-2
-1
0
1
2
3
4
1
2
3
4

12. 
-5
-2.5
0
2.5
5
7.5
x
y
-2.5
-5

13. 
5
y
2.5
-7.5
-5
-2.5
0
2.5
5
7.5
x
-2.5
-5

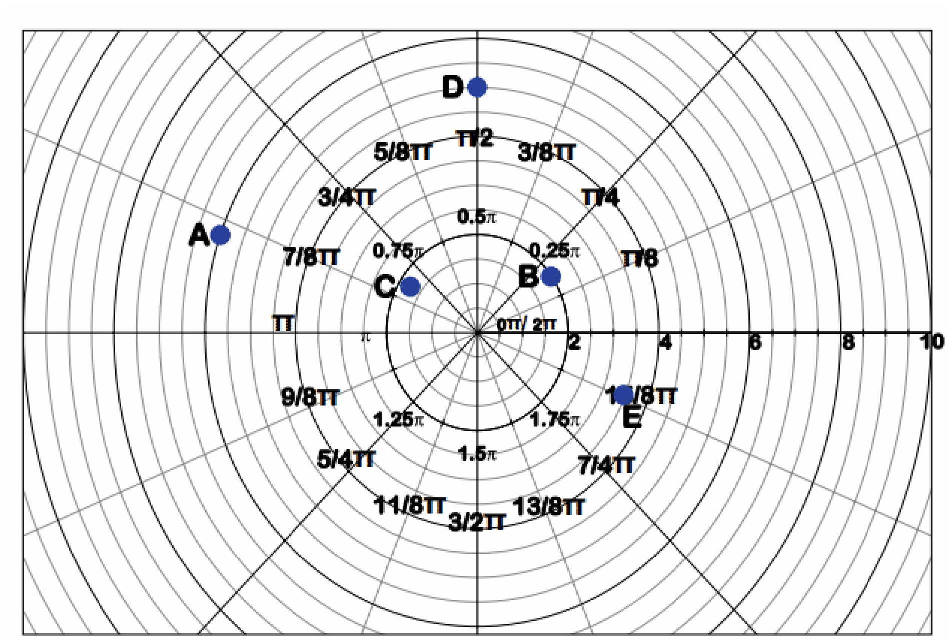
14. 
-1.6
-0.8
0
0.8
1.6
2.4
3.2
4
4.8
x
y
-2.5
-5

15. 

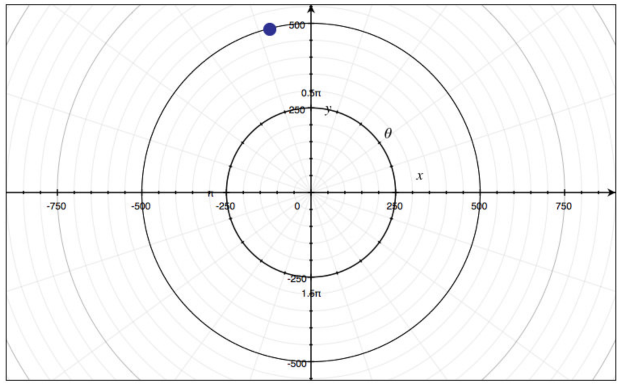
16.12 Answers - Ch 12: Polar Coordinates and Parametric Equations

Section 12.2: Polar Coordinate System

1. Because a given point may have multiple descriptions
2. If $r < 0$, you extend to the left to measure the distance. If $\theta > 360$, you circle around and continue.
3. See point A below.



4. See point B above.
5. See point C above.
6. See point D above.
7. See point E above.
8. (-1.5, -190°) and (1.5, 0.945π)
9. $\left(5, \frac{5\pi}{3}\right)$ and (-5, -60°)
10. (-3, -55) and (3, 1.7π)
11. (4, -150°) and (-4, $\frac{5\pi}{6}$)
12. (500, 105°)

13. 

14. a. 4.189 x 10^7

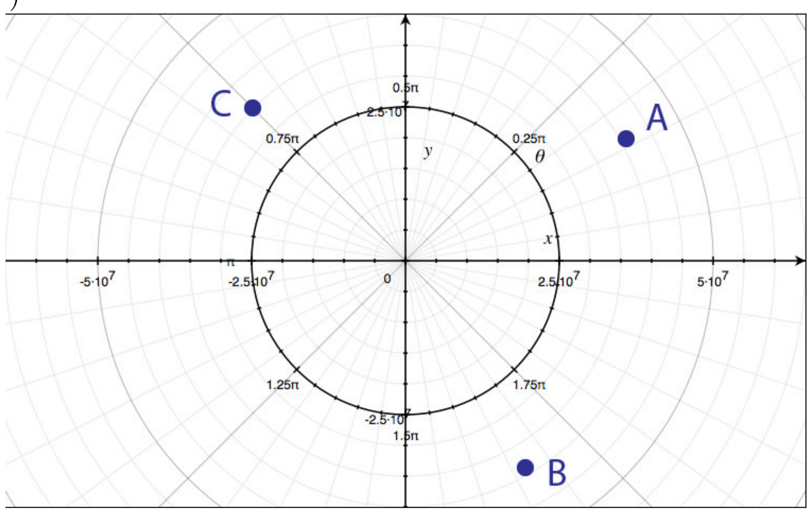
    b. 3.837 x 10^7

    c. 3.004 x 10^7

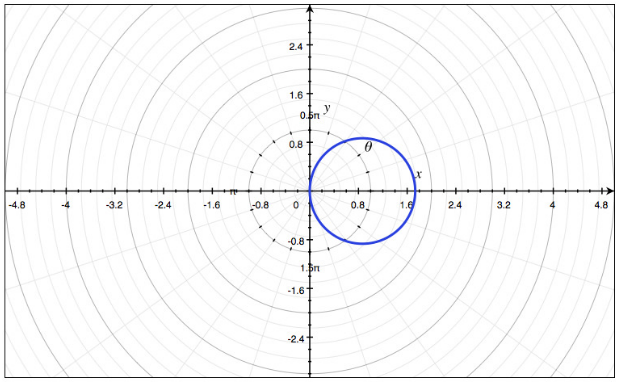
15. a. (4.189 x 10^7, 30°)

    b. (3.837 x 10^7, -60°)

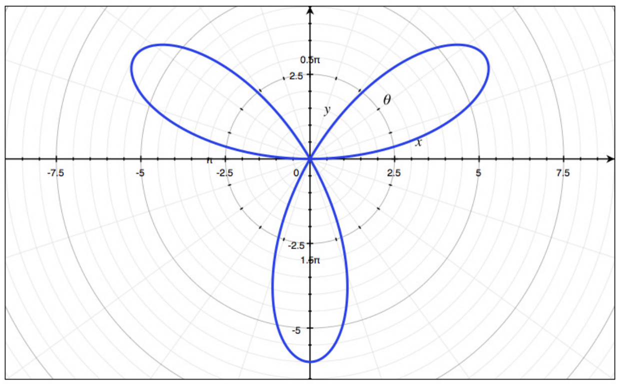
    c. (3.004 x 10^7, 135°)

16. 

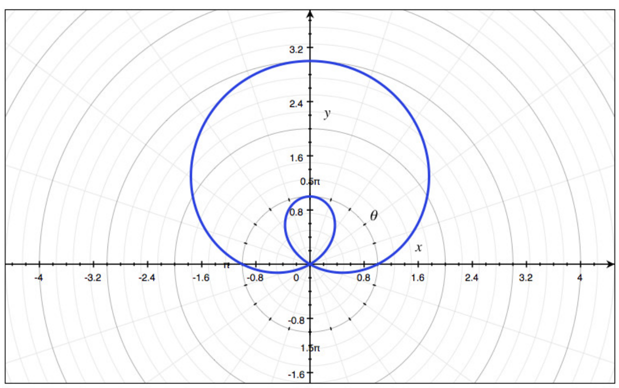
Section 12.3: Polar Equations



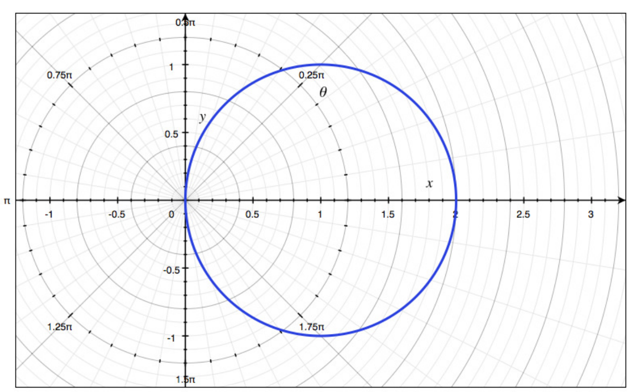
1.



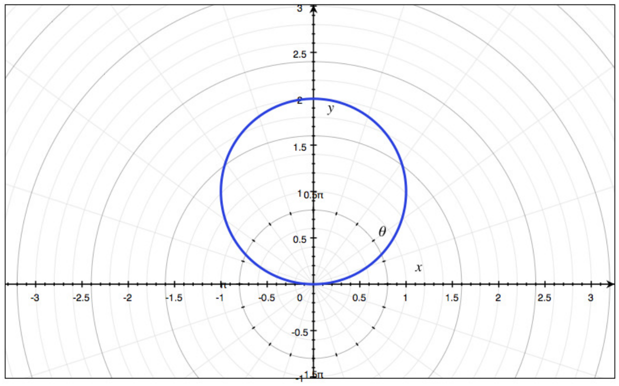
2.



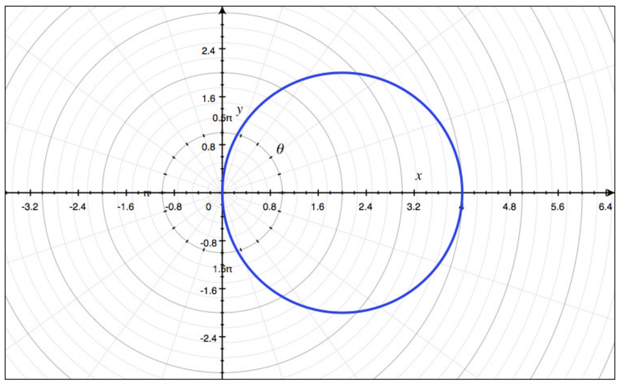
3.



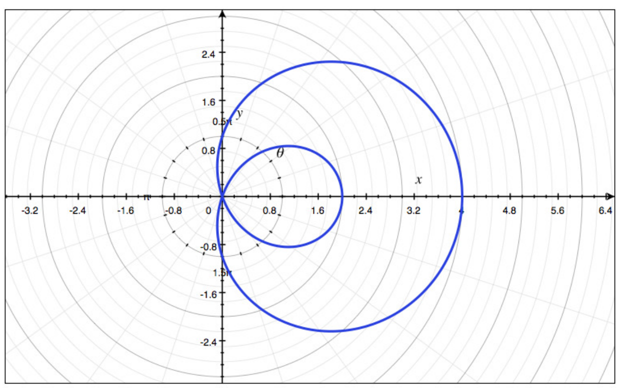
4.


3
2.5
2
y
1.5
0.5π
θ
0.5
x
-3 -2.5 -2 -1.5 -1 -0.5 0 0.5 1 1.5 2 2.5 3
-0.5
1.5π

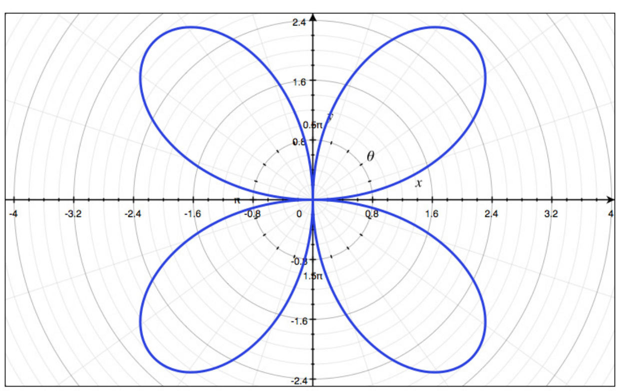
5.


2.4
1.6
0.5π
y
0.8
θ
x
-3.2 -2.4 -1.6 -0.8 0 0.8 1.6 2.4 3.2 4.8 5.6 6.4
-0.8
1.5π
-1.6
-2.4

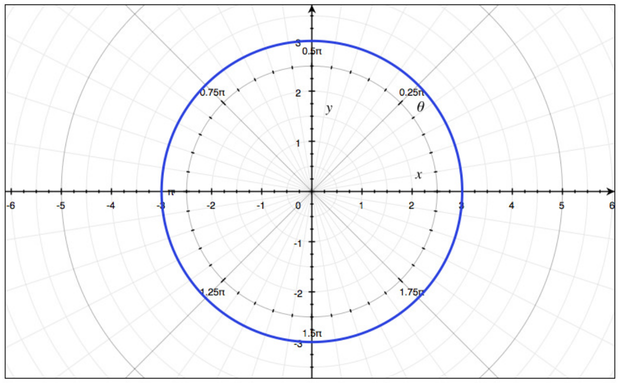
6.


2.4
1.6
0.5π
y
0.8
θ
x
-3.2 -2.4 -1.6 -0.8 0 0.8 1.6 2.4 3.2 4.8 5.6 6.4
-0.8
1.5π
-1.6
-2.4

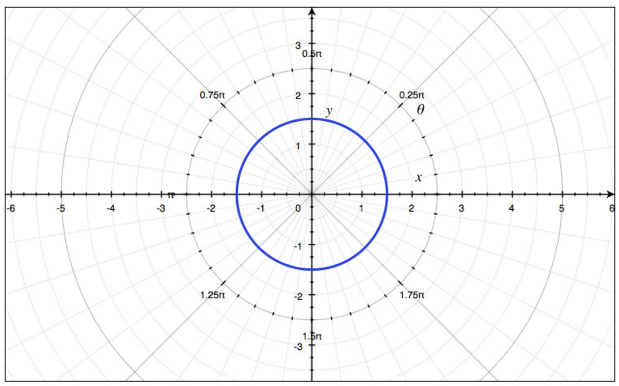
7.


2.4
1.6
0.5π
0.8
θ
x
-4 -3.2 -2.4 -1.6 -0.8 0 0.8 1.6 2.4 3.2 4
-0.8
1.5π
-1.6
-2.4

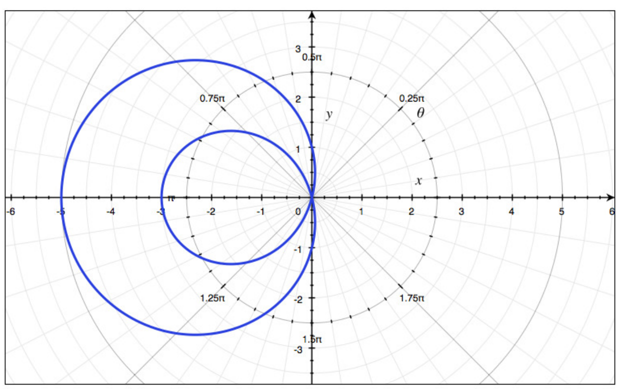
8.


0.5π
0.75π
0.25π
y
θ
x
π
1.25π
1.75π
1.5π
-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6
3 2 1 -1 -2 -3

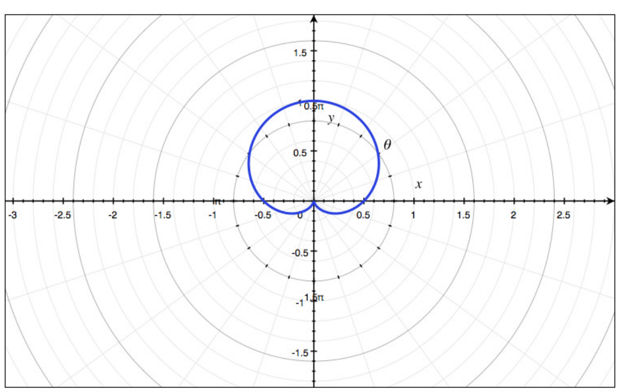
9.


0.5π
0.75π
0.25π
y
θ
x
π
1.25π
1.75π
1.5π
-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6
3 2 1 -1 -2 -3

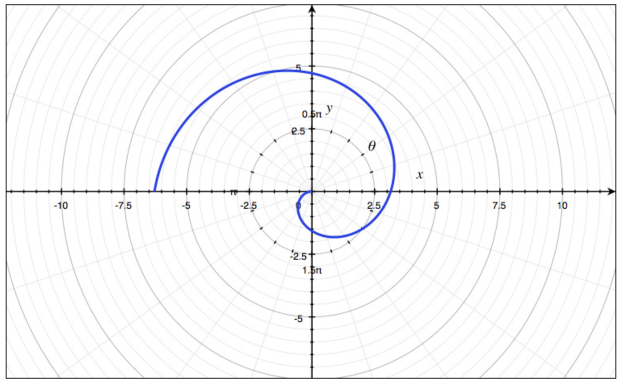
10.


0.5π
0.75π
0.25π
y
θ
x
π
1.25π
1.75π
1.5π
-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6
3 2 1 -1 -2 -3

11.

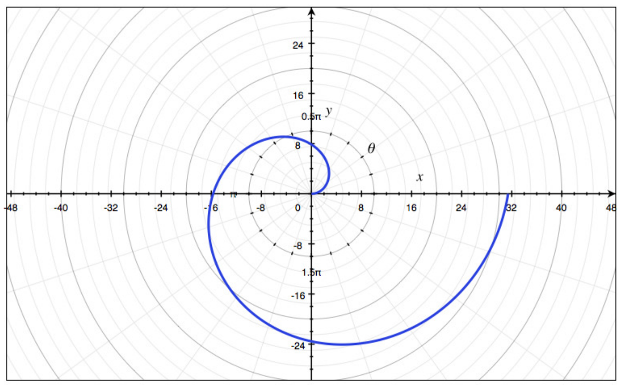

0.5π
y
θ
x
π
1.5π
-3 -2.5 -2 -1.5 -1 -0.5 0 0.5 1 1.5 2 2.5
1.5 0.5 -0.5 -1 -1.5

12.



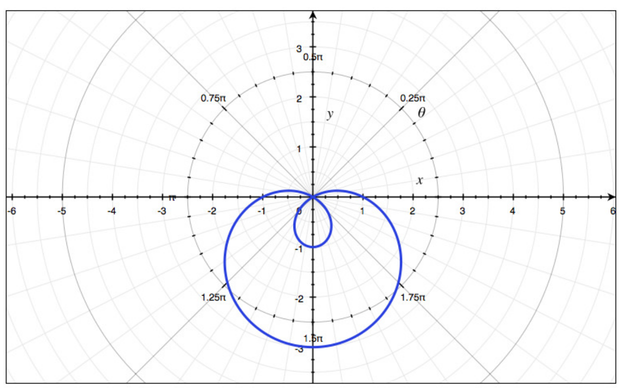
5
2.5
0.5π
y
θ
x
π
-10
-7.5
-5
-2.5
0
2.5
5
7.5
10
-2.5
1.5π
-5

13.



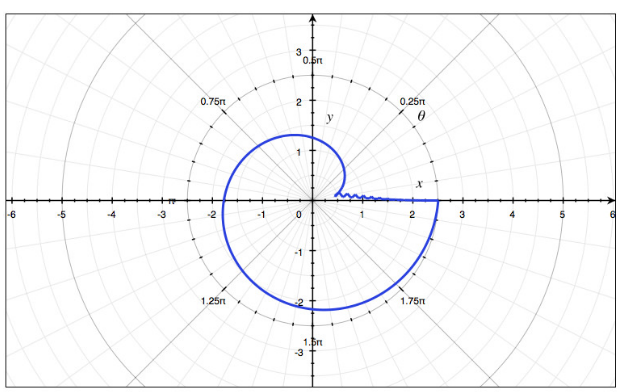
24
16
0.5π
y
8
θ
x
-48
-40
-32
-24
-16
-8
0
8
16
24
32
40
48
-8
1.5π
-16
-24

14.



3
0.5π
0.75π
0.25π
2
y
θ
1
x
π
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
-1
1.25π
-2
1.75π
1.5π
-3

15.

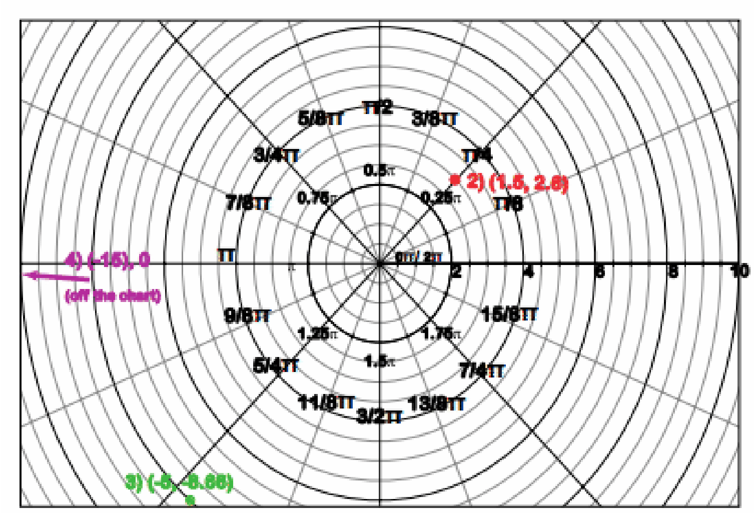


3
0.5π
0.75π
0.25π
2
y
θ
1
x
π
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
-1
1.25π
-2
1.75π
1.5π
-3

16.

Section 12.4: Polar and Cartesian Transformation

1. (-5, 0)
2. (1.5, 2.6) Point plotted below.

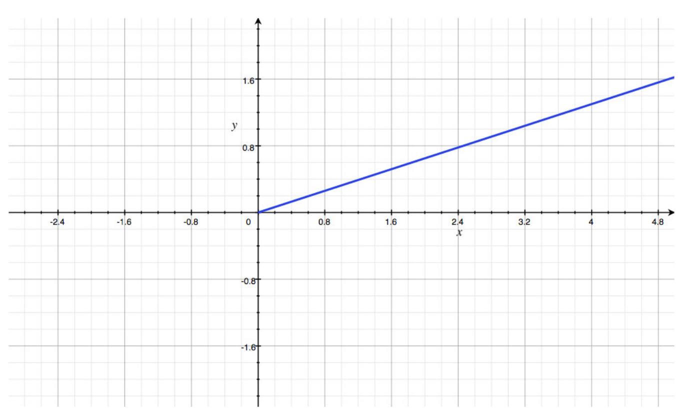


3. (-5, -8.66) Point plotted above.
4. (-15, 0) Point noted above.
5. $\left(\sqrt{50}, -\frac{\pi}{4}\right)$ or (7.07, -0.79)

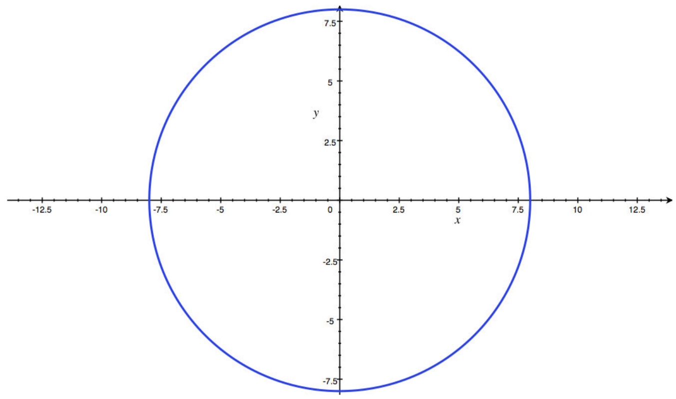
   $\left(\sqrt{50}, \frac{7\pi}{4}\right)$ or (7.07, 5.50)
6. $\left(10, \frac{\pi}{2}\right)$ or (10, 1.57)

   $\left(10, -\frac{3\pi}{2}\right)$ or (10, -4.71)
7. (10, 2.50)

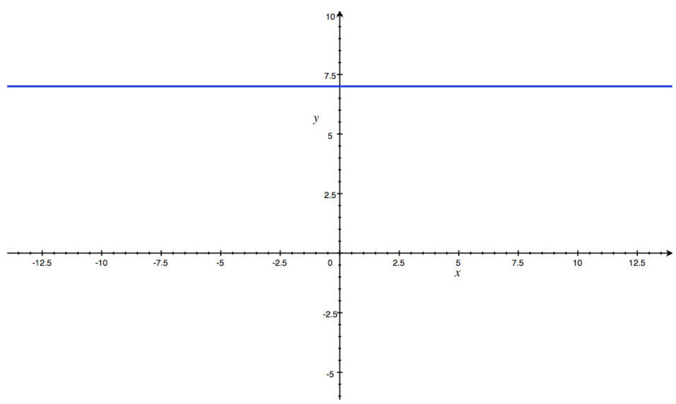
   (10, -3.79)
8. Approximate equation is $y = .33x$. This is a line with domain $x \geq 0$. It starts at the origin and has a slope of about 0.325.



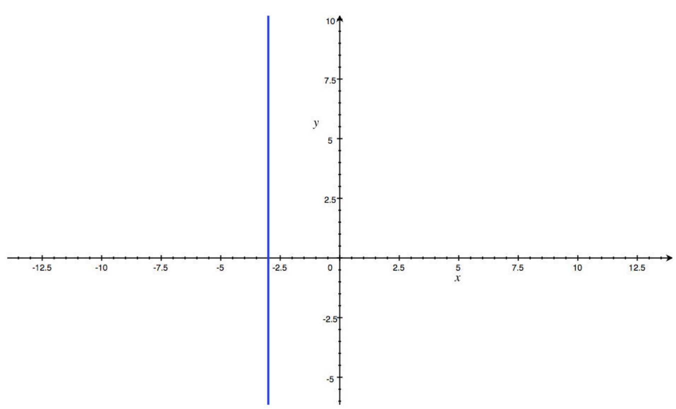
9. $x^2 + y^2 = 64$. This is a circle centered at the origin with radius 8.



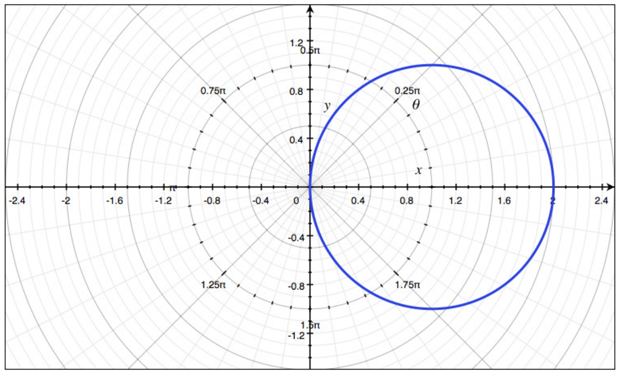
10. $y = 7$. This is a horizontal line through (0, 7).



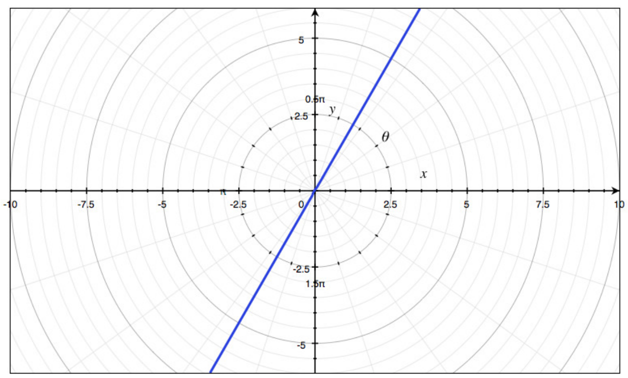
11. $x =$ -3. This is a vertical line through (-3, 0).



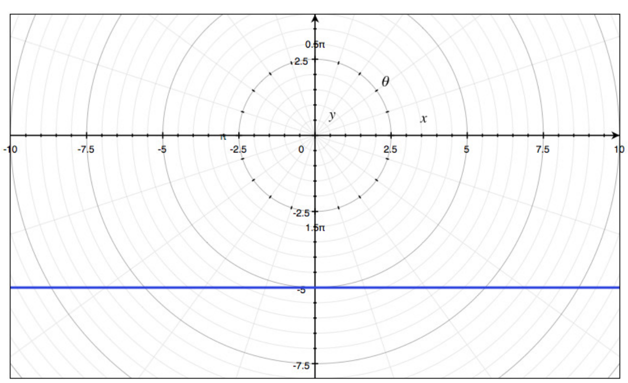
12. $r = 2\cos\theta$. This is a circle with center at (1, 0) and radius 1.



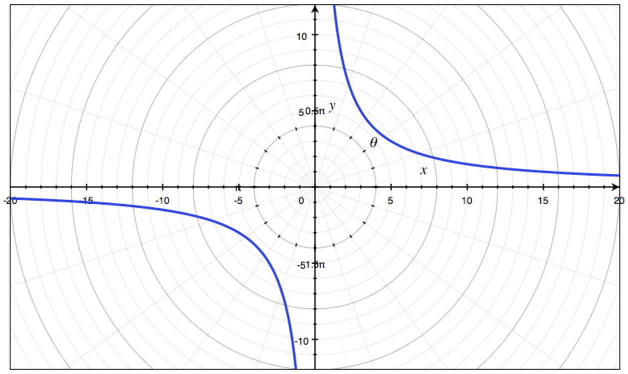
13. $\sin\theta = \sqrt{3}\cos\theta$. This is a line with slope equal to the square root of 3.



14. $r\sin\theta =$ -5. This is a horizontal line through (0, -5).



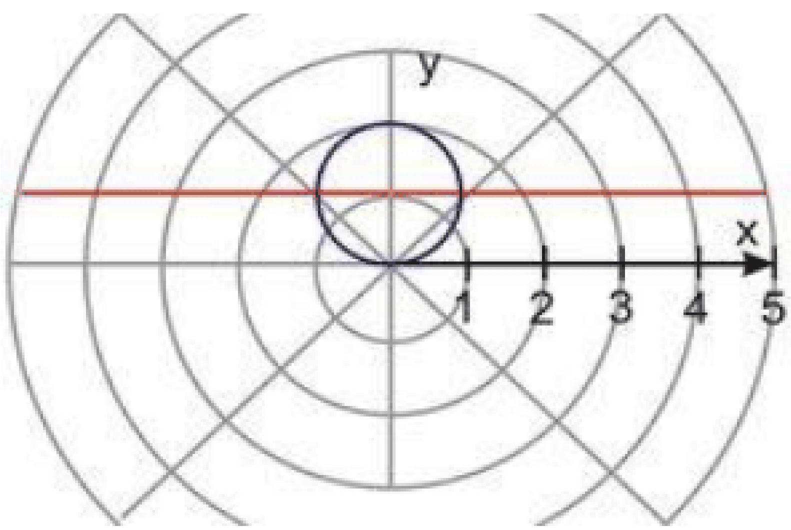
15. $r\cos\theta \times r\sin\theta = 15$ or $r^2\cos\theta\sin\theta = 15$. This is a rational function centered at the origin.



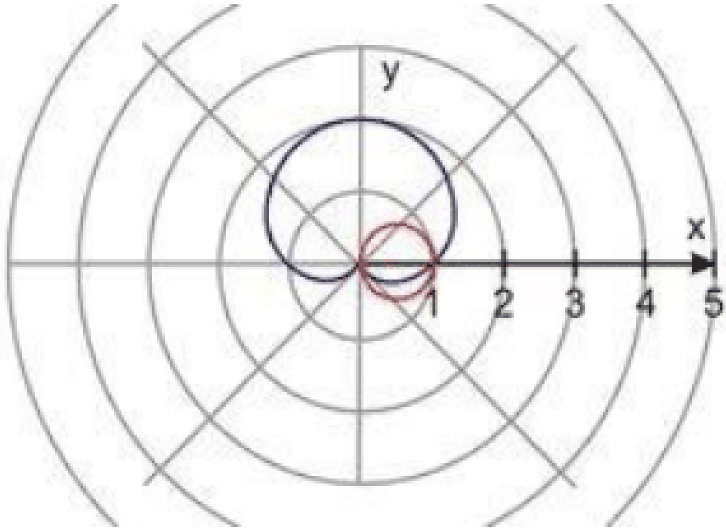
16. $\sqrt{\frac{4,380}{\pi}} = 132\cos(5\theta)$
17. $l = \frac{\sqrt{\frac{7,542}{\pi}}}{\cos 5\theta}$
18. $y = \frac{\frac{6}{17} - x\cos 25^\circ}{\sin 25^\circ}$ or $y \approx$ -2.14$x + 0.835$

Section 12.5: Systems of Polar Equations

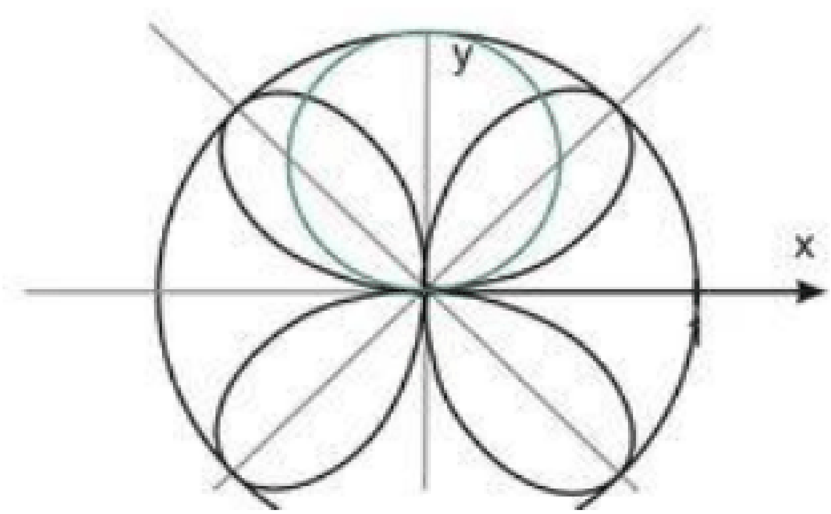
1. They intersect twice.
2. Once in the 1st and once in the 4th quadrant
3. They intersect at $(1, \frac{\pi}{3})$ and $(1, \frac{5\pi}{3})$.
4. 3 points of intersection
5. Points of intersection are $(0,0), (\frac{1}{2}, \frac{\pi}{2})$, and $(\frac{1}{2}, \frac{5\pi}{3})$.
6. $(2,0)$
7. $(0,0)$
8. $\left(2 + \sqrt{2}, \frac{3\pi}{4}\right)$ and $\left(2 - \sqrt{2}, \frac{7\pi}{4}\right)$
9. $(\frac{3}{2}, \frac{\pi}{3}), (\frac{3}{2}, \frac{5\pi}{3})$
10. $(\sqrt{2}, \frac{\pi}{4}), (\sqrt{2}, \frac{3\pi}{4})$



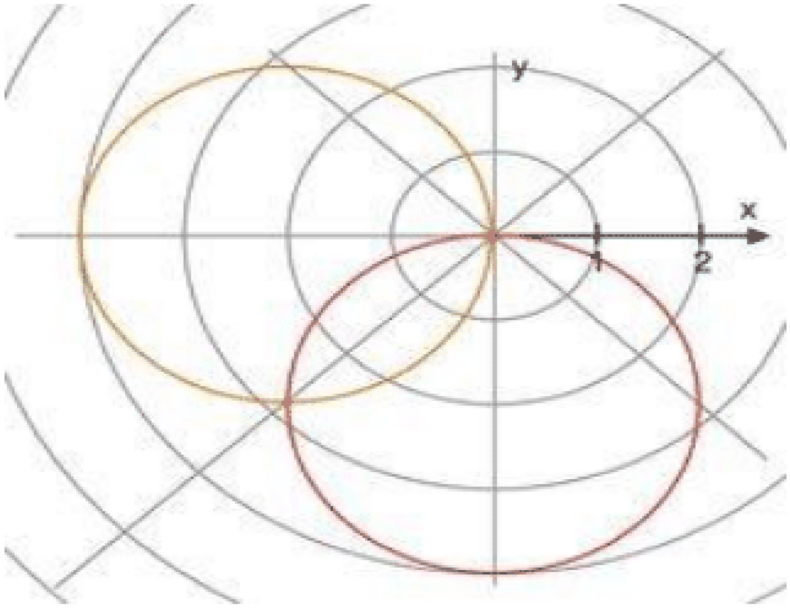
11. $(0,0), (1,0)$



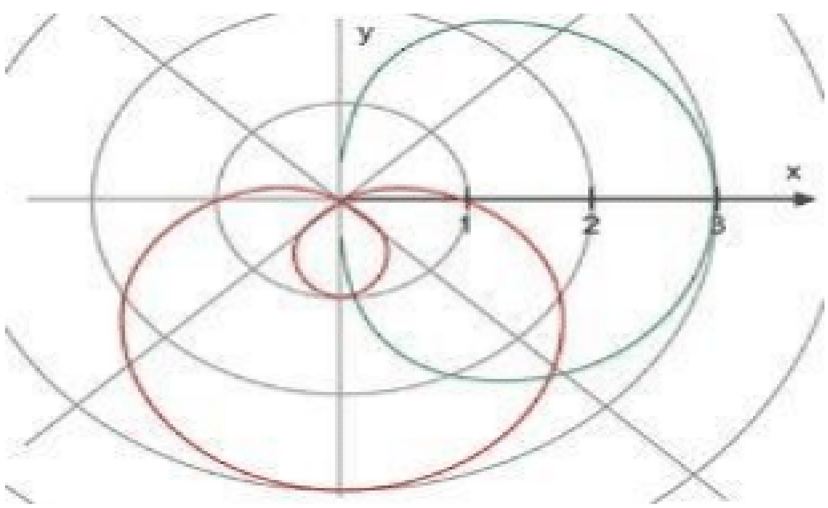
12. $(0,0),(\frac{\sqrt{3}}{2},\frac{2\pi}{3}),(\frac{\sqrt{3}}{2},\frac{\pi}{3})$



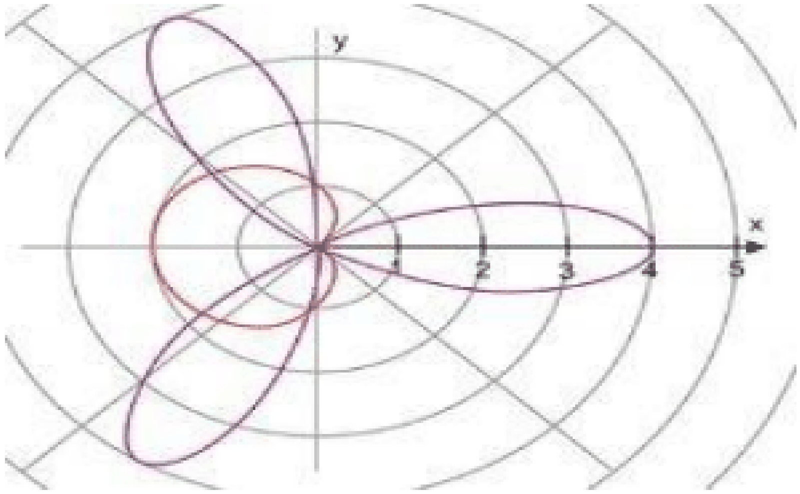
13. $(0,0),(2\sqrt{2},\frac{5\pi}{4})$



14. (1, 276°), (2.44, 313°)

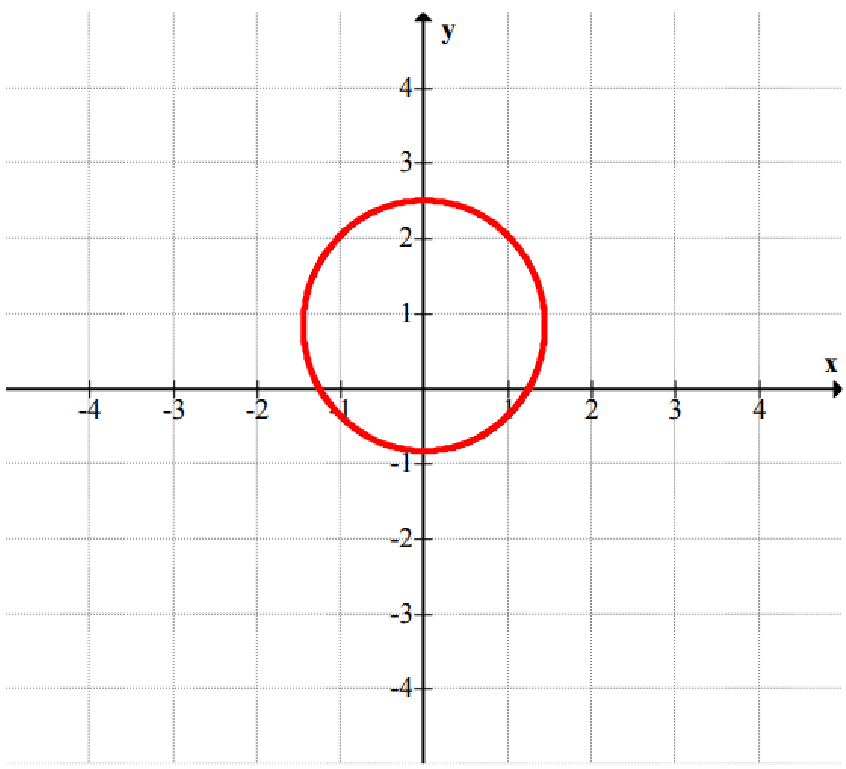


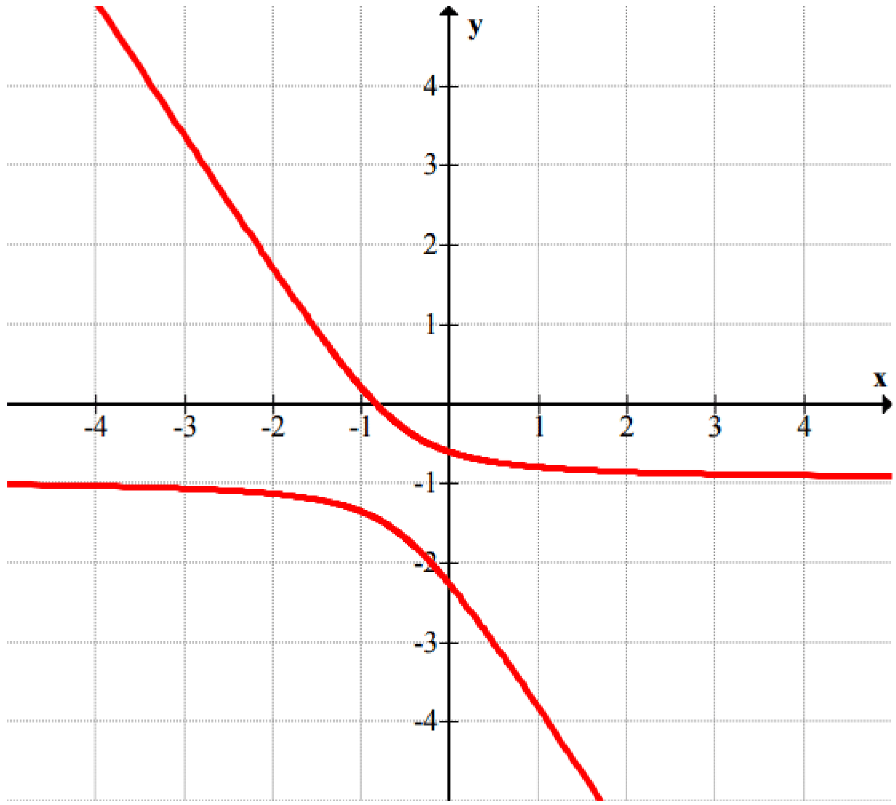
15. (0, 0), (1.08, 95°), (1,77, 142°), (1.77, 218°), (1.08, 265°)



Section 12.6: Polar Equations of Conics

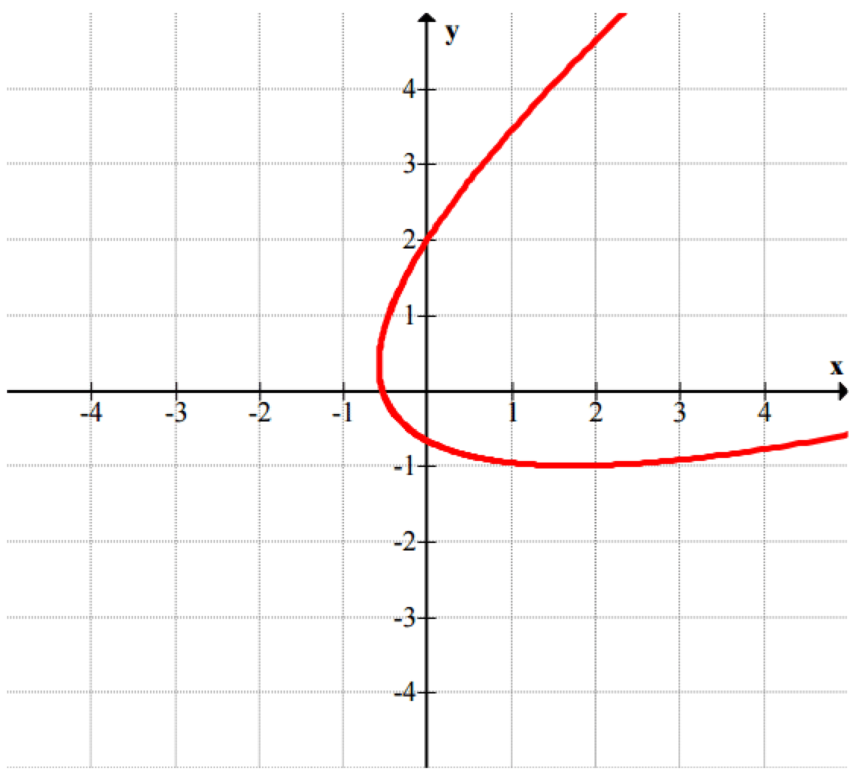
1. $8x^2 - 10x + 9y^2 - 25 = 0$; ellipse
2. $3x^2 - 8x + 4y^2 - 16 = 0$; ellipse
3. $3x^2 - 4x + 4y^2 - 4 = 0$; ellipse
4. $-12x^2 - 24x + 4y^2 - 9 = 0$; hyperbola
5. $x^2 - 5x + y^2 = 0$; circle
6.

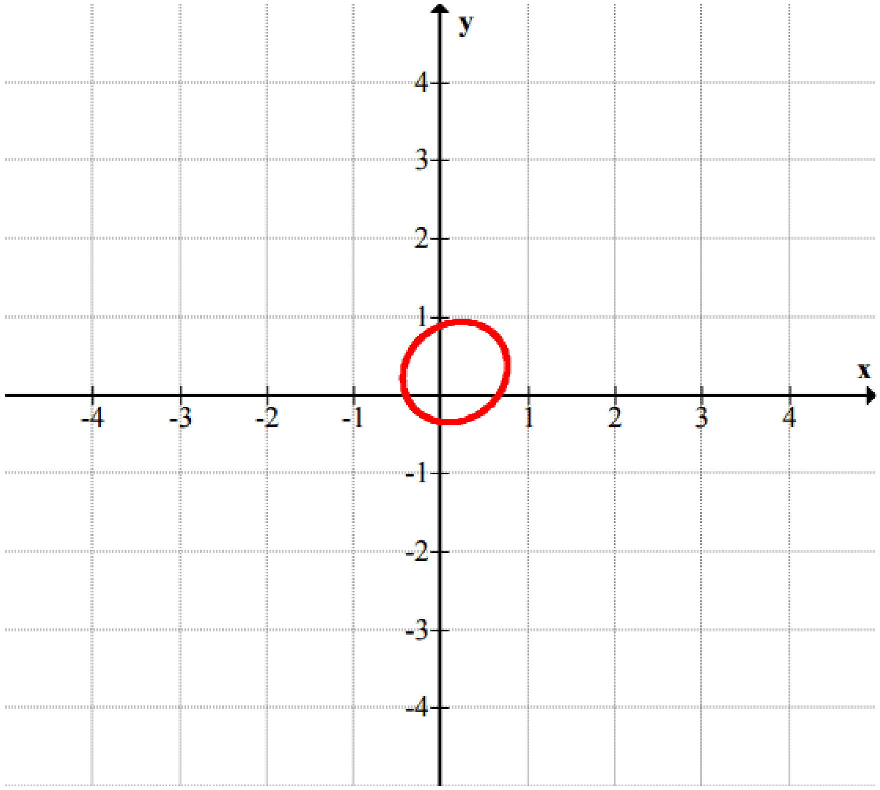




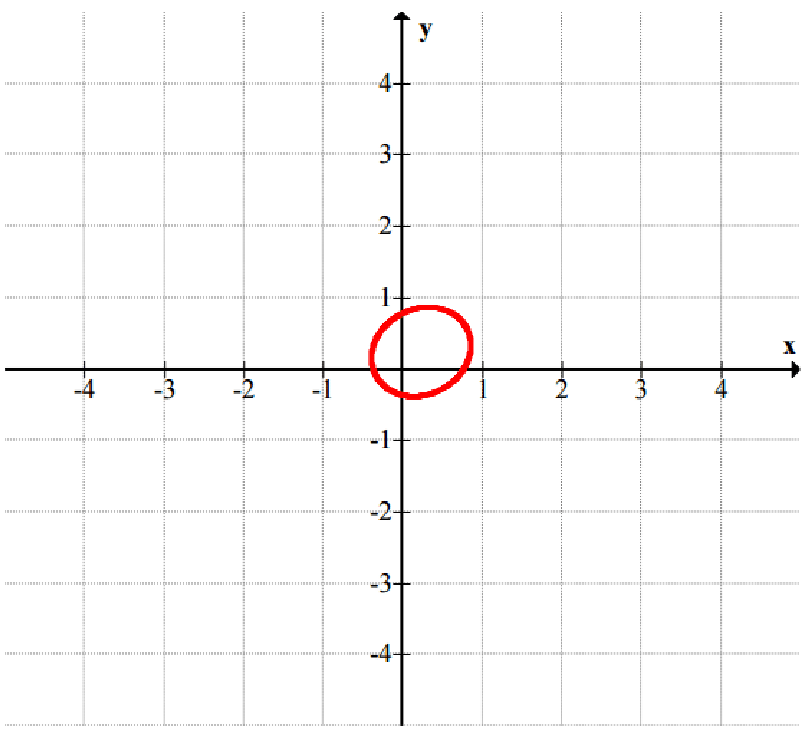
7.

8.





9.



10.

11. $r = \frac{3}{2-\cos\theta}$

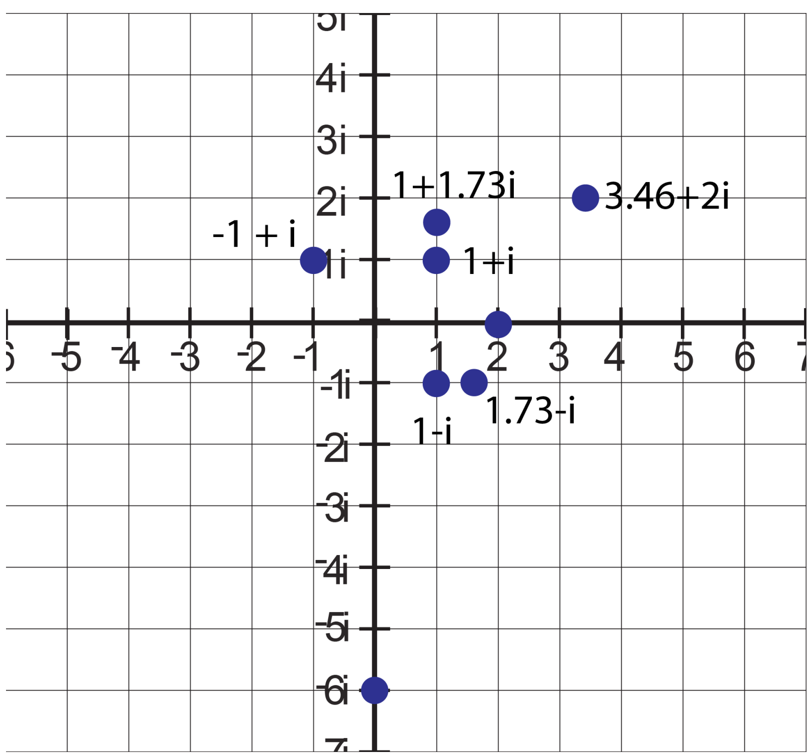
12. $r = 10\cos\theta - 24\sin\theta$

13. $r = -2\sin\theta$

14. $r = 2\cos\theta$

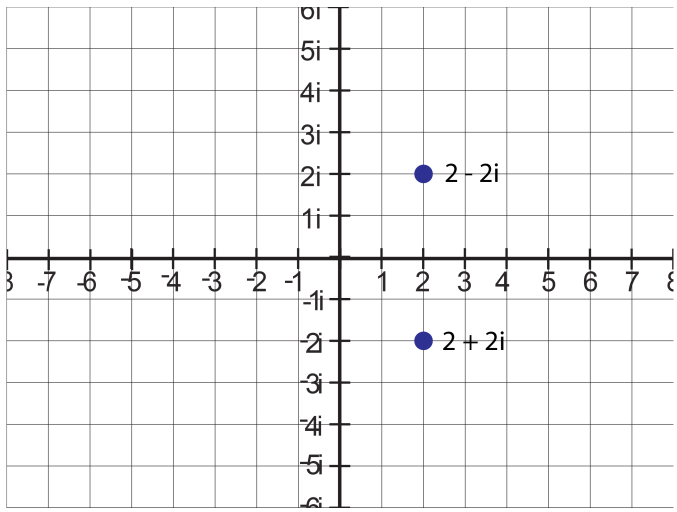
15. $r = \frac{1}{1-2\cos\theta}$
16. $e = 1$, parabola
17. $e = \frac{1}{2}$, ellipse
18. $e = 1$, parabola
19. $e = 2$, hyperbola
20. Answers will vary. One example: On the rectangular grid, you graph points based on x and y values (how far something is right/left or above/below the origin). On the polar grid, you graph points based on angles and the length of a radius.

Section 12.7: Polar Form of Complex Numbers



1. See graph above. $\left(\sqrt{2}, 45^\circ\right)$
2. See graph above. $\left(\sqrt{2}, 135^\circ\right)$
3. See graph above. $(6, 270^\circ)$
4. See graph above. $\left(\sqrt{2}, 45^\circ\right)$
5. See graph above. $\left(\sqrt{2}, \text{-}45^\circ\right)$
6. See graph above. $(2, 0^\circ)$
7. See graph above. $(2, 60^\circ)$
8. See graph above. $(2, \text{-}30^\circ)$
9. See graph above. $(4, 30^\circ)$
10. $4.25 + 4.25i$
11. $\text{-}7.5 + 13i$
12. $6 + 10.4i$

13. $\left(\sqrt{x^2+y^2}, \frac{1}{sin}\left(\frac{y^2}{r^2}\right)\right)$
$(\cos\theta + i\sin\theta)$ using θ from above
14. Zeros: 2 + 2i, 2 - 2i



15. (0, 0°)
16. $\left(\frac{\sqrt{3}-1}{2}, 0°\right)$
17. $\left(\sqrt{13}, 213.69°\right)$
18. (4, -30°)

Section 12.8: Product and Quotient Theorems

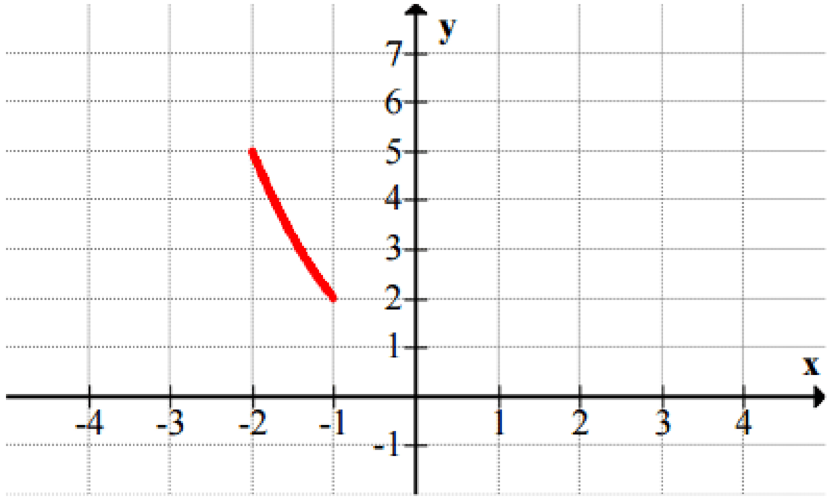
1. $4\sqrt{2}(\cos 15° + i\sin 15°)$
2. $8cis(60°)$
3. $\frac{1}{3}cis(\text{-}120°)$
4. 60°
5. $33.81 + 9.06i$
6. $.36 + 1.35i$
7. $\frac{3}{16}$
8. $0 + 40i$
9. $1.4 + \frac{4}{5}i$
10. $\frac{1}{2} + \frac{5}{16}i$
11. $\text{-}32 + 32i\sqrt{3}$
12. $0 + 125i$
13. $2\sqrt{3} + 2 + (2\sqrt{3} - 2)i$
14. $4 + 4\sqrt{3}i$
15. $3 + 2.6i$
16. $\text{-}\frac{1}{6} - \frac{2}{7}i$
17. $\text{-}\frac{23}{40} - \frac{27}{56}i$
18. $\left(\frac{32}{7}, 30°\right)$
19. $(84, 37°)$

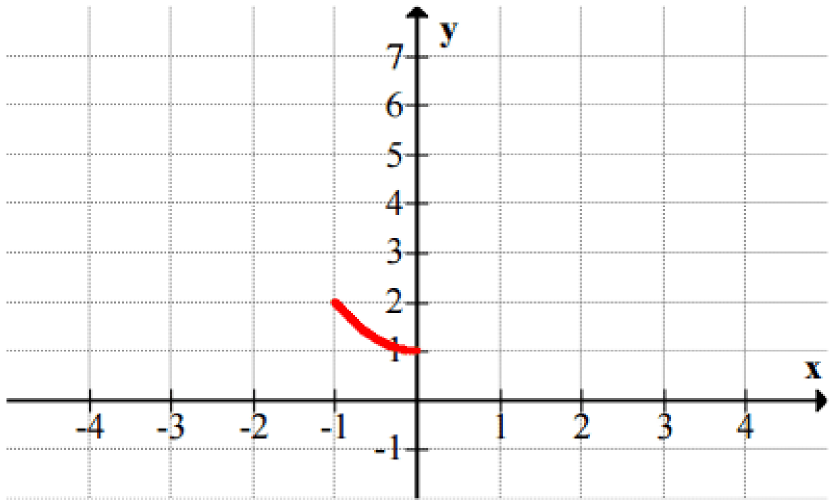
Section 12.9: Powers and Roots of Complex Numbers

1. $\frac{-1}{2}+\frac{5}{2}i$
2. 37
3. $(\frac{1}{2}-\frac{i\sqrt{3}}{2})$
4. $4\sqrt{2}(\cos 15^\circ + i\sin 15^\circ)$
5. $8cis(60^\circ)$
6. $4cis(\frac{9\pi}{40})$
7. $\frac{1}{3}cis\,(\text{-}120^\circ)$
8. $\frac{3}{4}cis\,(\text{-}140^\circ)$
9. $\text{-}\frac{27}{2}-\frac{27\sqrt{3}}{2}i$
10. $\text{-}2\sqrt{2}-2\sqrt{2}i$
11. -64
12. $(\sqrt[6]{2})cis15^\circ,\ \sqrt[6]{2}cis135^\circ,\ \sqrt[6]{2}cis255^\circ$
13. $2cis67.5^\circ, 2cis157.5^\circ,\ 2cis247.5^\circ, 2cis337.5^\circ$
14. $cis18^\circ, cis90^\circ, cis162^\circ, cis234^\circ, cis306^\circ$

Section 12.10: Parameters and Parameter Elimination

1. $y=\frac{4}{9}x^2+\frac{2}{9}x-\frac{2}{9}$
2. $x=\frac{3}{4}y^2+\frac{9}{2}y+\frac{15}{4}$
3. $y=x^2$
4. $y=x^3+15x^2+75x+126$
5. $y=x^2-8x+11$

6. 
y
7
6
5
4
3
2
1
x
-4
-3
-2
-1
1
2
3
4
-1

7. 
y
7
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4
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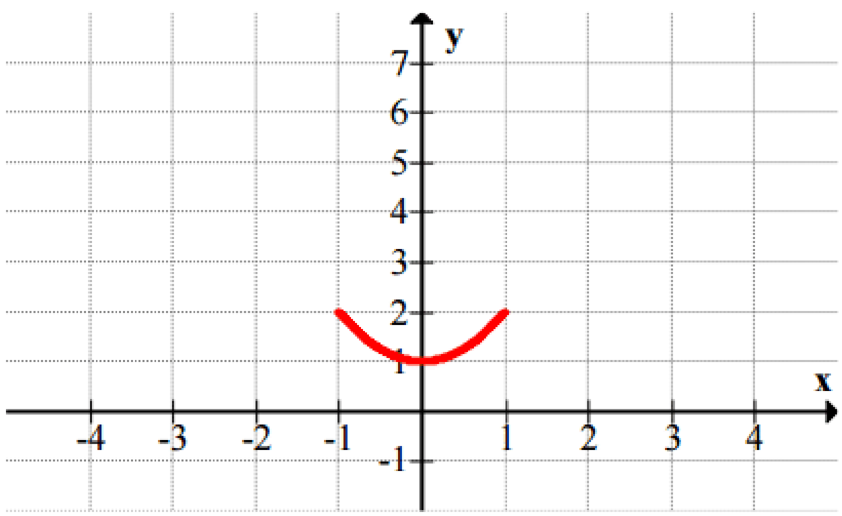
x
-4
-3
-2
-1
1
2
3
4
-1

8. 
y
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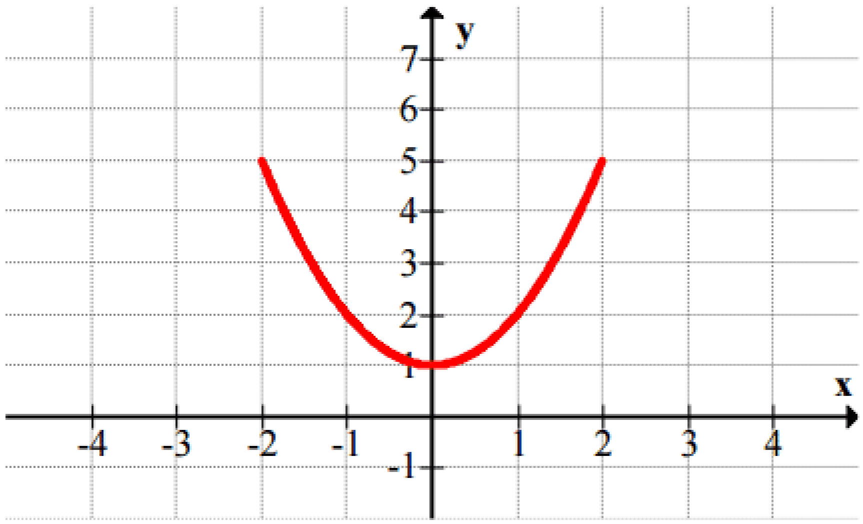
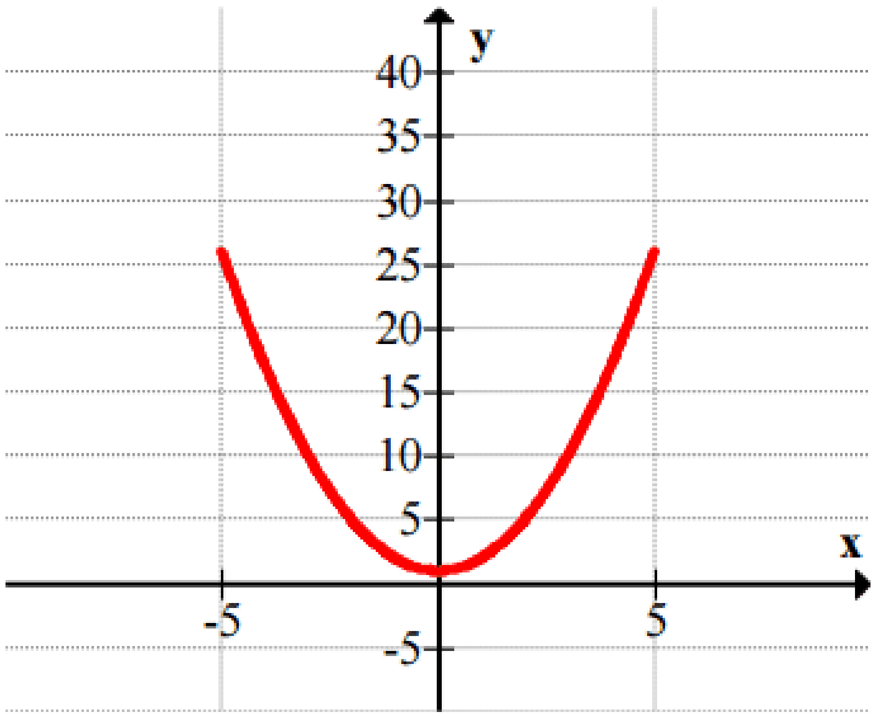
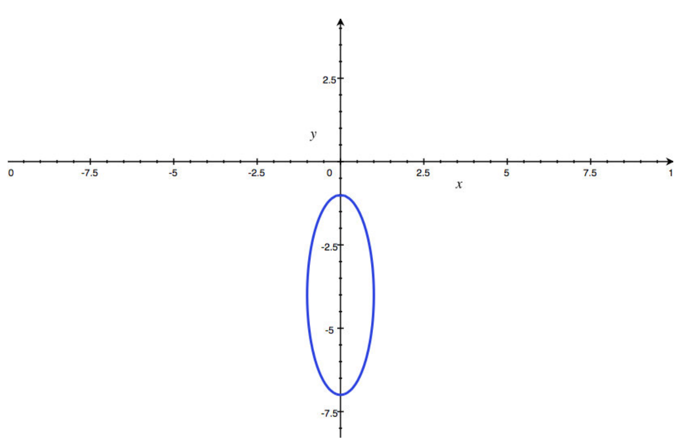
x
-4
-3
-2
-1
1
2
3
4
-1

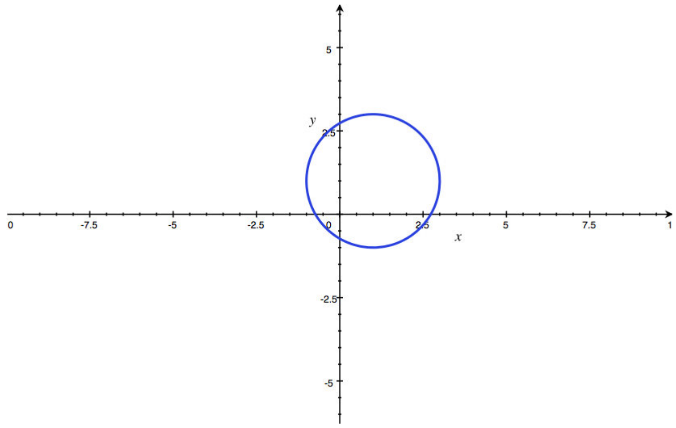
9. 

10. 

11. $x^2 + \left(\frac{y+4}{3}\right)^2 = 1$



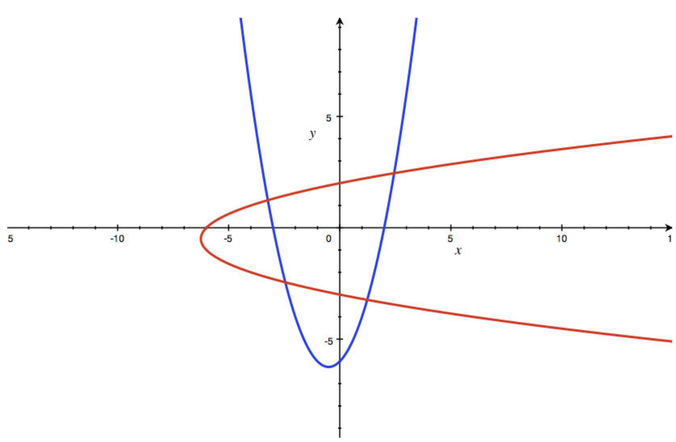
12. $(x-1)^2 + (y-1)^2 = 4$



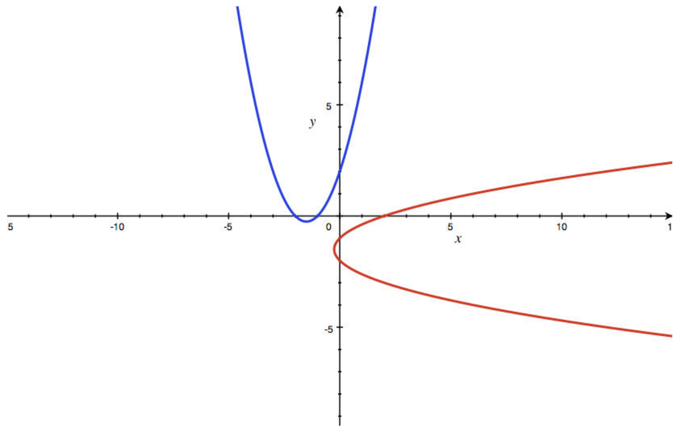
13. $x = 2 + 3\cos t$; $y = 4 + 3\sin t$
14. $x = 2\cos t$; $y = 5\sin t$
15. $x = 3\cos t + 4$; $y = 6\sin t - 1$

Section 12.11: Parametric Inverses

1. $x = t^2 + 2$; $y = t - 4$
2. Function
3. Neither
4. $x = 4 - t$; $y = t^2$
5. Inverse
6. Relation
7. $x = t^2 - 3$; $y = 2t - 1$
8. Inverse
9. Function
10. $x = t^2 - 2t$; $y = 3t + 14$
11. When $t =$ -2 at $(8, 8)$ and when $t = 7$ at $(35, 35)$
12. $x = 4t - 4$; $y = t^2$
13. When $t = 2$ at $(4, 4)$
14. $x = t$; $y = t^2 + t - 6$

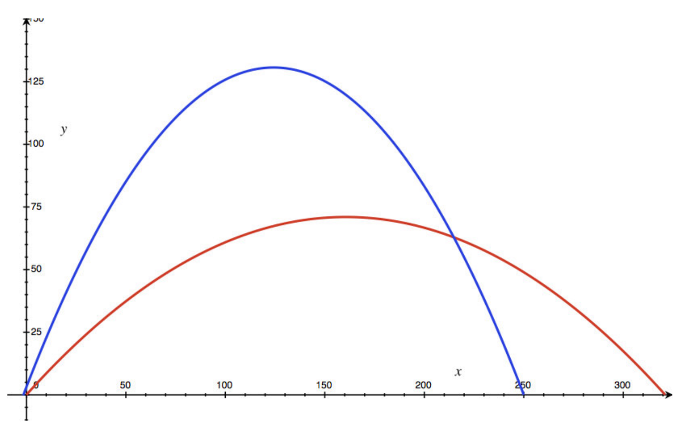


15. $x = t$; $y = t^2 + 3t + 2$



Section 12.12: Applications of Parametric Equations

1. $x = \text{-}50\sin\frac{2\pi}{5}t$; $y = \text{-}50\cos(\frac{2\pi}{5}t) + 53$
2. $(\text{-}29.4, 93.1)$
3. $(47.55, 37.55)$
4. $x = \text{-}40\sin(\frac{\pi}{3}(t+1))$; $y = \text{-}40\cos(\frac{\pi}{3}(t+1)) + 43$
5. $(0, 83)$
6. $(34.6, 23)$
7. $x = t \cdot 73.33 \cdot \cos\frac{\pi}{4}$; $y = \frac{1}{2} \cdot (\text{-}32) \cdot t^2 + t \cdot 73.33 \cdot \sin(\frac{\pi}{4}) + 5$
8. $(103.7, 44.7)$
9. 172.7 feet in about 3.33 seconds
10. $x = t \cdot 102.67 \cdot \cos(\frac{\pi}{3}) + 8.8t$; $y = \frac{1}{2} \cdot (\text{-}32) \cdot t^2 + t \cdot 102.67 \cdot \sin(\frac{\pi}{3}) + 7$
11. $(120.27, 120.83)$
12. 338.56 feet in about 5.63 seconds
13. $x_1 = \text{-}t \cdot 105.6 \cdot \cos(\frac{\pi}{3}) + 250 + 8.8t$; $y_1 = \frac{1}{2} \cdot (\text{-}32) \cdot t^2 + t \cdot 105.6 \cdot \sin(\frac{\pi}{3})$
14. $x_2 = t \cdot 95.33 \cdot \cos(\frac{\pi}{4}) + 8.8t$; $y_2 = \frac{1}{2} \cdot (\text{-}32) \cdot t^2 + t \cdot 95.33 \cdot \sin(\frac{\pi}{4})$
15. While the graphs intersect, each ball passes through the point of intersection at a different time.



16.13 Answers - Ch 13: Sequences and Series

Section 13.2: Recursion

1. $a_1 = 3;\ a_k = a_{k-1} + 4$
2. $a_1 = 3;\ a_k = 3a_{k-1}$
3. $a_1 = 3;\ a_k = a_{k-1} + 3$
4. $a_1 = 3;\ a_k = 2a_{k-1}$
5. $a_1 = 1;\ a_k = 4a_{k-1}$
6. 2, 8, 40, 208, 1,088, 5,696
7. 4, 18, 56, 202, 684, 2,378
8. 2, 5, 7, 12, 19, 31, 50, 81, 131, 212
9. 550
10. $a_1 = 1;\ a_k = 3a_{k-1} + 1$
11. $a_1 = 1;\ a_k = 3a_{k-1} + 2$
12. $a_1 = 2;\ a_k = 5a_{k-1} + 1$
13. $a_1 = 2;\ a_2 = 3;\ a_k = a_{k-1} \cdot a_{k-2}$
14. $a_1 = 4;\ a_2 = 6;\ a_k = a_{k-1} + a_{k-2} + 1$
15. $a_1 = 7;\ a_2 = 13;\ a_k = 2(a_{k-1} + a_{k-2})$

Section 13.3: Arithmetic and Geometric Sequences

1. 17, 21, 25
2. $a_k = 1 + (k-1) \cdot 4$
3. 597
4. $\frac{4}{27}, \frac{4}{81}, \frac{4}{243}$
5. $a_k = 12 \cdot \left(\frac{1}{3}\right)^{k-1}$
6. $\frac{12}{3^{16}}$ or $\frac{4}{3^{15}}$
7. $\frac{2}{125}, -\frac{2}{625}, \frac{2}{3,125}$
8. $a_k = 10 \cdot \left(-\frac{1}{5}\right)^{k-1}$
9. $-\frac{10}{5^{11}}$ or $-\frac{2}{5^{10}}$
10. $\frac{15}{2}, \frac{17}{2}, \frac{19}{2}$
11. $a_k = \frac{7}{2} + (k-1) \cdot 1$
12. $\frac{633}{2}$
13. $a_k = k^3 + 3$
14. Linear functions can be written in the form $f(x) = mx + b$, while arithmetic sequences can be defined as $a_k = d(k-1) + a_1$. The common difference in an arithmetic sequence is like the slope, and the 1st term of the sequence is like the y-intercept. The reason for the $k-1$in the equation for the sequence has to do with the fact that we start sequences at term 1 instead of term 0. One difference between linear functions and arithmetic sequences is that arithmetic sequences are discrete (they exist only as specific values of the sequence), while linear functions are continuous.
15. Exponential functions can be written in the form $f(x) = a \cdot b^x$, while geometric sequences can be defined as $a_k = a_1 \cdot r^{k-1}$. The common ratio in a geometric sequence is like the $b$value in an exponential function, and

the 1st term in the sequence is like the y-intercept. The reason for the $k-1$ in the equation for the sequence has to do with the fact that we start sequences at term 1 instead of term 0. One difference between exponential functions and geometric sequences is that geometric sequences are discrete (they exist only at the specific values of the sequence), while exponential functions are continuous.

16. a. $\frac{1}{4}$ of a caramel

    b. $1, 1\frac{1}{4}$, and $1\frac{1}{2}$ caramels
17. 2^{10} or 1,024 aliens
18. 2^{15} or 32,768MB

Section 13.4: Sigma Notation

1. -1 + 1 + 3 + 5 + 7 = 15
2. 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 = 511
3. 6 + 18 + 54 + 162 = 240
4. 3 + 7 + 11 + 15 + 19 + 23 + 27 + 31 + 35 + 39 = 210
5. 9 + 36 + 144 + 576 + 2,304 + 9,216 + 36,864 + 147,456 + 589,824 + 359,296 + 9,437,184 = 12,582,909
6. $\sum_{i=0}^{7} 3i+1$
7. $\sum_{i=0}^{4} 2i+3$
8. $\sum_{i=0}^{7} -i+8$
9. $\sum_{i=0}^{3} i+5$
10. $\sum_{i=0}^{\infty} 3 \cdot 2^i$
11. $\sum_{i=0}^{3} 10 \cdot \left(\frac{1}{2}\right)^i$
12. $\sum_{i=0}^{\infty} 4(-2)^i$
13. $\sum_{i=0}^{\infty} 2i+2$
14. $\sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^{i+1}$
15. $\sum_{i=0}^{\infty} 2 \cdot \left(\frac{1}{3}\right)^{i+1}$

Section 13.5: Arithmetic Series

1. 1,128
2. 11,016
3. 6,885
4. 1,584.3333 or $\frac{4,753}{3}$
5. 1,502.6666 or $\frac{4,508}{3}$
6. 15,652
7. 140,868
8. 116,316
9. 3,535
10. -47,429
11. -43,878
12. 8,415
13. -69,636
14. 384,750
15. -717,778.5

Section 13.6: Geometric Series

1. 163,835
2. 715,827,882
3. 9.999694824
4. 17.99999875
5. 2,391,484.3333
6. Divergent
7. Divergent
8. Convergent; 10
9. Convergent; 18
10. Divergent
11. Convergent; 9
12. $202,840.40
13. $3,144.07
14. $146,019.62
15. The terms in an arithmetic series can never approach zero, as they can in a geometric series.

Section 13.7: Induction Proofs

1. a. $4(5) = 20$ and $2^5 = 32$; $20 < 32$.

   b. Assume $4k < 2^k$.

   c. Prove $4(k+1) < 2^{k+1}$.
2. a. $8^1 - 3^1 = 8 - 3 = 5$, which is divisible by 5.

   b. Assume $8^k - 3^k$ is divisible by 5.

   c. Prove $8^{k+1} - 3^{k+1}$ is divisible by 5.
3. a. $7^1 - 1 = 6$, which is divisible by 6.

   b. Assume $7^k - 1$ is divisible by 6.

   c. Prove $7^{k+1} - 1$ is divisible by 6.
4. a. $2^2 = 4 \geq 4 = 2(2)$.

   b. Assume $k^2 \geq 2k$.

   c. Prove $(k+1)^2 \geq 2(k+1)$.
5. a. $4^1 + 5 = 9$, which is divisible by 3.

   b. Assume $4^k + 5$ is divisible by 3.

   c. Prove $4^{k+1} + 5$ is divisible by 3.

6. a. $0^2+1^2=\frac{(1)(1+1)(2(1)+1)}{6}$ (both sides equal 1)

   b. Assume $0^2+\ \dots\ +k^2=\frac{(k)(k+1)(2k+1)}{6}$.

   c. Prove $0^2+\ \dots\ +k^2+(k+1)^2=\frac{(k+1)(k+2)(2(k+1)+1)}{6}$.

TABLE 16.1:

Inductive step:	$4k<2^k$
Add 4 to both sides:	$4k+4<2^k+4$
Logical reasoning:	$4<2^k$ since $k>5$; therefore, $2^k+4<2^k+2^k$
Combine previous steps:	$4k+4<2^k+4<2^k+2^k$
Rewrite inequality:	$4(k+1)<2^k+4<2^{k+1}$

7.

TABLE 16.2:

Start with the statement you are trying to prove is divisible by 5:	$8^{k+1}-3^{k+1}$
Add and subtract $3\cdot 8^k$:	$8^{k+1}-3\cdot 8^k+3\cdot 8^k-3^{k+1}$
Look for common factors:	$8^k(8-3)+3(8^k-3^k)$
Simplify:	$8^k(5)+3(8^k-3^k)$
	Since both parts of the expression are divisible by 5, the whole expression is divisible by 5.

8.

TABLE 16.3:

Start with the statement you are trying to prove is divisible by 6:	$7^{k+1}-1$
Add and subtract 7:	$7^{k+1}-7+7-1$
Look for common factors:	$7(7^k-1)+(7-1)$
Simplify:	$7(7^k-1)+(6)$
	Since both parts of the expression are divisible by 6, the whole expression is divisible by 6.

9.

TABLE 16.4:

Statements we know:	$k^2\geq 2k;\ 2k\geq 1$
Reasoning: = k^2 + 2k +1 $\geq 2k+2k+1\geq 2k+1+1=2k+2=2(k+1)$	$(k+1)^2$
Therefore:	$(k+1)^2\geq 2(k+1)$

10.

TABLE 16.5:

Start with the statement you are trying to prove is divisible by 3:	$4^{k+1}+5$

TABLE 16.5: (continued)

Add and subtract 20:	$4^{k+1}+20-20+5$
Look for common factors:	$4(4^k+5)-15$
Simplify:	$4(4^k+5)-3(5)$
	Since both parts of the expression are divisible by 3, the whole expression is divisible by 3.

11.

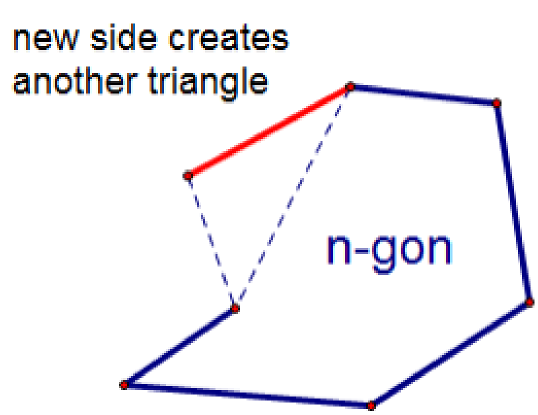
TABLE 16.6:

Inductive step:	$0^2+\ ...\ +k^2=\frac{(k)(k+1)(2k+1)}{6}$
Add $(k+1)^2$ to both sides:	$0^2+\ ...\ +k^2+(k+1)^2=\frac{(k)(k+1)(2k+1)}{6}+(k+1)^2$
Manipulate the right side:	$0^2+\ ...\ +k^2+(k+1)^2=\frac{(k)(k+1)(2k+1)+6(k+1)^2}{6}$
Simplify the numerator of the right side:	$0^2+\ ...\ +k^2+(k+1)^2=\frac{2n^3+9n^2+13n+6}{6}$
Notice that the numerator of the right side that we want is $(k+1)(k+2)(2(k+1)+1)$,which simplifies to $2n^3+9n^2+13n+6$	$0^2+\ ...\ +k^2+(k+1)^2=\frac{(k+1)(k+2)(2(k+1)+1)}{6}$

12.
13. You cannot find a base case to prove it's true.
14. The base case is necessary for showing that the statement is true at least once. The inductive step then shows that for every number for which the statement is true, the statement is also true for the following number. Because the base case is true, then the statement must be true for all numbers. Without the base case, you haven't proven anything!
15. Base case: For n = 3, the sum of the angles in a triangle is 180° by the Triangle Sum Theorem from geometry.

Inductive step: Assume an n-gon has interior angles with a sum of $180(n-2)$.

Proof: We want to show that the sum of the interior angles of an $n+1$-gon is $180(n+1-2)$. An $n+1$-gon will have one additional side compared to an n-gon. This means that one additional triangle can be drawn connecting exterior vertices. This logic is shown in the picture below.



The additional triangle adds another 180 degrees to the sum of the interior angles. This means an $n+1$-gon has interior angles with a sum of $180(n-2)+180$. This can be rewritten as $180(n+1-2)$.

16.14 Answers - Ch 14: Probability and Statistics

Section 14.2: Counting with Permutations and Combinations

1. 840
2. 122,391,522
3. 132,600
4. 84
5. 32,760
6. Permutation/decision chart; $41 \cdot 40 \cdot 39 = 63{,}960$
7. Combination; ${}_5C_2 = 10$
8. Permutation/decision chart; $4 \cdot 3 \cdot 2 = 24$
9. Combination; ${}_{12}C_3 = 220$
10. Permutation/decision chart; $14 \cdot 13 \cdot 12 = 2{,}184$
11. Combination; ${}_{10}C_3 = 120$
12. Combination; ${}_{12}C_4 = 495$
13. Permutation/decision chart; $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 95{,}040$
14. This problem is more complicated because you are allowed to repeat. It is like a combination problem (because order does not matter) where you are allowed to repeat.
15. Students should feel confident using the graphing calculator for permutations and combinations.

Section 14.3: Binomial Theorem

1. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$
2. $x^5 - 15x^4y + 90x^3y^2 - 270x^2y^3 + 405xy^4 - 243y^5$
3. $128x^7 + 1{,}792x^6y + 10{,}752x^5y^2 + 35{,}840x^4y^3 +$
   $71{,}680x^3y^4 + 86{,}016x^2y^5 + 57{,}344xy^6 + 16{,}384y^7$
4. -280
5. 56
6. -960
7. 2,400
8. 112
9. 70,000
10. 512
11. 4,096
12. 256
13. $(3x-1)^5$
14. $(x-y)^7$
15. $(2x-y)^7$

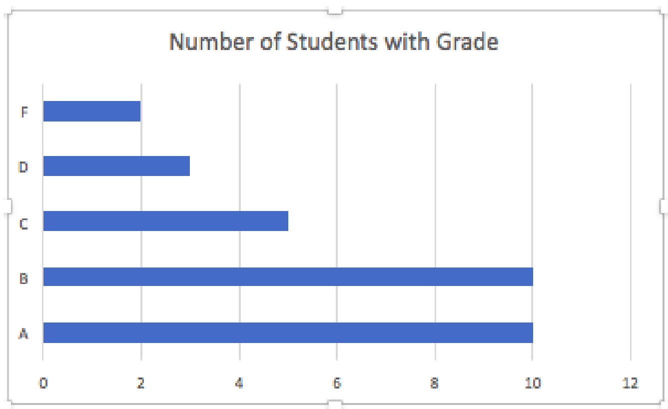
Section 14.4: Basic Probability

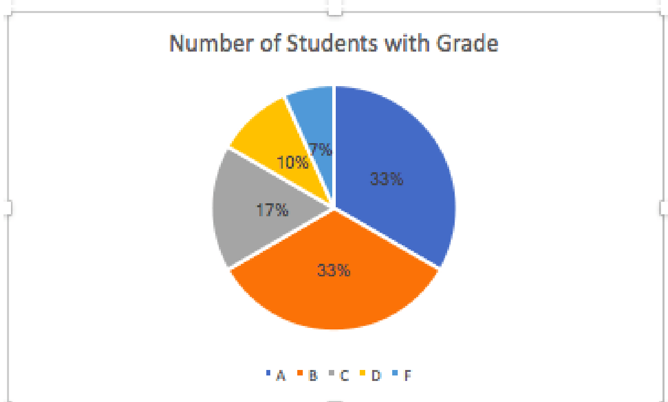
1. $\frac{1}{13}$
2. $\frac{16}{52} = \frac{4}{13}$
3. There should be 8 total outcomes: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT.
4. $\frac{1}{2}$
5. $\frac{1}{8}$
6. $\frac{7}{8}$
7. $\frac{7}{26}, \frac{6}{25}, \frac{5}{24}, \frac{4}{23} = \frac{7}{2{,}990} \approx 0.23\%$
8. $\frac{7}{26}, \frac{9}{25}, \frac{10}{24}, \frac{9}{23} = \frac{189}{11{,}960} \approx 1.58\%$
9. $(0.55)^{40} \approx 0.000000004\%$
10. $(0.45)^{40} \approx 0.0000000000013\%$
11. $\frac{1}{2}$
12. $\frac{1}{6}$
13. $\frac{35}{36}$
14. $\frac{1}{24}$
15. There are 10,240 ways to have 5 cards in a row, but 40 of those ways are straight flushes or royal flushes. Therefore, there are only 10,200 true flushes. There are ${}_{52}C_5 = 2{,}598{,}960$ total poker hands. The probability is $\frac{10{,}200}{2{,}598{,}960} = 0.392\%$.

Section 14.5: Expected Value and Payoffs

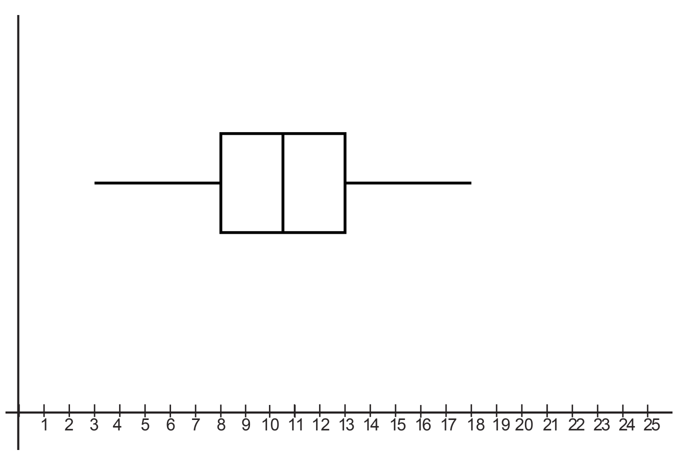
1. To calculate expected value, multiply the value of each outcome by the probability of that outcome and find the sum of all these products.
2. False. $0.50 is your expected average amount of winnings if you were to play the game many times.
3. True
4. $\frac{1}{6}$ or 17 cents
5. 36 cents
6. They should charge people more than 65 cents (the expected value) to play.
7. 18.02
8. 81.3%
9. 70.5%
10. 93.0%
11. 79.5%
12. 76.3%
13. Answers vary.
14. You should charge more than the $1.15 to theoretically make a profit by the end of the night.
15. Answers vary.
16. Casinos need to design games that people have a chance of winning, but which ultimately enable the casino to make money. Expected value helps determine what their profit will be on average for each game, and can help them determine what should be charged for each game.

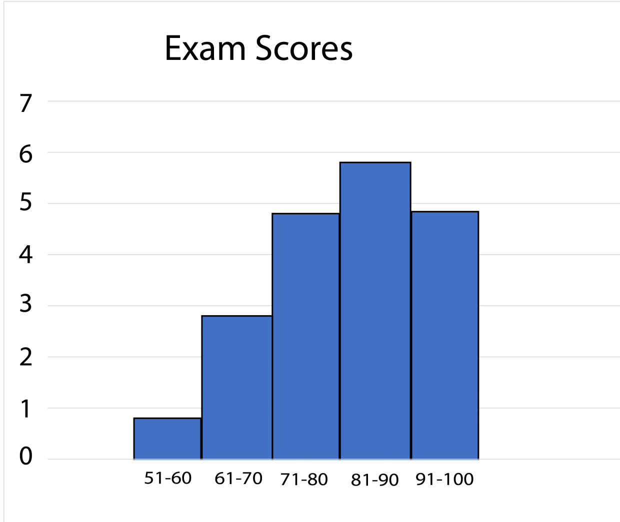
Section 14.6: Graphic Displays of Data

1. 

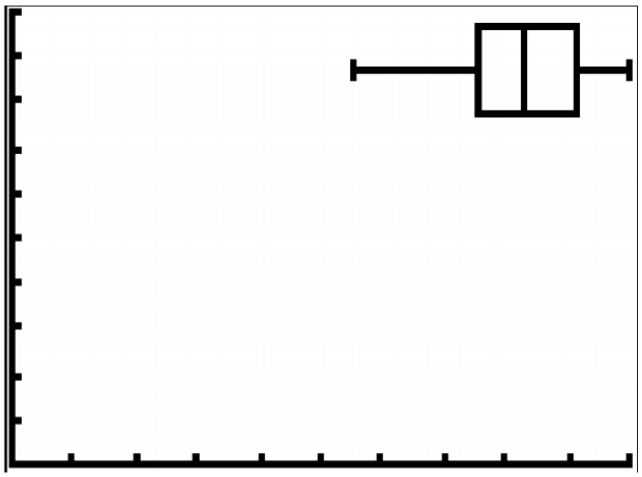
2. 

3. Answers will vary.

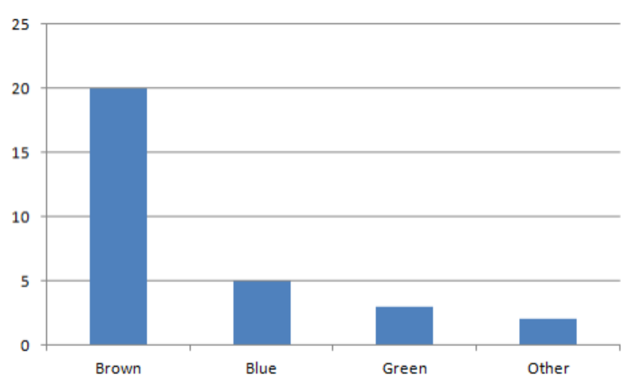
4. 

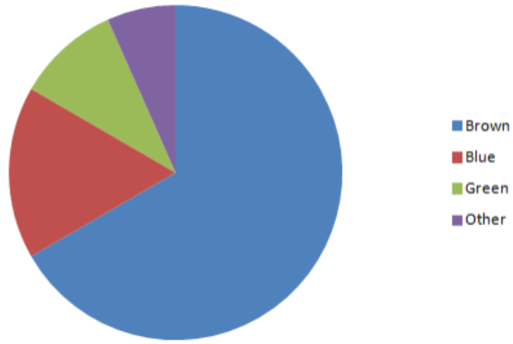
5. 

6. {55, 75, 82.5, 91.5, 100}

7. 

8. Answers vary. A student might notice there is a bigger range in the 1st half of the data than in the 2nd half of the data.
9. Answers vary. A student might notice that more of the data is in the left side of the graph than in the right side of the graph.

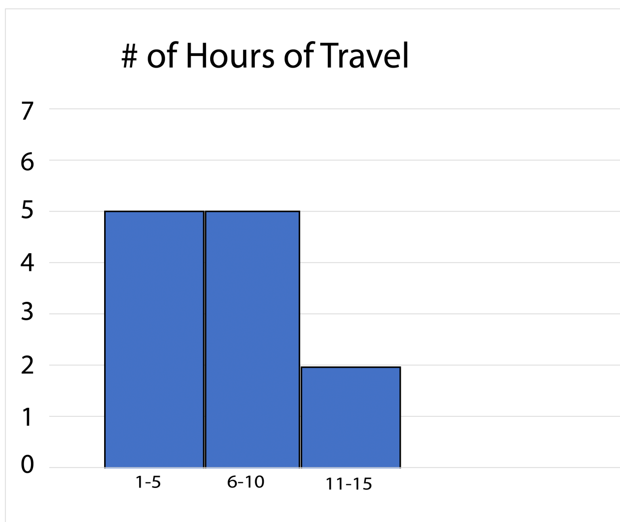
10. 

11. 

12. Answers vary.
13. Bar graph or pie chart
14. Histograph or boxplot

TABLE 16.7:

Interval	Frequency
1-5	5
6-10	5
11-15	2

15.
16. 

17. Answers will vary.
18. Answers will vary.

Section 14.7: Mean, Median, and Mode

1. Mean: 1.76; Median: 1; Mode: 1
2. Because there are two large numbers at the end that contribute more heavily to the mean, but don't impact the median.
3. The median or mode would make the most sense in this case, because they are not impacted by the outliers of 10 siblings. Also, you can't have 1.76 siblings, so 1.76 siblings would not make sense as the "typical number of siblings."
4. 885

5. 16
6. 154 miles
7. 5
8. Mean: 19.2; Median: 19; Mode: 15
9. Mean: 7.083; Median: 6; Mode: 5
10. Mean: 3.889; Median: 4; Mode: none
11. Mean: 69.75; Median: 88.5
12. Mean: 31.67; Median: 19.5
13. Number 12, because there was one number much larger than the rest in the data.
14. Number 11, because there was one number much smaller than the rest in the data.
15. In Number 11, the outlier is 12. In Number 12, the outlier is 98. If you were to remove the outliers, the mean would be closer to the median in each case.

Section 14.8: Five-Number Summary

1. Minimum: 0.08

   Q1: 0.18

   Q2: 0.235

   Q3: 0.27

   Maximum: 0.32
2. Minimum: 77

   Q1: 79.5

   Q2: 86.5

   Q3: 90.5

   Maximum: 99
3. Minimum: 53

   Q1: 79.5

   Q2: 84.5

   Q3: 92

   Maximum: 98
4. Minimum: 51

   Q1: 72

   Q2: 85

Q3: 91

Maximum: 96

5. Minimum: 185

   Q1: 220.5

   Q2: 281

   Q3: 363.5

   Maximum: 518
6. Minimum: 33

   Q1: 38

   Q2: 40

   Q3: 48.5

   Maximum: 71
7. Minimum: 3

   Q1: 6

   Q2: 12

   Q3: 16

   Maximum: 21
8. Minimum: 6

   Q1: 16.5

   Q2: 26

   Q3: 37

   Maximum: 49
9. Minimum: 5

   Q1: 9

   Q2: 10.5

   Q3:12

   Maximum: 17
10. Minimum: 49

    Q1: 50

Q2: 53

Q3: 57

Maximum: 67

11. Minimum: 5

    Q1: 18.5

    Q2: 20.5

    Q3: 22.5

    Maximum: 24

12. Minimum: 620

    Q1: 800

    Q2: 850

    Q3: 900

    Maximum: 1,070

13. Minimum: 12

    Q1: 13.5

    Q2: 17

    Q3: 22.5

    Maximum: 42

14. Minimum: 15

    Q1: 17.5

    Q2: 21

    Q3: 32.5

    Maximum: 55

15. Minimum: 120

    Q1: 122

    Q2: 124.5

    Q3: 129

    Maximum: 149

Section 14.9: Variance

1. Standard deviation and variance are both measures of spread, but variance is a larger number. Standard deviation is the square root of variance.
2. Dataset A has data that is more varied and spread out than the data in Dataset B.
3. 33.2967
4. 16.5432
5. 254.568
6. 33.2967
7. 16.5432
8. 254.568
9. 4
10. c
11. If variance is large, data will be spread out on a histogram. There might be a lot of data at low values and a lot of data at high values, with not much in between. If variance is small, all data will be close together near the mean.
12. No, you can only calculate the variance for quantitative data, not categorical data. Bar graphs only show categorical data.
13. Mean: 81.5; Sample Variance: 149.105; Sample Standard Deviation: 12.2109
14. Mean: 5.15; Population Variance: 12.0275; Population Standard Deviation: 3.46807
15. It is often not realistic or possible to find data from the whole population, and then you have to be satisfied with having only a sample of the population. For example, it would be impossible to find a piece of information from every person in the world, but you might be able to get data samples from every country.

Section 14.10: The Normal Curve

1. 0
2. 1
3. 84.13%
4. 15.87%
5. 2.28%
6. 95.45%
7. 64.69%
8. 0
9. 90.31%
10. 1.16%
11. 1.64 standard deviations below the mean.
12. 49.38%
13. Approximately 68% of adult women are between 63 and 67 inches tall.
14. 69.15%
15. 0.644%

Section 14.11: Linear Correlation

1. Negative
2. No correlation
3. Positive
4. Very strong positive correlation. The data is perfectly linear with a positive slope. As the value of one variable increases, the value of the other variable increases.
5. Mild negative correlation. The data is somewhat linear with a negative slope. As the value of one variable increases, the value of the other variable tends to decrease.
6. Very strong negative correlation. The data is perfectly linear with a negative slope. As the value of one variable increases, the value of the other variable decreases.
7. No correlation. There is no relationship between the two variables. As the value of one variable increases, there is no pattern to what happens to the value of the other variable.
8. Strong positive correlation. The data is strongly linear with a positive slope. As the value of one variable increases, the value of the other variable increases.
9. $\hat{y} = 0.0974125588 + 0.005546124x$; $r = 0.94668$
10. 2.87. This seems to fit with the data.
11. No. Just because the two variables are correlated doesn't mean that a high SAT score would *cause* a high GPA.
12. $\hat{y} = 9.1513x + 168.692$; $r = 0.98082$
13. $425
14. Answers vary.
15. $\hat{y} = 5659.45 - 90x$; $r =$ -0.8491
16. 4,579
17. Answers vary.
18. Answers vary. Possible answer: The correlation coefficient measures how strong the correlation is between two variables and whether the correlation is positive or negative.
19. Answers vary. Possible answer: If you have a larger sample size, you will have more data points and will be closer to having data from the full population.

Section 14.12: Modeling with Regression

1. Two good choices are natural log regression and logistic regression. With natural log regression, the equation is $\hat{y} = 11.4146888 ln(x) + 27.61585$, and with logistic regression, the equation is $\hat{y} = \frac{65.8877}{1+1.422e^{-.210439}}$.
2. The logistic function is a better fit because it levels off as the height of the women levels off.
3. The natural log regression does not have a y-intercept, and the logistic equation has a y-intercept as 26.7. This would be the height of a baby when it was born. The typical length of a baby at birth is 20 inches, so the model is a bit unreasonable.
4. With the logistic equation, the predicted height is 64.88768 inches. This is reasonable because women aren't growing anymore when they are 70. To be perfectly realistic, the model should level off for awhile in the middle and then start to decrease, as women tend to actually slowly lose height as they get older, after about age 40.
5. The equation is $\hat{y} = 58.9 \cdot 0.514^x$.
6. Exponential regression makes sense because she is losing approximately half her height with each sip. This is exponential decay with a common ratio of $\frac{1}{2}$.
7. About 5 sips.
8. 1.0867 inches.
9. The equation is $\hat{y} = \frac{385.3269}{1+74{,}425{,}077e^{-3.55923x}}$.

10. The logistic model is appropriate because there is a maximum for how many people can know the rumor (400 students), so the model needs to level off around 400.
11. The model says 1 person will know the rumor after about 3.4 days. This doesn't fit with the actual data, which found that after 3 days, 29 people knew the rumor. After 5 days, the model tends to fit the data much better.
12. Sine regression works very well. The equation is $\hat{y} = 4.033 \cdot \sin(0.51745x + 0.0307) + 8.94582$.
13. The predicted depth is 5.392634697 feet. The actual depth is 5.4 feet. The residual is 0.073653035.
14. The cubic regression equation is $\hat{y} = 0.052x^3 - 0.92837995x^2 + 3.737x + 8.6615$. This model fits the data points well, but doesn't make as much sense out of the domain of 0 to 10 hours. The cubic model shows the depth continually increasing after 10 hours, which it of course wouldn't actually do. The sine model makes more sense because it is periodic, just as the depth of the water will be.
15. Modeling with regression allows you to quickly make predictions and generalizations about relationships between variables. When there are a lot of data, modeling helps to summarize visually and algebraically.

16.15 Answers - Ch 15: A Preview of Calculus

Section 15.2: Limit Notation

1. $\lim_{x\to\infty} \frac{2x^4+4x^2-1}{5x^4+3x+9} = \lim_{x\to-\infty} \frac{2x^4+4x^2-1}{5x^4+3x+9} = \frac{2}{5}$
2. $\lim_{x\to\infty} \frac{8x^3+4x^2-1}{2x^3+4x+7} = \lim_{x\to-\infty} \frac{8x^3+4x^2-1}{2x^3+4x+7} = 4$
3. $\lim_{x\to\infty} \frac{x^2+2x^3-3}{5x^3+x+4} = \lim_{x\to-\infty} \frac{x^2+2x^3-3}{5x^3+x+4} = \frac{2}{5}$
4. $\lim_{x\to\infty} \frac{4x+4x^2-5}{2x^2+3x+3} = \lim_{x\to-\infty} \frac{4x+4x^2-5}{2x^2+3x+3} = 2$
5. $\lim_{x\to\infty} \frac{3x^2+4x^3+4}{6x^3+3x^2+6} = \lim_{x\to-\infty} \frac{3x^2+4x^3+4}{6x^3+3x^2+6} = \frac{2}{3}$
6. $\lim_{x\to3} 2x^2+1 = 19$
7. $\lim_{x\to-\infty} e^x = 0$
8. $\lim_{x\to\infty} \frac{1}{x} = 0$
9. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{4}\right)^i = \frac{1}{3}$
10. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i}$ does not exist
11. $\lim_{n\to\infty} \sum_{i=0}^{n} \left(\frac{1}{2}\right)^i = 2$
12. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{9}{10^i} = 1$
13. The limit of $f(x) = \frac{5x^2-4}{x+1}$ as x approaches 0 is -4.
14. The limit of $y = \frac{x^3-1}{x-1}$ as x approaches 1 is 3.
15. Yes, it's possible as long as a does not equal positive or negative infinity. Number 6 is an example of this.

Section 15.3: Graphs to Find Limits

1. $\lim_{x\to-\infty} f(x) = 0$
2. $\lim_{x\to\infty} f(x) = DNE$
3. $\lim_{x\to2} f(x) = DNE$
4. $\lim_{x\to0} f(x) = 1$
5. $f(0) = 2$
6. $f(2) = 6$
7. $\lim_{x\to-\infty} g(x) = DNE$
8. $\lim_{x\to\infty} g(x) = 1$
9. $\lim_{x\to2} g(x) = -2$
10. $\lim_{x\to0} g(x) = DNE$
11. $\lim_{x\to4} g(x) = 0$
12. $g(0) = 2$
13. $g(2) = 4$
14. Answers vary.

15. Answers vary.

Section 15.4: Tables to Find Limits

1. 10
2. -5
3. DNE
4. $\frac{1}{2\sqrt{2}} \approx 0.35355$
5. DNE
6. 9
7. -6
8. $\frac{1}{2\sqrt{5}} \approx 0.2236$
9. $\frac{1}{6}$
10. 5
11. -6
12. $\frac{1}{4}$
13. $\frac{1}{4}$
14. $\frac{2}{15}$
15. DNE

Section 15.5: Substitution to Find Limits

1. 10
2. -5
3. -7
4. -$\frac{1}{5}$
5. 3
6. 9
7. -6
8. -$\frac{1}{3}$
9. 7
10. 5
11. -6
12. DNE
13. 6
14. $\frac{2}{15}$
15. DNE

Section 15.6: Rationalization to Find Limits

1. $\frac{1}{6}$
2. $\frac{1}{4}$

3. $\frac{1}{4}$
4. $\frac{1}{2\sqrt{3}}$
5. $\frac{5}{8}$
6. $-\frac{1}{4}$
7. $\frac{1}{2\sqrt{7}}$
8. 8
9. $4\sqrt{3}$
10. $\frac{1}{6}$
11. $\frac{1}{2}$
12. 2
13. 64
14. -144
15. If the function is a rational expression with a square root somewhere, there is a good chance that rationalizing will help you to evaluate the limit.

Section 15.7: Continuity

1. 1
2. 1
3. Yes
4. 11
5. 11
6. No, because $g(-2) \neq 11$.
7. -3
8. -3
9. -2
10. No, because $h(0) \neq -3$
11. Answers vary. Any continuous interval that does not include 0 should work.
12. Answers vary. Possible answer: [-1, 1].
13. Since $\lim_{x \to -2^-} 3x+1 = -5$ and $\lim_{x \to -2^+} -2x-1 = 3$, there is no value of k that makes the function continuous.
14. 1
15. -3

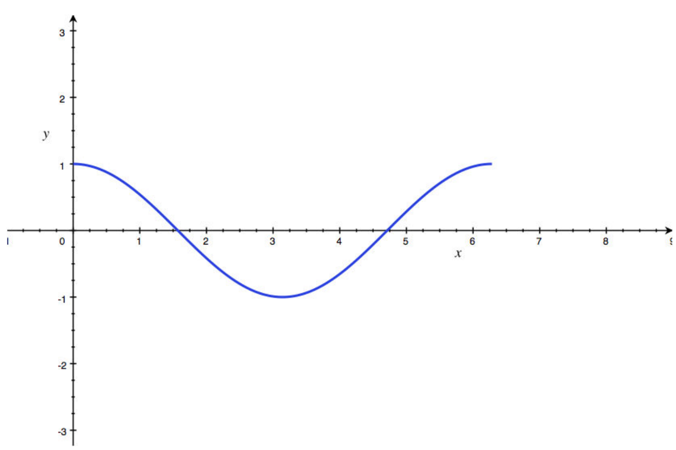
Section 15.8: Intermediate and Extreme Value Theorems

1. $f(-1) = \cos(-1) - 1 = -0.4596$ and $f(1) = \cos(1) + 1 = 1.54$; therefore, there must exist a c such that $f(c) = 0$ because $-0.4596 < 0 < 1.54$.
2. $f(1) = ln(1) - e^{-1} - 1 = -1.37$ and $f(3) = ln(3) - e^{-3} - 1 = 0.4883$; therefore, there must exist a c such that $f(c) = 0$ because $-1.37 < 0 < 0.4883$.
3. $f(1) = 2(1)^3 - 5(1)^2 - 10(1) + 5 = -8$ and $f(0) = 2(0)^3 - 5(0)^2 - 10(0) + 5 = 5$; therefore, there must exist a c such that $f(c) = 0$ because $-8 < 0 < 5$.
4. $f(x) = x^3 - x - 1$. $f(0) = -1$ and $f(2) = 5$; therefore, there must exist a c such that $f(c) = 0$ because $-1 < 0 < 5$.
5. $f(-2) = (-2)^2 - \cos(-2) = 4.4$ and $f(0) = (0)^2 - \cos(0) = -1$; therefore, there must exist a c such that $f(c) = 0$ because $-1 < 0 < 4.4$.

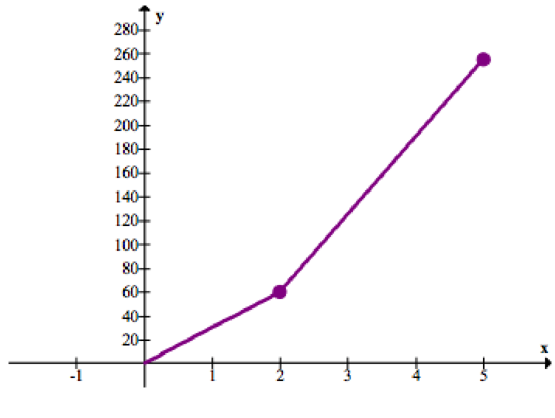
6. $f(x) = x^5 - 2x^3 - 2$. $f(1) =$ -3 and $f(2) = 14$; therefore, there must exist a c such that $f(c) = 0$ because -3 $< 0 < 14$.
7. $f(x) = 3x^2 + 4x - 11$. $f(1) =$ -4 and $f(2) = 9$; therefore, there must exist a c such that $f(c) = 0$ because -4 $< 0 < 9$.
8. $f(x) = 5x^4 - 6x^2 - 1$. $f(1) =$ -2 and $f(2) = 55$; therefore, there must exist a c such that $f(c) = 0$ because -2 $< 0 < 55$.
9. $f(x) = 7x^3 - 18x^2 - 4x + 1$. $f(\text{-}1) =$ -20 and $f(0) = 1$; therefore, there must exist a c such that $f(c) = 0$ because -20 $< 0 < 1$.
10. $f(1) = \frac{1}{3}$ and $f(2) =$ -1; therefore, there must exist a c such that $f(c) = 0$ because -1 $< 0 < \frac{1}{3}$.
11. $f(\text{-}1) =$ -1 and $f(0) = \frac{1}{4}$; therefore, there must exist a c such that $f(c) = 0$ because -1 $< 0 < \frac{1}{4}$.
12. False
13. True
14. True
15. Functions must be continuous over given intervals in order for the theorems to apply.

Section 15.9: Instantaneous Rate of Change

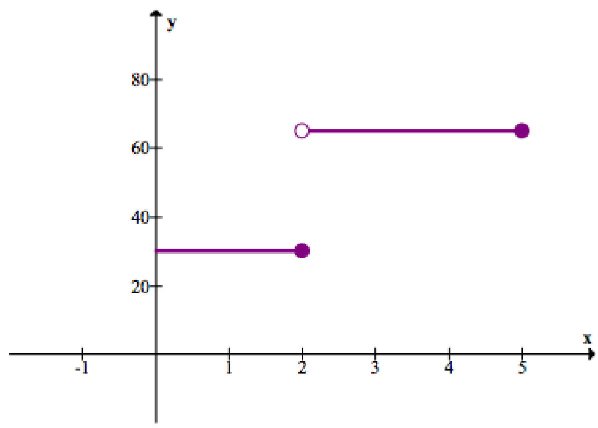
1. The slope appears to be 2.
2. The limit is 2, which is the same as what the slope appeared to be in Number 1.
3. The slope appears to be 6.
4. The limit is 6, which is the same as what the slope appeared to be in Number 3.
5. The slope appears to be 3.
6. The limit is 3, which is the same as what the slope appeared to be in Number 5.
7. The slope appears to be 6.
8. $\lim_{x \to 1} \left(\frac{2x^3 - 1 - 1}{x - 1} \right)$
9. The slope at 0 is 0. The slope at $\frac{\pi}{2}$ is -1. The slope at π is 0. The slope at $\frac{3\pi}{2}$ is 1. The slope at 2π is 0.



10. The derivative of the cosine function is the negative sine function.
11. The slope is 2 at every point. The derivative of the function is $y = 2$.
12. Distance vs. Time



Rate vs. Time



13. A tangent line is a line that "just touches" a curve. The slope of the tangent line at a given point is the derivative of the function at that point.
14. Instantaneous rate of change is the speed at a given point. Speed is shown as slope in functions; therefore, the slope of the tangent line will be the speed or instantaneous rate of change at that point.
15. We can't calculate a slope with a denominator of 0, but we can use limits to find the limit of the slope as the denominator approaches 0.

Section 15.10: Area Under a Curve

1. 176
2. 60
3. 8.79
4. 8.86
5. -0.33
6. -0.59
7. -0.72
8. The car is going at a constant speed of 25 mph for 3 hours, and then instantly starts going 65 mph for the next 2 hours.
9. 205 miles

10. The car accelerates steadily from 0 to 75 meters per second in the first 3 seconds, and then stays at 75 meters per second for the next 2 seconds.
11. 262.5 feet
12. The runner increases in speed from 0 feet per second to 16 feet per second, then slows back down to 0 feet per second.
13. The exact answer is $\frac{256}{3} \approx 85.33$.
14. Integrals are areas under a curve. They can be calculated by finding the sum of the areas of an infinite number of rectangles.

16.16 References

1. CK-12 Foundation. .
2. CK-12 Foundation. .
3. CK-12 Foundation. .
4. CK-12 Foundation. .
5. CK-12 Foundation. .
6. CK-12 Foundation. .
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106. CK-12 Foundation. .
107. CK-12 Foundation. .
108. CK-12 Foundation. .
109. CK-12 Foundation. .
110. CK-12 Foundation. .
111. CK-12 Foundation. .
112. Supreme Aryal. http://www.texample.net/tikz/examples/unit-circle/ .
113. ck-12 foundation. ck12 foundation .
114. ck-12 foundation. ck12 foundation .
115. ck-12 foundation. ck-12 foundation .
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CHAPTER 17 Appendix G: Glossary

Chapter Outline

17.1 CK-12 College Precalculus Glossary

30-60-90:
A 30-60-90 triangle is a special right triangle with angles of 30°, 60°, and 90°.

45-45-90 Triangle:
A 45-45-90 triangle is a special right triangle with angles of 45°, 45°, and 90°.

A

Absolute Deviation:
The sum total of how different each number is from the mean.

Absolute Extrema:
The points with the y-values that are the highest or lowest of the entire function.

Amplitude:
One-half of the difference between the minimum and maximum values of a wave, and can be related to the radius of a circle.

Angle of Depression:
The angle formed by a horizontal line and the line of sight down to an object, when the image of the object is located beneath the horizontal line.

Angle of Elevation:
The angle formed by a horizontal line and the line of sight up to an object, when the image of the object is located above the horizontal line.

Angular Velocity:
For a rotating object, the change in angle of the object divided by the change in time.

Arc Length:
Found using the formula $s = r \cdot \theta$, where s is the length of the arc, r is the radius, and θ is the measure of the angle in radians.

Arithmetic Growth:
Occurs when a quantity increases by the same amount in each given time period (repeated addition).

Arithmetic Sequence:
Has a common difference between each two consecutive terms, which are also known as arithmetic progressions.

Arithmetic Series:
The sum of an arithmetic sequence, a sequence with a common difference between each two consecutive terms.

ASA:
Angle-side-angle, refers to two known angles in a triangle with one known side between the known angles.

Asymptotes:
A line on the graph of a function representing a value toward which the function may approach, but does not reach (with certain exceptions).

Asymptotic:
A function is asymptotic to a given line if the given line is an asymptote of the function.

Augmented Matrix:
A matrix formed when two matrices are joined together and operated on as if they were a single matrix.

Average Rate of Change:
Of a function, the change in y-coordinates of a function, divided by the change in x-coordinates.

B

Bar Chart
A graphic display of categorical variables that uses bars to represent the frequency of the count in each category.
Base Case
In an induction proof, the anchor step. It is the 1st domino to fall, creating a cascade and thus proving the statement true for every number greater than the base case.
Bearing:
How direction is measured at sea. North is 0°, east is 90°, south is 180°, and west is 270°.
Bimodal:
If there are two numbers that occur equally frequently in a set of data, then the data is said to be bimodal.
Binomial Expansion:
The process of raising a binomial such as (x+2) to a power. The process can be time-consuming when completed manually, particularly with higher exponents.
Binomial Theorem:
An efficient formula for calculating the expansion of binomials. It states that $(x+y)^n = \sum_{r=0}^{n} \left(\binom{n}{r} x^{n-r} y^r \right)$.
Bivariate Data:
Consists of two paired sets of data.
Bounds on Zeros Theorem:
States that if f is continuous on [a, b] and there is a sign change between $f(a)$ and $f(b)$ (that is, $f(a)$ is positive and $f(b)$ is negative, or vice versa), then there is a $c \in (a,b)$, such that $f(c) = 0$.
Boxplot:
Graphic display of quantitative data that illustrates the five-number summary.

C

Carrying Capacity:
The maximum sustainable population that the environmental factors will support. In other words, it is the population limit.
Categorical Variable:
A variable that can take on one of a limited number of values. Examples of categorical variables are TV stations, the state someone lives in, and eye color.
Center:
Of a circle, the point that defines the location of the circle. All points on the circle are equidistant from the center of the circle.
Circle:
The set of all points at a specific distance from a given point in two dimensions.
Closed Interval:
Interval with square or box brackets, [ ], in which the interval includes the endpoints.
Cofunction:
Cofunctions are functions that are identical except for a reflection and horizontal shift. Examples include sine and cosine, tangent and cotangent, and secant and cosecant.
Combination:
Distinct arrangements of a specified number of objects without regard to order of selection from a specified set.
Common Difference:

Every arithmetic sequence has a common or constant difference between consecutive terms. For example, in the sequence 5, 8, 11, 14, ..., the common difference is 3.

Common Logarithm:
A log with base 10. The log is usually written without the base.

Common Ratio:
Every geometric sequence has a common ratio, or a constant ratio between consecutive terms. For example, in the sequence 2, 6, 18, 54, ..., the common ratio is 3.

Complement:
A mutually exclusive pair of events are complements to each other. For example, if the desired outcome is *heads* on a flipped coin, the complement is *tails*.

Completing the Square:
A common method for rewriting quadratics. It refers to making a perfect square trinomial by adding the square of one half of the coefficient of the x-term.

Compound Interest:
Interest earned on the total amount at the time it is compounded, including previously earned interest.

Conic:
Conic sections are those curves that can be created by the intersection of a double cone and a plane. They include circles, ellipses, parabolas, and hyperbolas.

Conjugate Pairs Theorem:
States that if $f(z)$ is a polynomial of degree n, with $n \neq 0$ and with real coefficients, and if $f(z_0) = 0$, where $z_0 = a + bi$, then $f(z_0^*) = 0$ where z_0^* is the complex conjugate of z_0.

Continuously Compounding:
Refers to a loan or investment with interest that is compounded constantly, rather than on a specific schedule. It is equivalent to infinitely many but infinitely small compounding periods.

Correlation Coefficient:
Standard quantitative measure of best fit of a line. It has the symbol r and has values from -1 to +1.

Cosecant:
Of an angle in a right triangle, a relationship found by dividing the length of the hypotenuse by the length of the side opposite to the given angle. This is the reciprocal of the sine function.

Cosine:
Of an angle in a right triangle, a value found by dividing the length of the side adjacent to the given angle by the length of the hypotenuse.

Cotangent:
Of an angle in a right triangle, a relationship found by dividing the length of the side adjacent to the given angle by the length of the side opposite to the given angle. This is the reciprocal of the tangent function.

Coterminal Angles:
Angles with the same terminal side but expressed differently, such as a different number of complete rotations around the unit circle, or angles being expressed as positive versus negative angle measurements.

Cramer's Rule:
A formula involving ratios of determinants for the solution of a system of linear equations.

Cross Product:
Of two vectors, a 3rd vector that is perpendicular to both of the original vectors.

D

Decreasing
A function is decreasing over an interval if its y-values are getting smaller over the interval. The graph will go down from left to right over the interval.

Degenerate Conic:
A conic that does not have the usual properties of a conic section. Since some of the coefficients of the general conic equation are zero, the basic shape of the conic is merely a point, a line, or a pair of intersecting lines.

Derivative:
Of a function, the slope of the line tangent to the function at a given point on the graph. Notations for derivatives include $f'(x)$, $\frac{dy}{dx}$, y', $\frac{df}{dx}$, and $\frac{df(x)}{dx}$.

Descartes's Rule of Signs:
A technique for determining the number of positive and negative real roots of a polynomial.

Descriptive Statistics:
In descriptive statistics, the goal is to describe the data that is found in a sample or given in a problem.

Determinant:
A single number descriptor of a square matrix. The determinant is computed from the entries of the matrix, and has many properties and interpretations explored in linear algebra.

Dihedral Angle:
An angle between two planes in three-dimensional space.

Directrix:
Of a parabola, the line that the parabola seems to curve away from. All points on a parabola are equidistant from the focus of the parabola and the directrix of the parabola.

Discontinuities:
The points of discontinuity for a function are the input values of the function where the function is discontinuous.

Discontinuous:
A function is discontinuous if the function exhibits breaks or holes when graphed.

Displacement Vector:
Models the movement between one point and another on a coordinate plane.

Domain:
Of a function, the set of x-values for which the function is defined.

Dot Product:
The inner or scalar product. The two forms of the dot product are $\vec{a} \cdot \vec{b} = \left\|\vec{a}\right\| \left\|\vec{b}\right\| \cos\theta$ and $\vec{a} \cdot \vec{b} = x_a x_b + y_a y_b$.

Double Angle Identity:
Relates the trigonometric function of two times an argument to a set of trigonometric functions, each containing the original argument.

E

e:
An irrational number that is approximately equal to 2.71828. As $n \to \infty$, $\left(1+\frac{1}{n}\right)^n \to e$.

Eccentricity:
A measure of how much the conic section deviates from being circular. The eccentricity of circles is 0, of ellipses is between 0 and 1, of parabolas is 1, and of hyperbolas is greater than 1. For ellipses and hyperbolas, $eccentricity = \frac{c}{a}$.

Ellipse:
Conic sections that look like elongated circles. An ellipse represents all locations in two dimensions that are the same distance from two specified points called foci.

Empirical Rule:
States that for data that are normally distributed, approximately 68% of the data will fall within 1 standard deviation of the mean, approximately 95% of the data will fall within 2 standard deviations of the mean, and approximately 99.7% of the data will fall within 3 standard deviations of the mean.

End Behavior:
A description of the trend of a function as input values become very large or very small, represented as the "ends" of a graphed function.

Even Function:
A function with a graph that is symmetric with respect to the y-axis, and has the property that $f(-x) = f(x)$.

Expected Value:
The return or cost you can expect on average, given many trials.

Explicit Formula:
Defines each term in a sequence directly, allowing one to calculate any term in the sequence without knowing the value of the previous terms.

Exponential Decay:
Occurs when a quantity decreases by the same proportion in each given time period.

Exponential Function:
A function whose variable is in the exponent. The general form is $y = a \cdot b^{x-h} + k$.

Exponential Growth:
Occurs when a quantity increases by the same proportion in each given time period.

Extreme Value Theorem:
States that in every interval [a, b] where a function is continuous, there is at least one maximum and one minimum. In other words, it must have at least two extreme values.

F

Factored Form:
The factored form of a quadratic function $f(x)$ is $f(x) = a(x-r_1)(x-r_2)$, where r_1 and r_2 are the roots of the function.

Factor Theorem:
States that if $f(x)$ is a polynomial of degree $n > 0$ and $f(c) = 0$, then $x-c$ is a factor of the polynomial $f(x)$.

Factorization Theorem:
States that if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n \neq 0$, and n is a positive integer, then $f(x) = a_n(x-c_1)(x-c_2)\cdots(x-c_0)$, where the numbers c_i are complex numbers.

First Quartile:
Also known as Q_1, the median of the lower half of a set of data.

Five-Number Summary:
Of a set of data, it is the minimum, 1st quartile, 2nd quartile, 3rd quartile, and maximum.

Foci:
Two points that define an ellipse or hyperbola. The sum of the distances from any point on an ellipse to the foci is constant. For every point on a hyperbola, the difference of the distances to each focus is constant.

Focus:
Of a parabola, the point that "anchors" a parabola. Any point on the parabola is exactly the same distance from the focus as from the directrix.

Frequency:
Of a trigonometric function, the number of cycles that a periodic function completed in 1 unit interval.

Function Composition:
Involves "nested functions" or functions within functions. Function composition is the application of one function to the result of another function.

Function Notation:
The notation used to describe a function, often written as $f(x)$.

Fundamental Counting Principle:
States that if an event can be chosen in p different ways and another independent event can be chosen in

q different ways, the number of different arrangements of the events is $p \times q$.

Fundamental Theorem of Algebra:

States that if $f(x)$ is a polynomial of degree $n \geq 1$, then $f(x)$ has at least one zero in the complex number domain. In other words, there is at least one complex number c such that $f(c) = 0$. The theorem can also be stated as follows: an nth degree polynomial with real or complex coefficients has, with multiplicity, exactly n complex roots.

G

Gauss-Jordan Elimination:

Putting a matrix into reduced row echelon form is a result of performing Gauss-Jordan elimination.

Geometric Sequence:

A sequence with a constant ratio between successive terms. Geometric sequences are also known as geometric progressions.

Geometric Series:

A geometric sequence written as an uncalculated sum of terms.

Global Extrema:

Of a function, the points with the y-values that are the highest or the lowest of the entire function.

Global Maximum:

Of a function, the largest value of the entire function. Symbolically, it is the highest point on the entire graph.

Global Minimum:

Of a function, the smallest value of the entire function. Symbolically, it is the lowest point on the entire graph.

H

Half Angle Identity:

Relates a trigonometric function of one-half of an argument to a set of trigonometric functions, each containing the original argument.

Half-Life:

Refers to the time required for a radioactive material to decay to one-half of its initial concentration.

Histogram:

A display that indicates the frequency of specified ranges of continuous data values on a graph in the form of immediately adjacent bars.

Horizontal Line Test:

Says that if a horizontal line drawn anywhere through the graph of a function intersects the function in more than one location, then the function is not one-to-one and not invertible.

Hyperbola:

A conic section formed when the cutting plane intersects both sides of a cone, resulting in two infinite U-shaped curves.

I

Identity:

A mathematical sentence involving the symbol "=" that is always true for variables within the domains of the expressions on either side.

Identity Matrix:
A matrix with zeros everywhere except along the diagonal, where there are ones.
Included Angle:
In a triangle, the angle between two known sides.
Increasing:
A function is increasing over an interval if its y-values are getting larger over the interval. The graph will go up from left to right over the interval.
Independent Events:
Two events are independent if the occurrence of one event does not impact the probability of the other event.
Induction:
A method of mathematical proof typically used to establish that a given statement is true for all positive integers.
Inductive Hypothesis:
In an induction proof, the inductive hypothesis is the step where you assume the statement is true for k.
Inductive Step:
In an induction proof, the inductive step is the proof. It is when you show the statement is true for $k+1$ using only the inductive hypothesis and algebra.
Inferential Statistics:
With inferential statistics, your goal is to use the data in a sample to draw conclusions about a larger population.
Instantaneous Rate of Change:
Of a curve at a given point, the slope of the line tangent to the curve at that point.
Integral:
Used to calculate the area under a curve or the area between two curves.
Intercepts:
Of a curve, the locations where the curve intersects the x and y axes. An x-intercept is a point at which the curve intersects the x-axis. A y-intercept is a point at which the curve intersects the y-axis.
Intermediate Value Theorem:
States that if $f(x)$ is continuous on some interval $[a,b]$ and n is between $f(a)$ and $f(b)$, then there is some $c \in [a,b]$ such that $f(c) = n$.
Interval Notation:
The notation $[a,b)$, where a function is defined between a and b. Use (or) to indicate that the end value is not included and [or] to indicate that the end value is included. Never use [or] with infinity or negative infinity.
Inverse Function:
Two functions are inverses of each other if their graphs are reflections of each other over the line $y = x$. Formally, $f(x)$ and $g(x)$ are inverse functions if $f(g(x)) = g(f(x)) = x$.
Inverse Properties of Logarithms:
Inverse properties of logarithms are $\log_b b^x = x$ and $b^{\log_b x} = x, b \neq 1$.
Irrational Number:
A number that cannot be expressed exactly as the quotient of two integers.

L

Law of Cosines:
A rule relating the sides of a triangle to the cosine of one of its angles. The law of cosines states that $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the angle across from side c.
Law of Sines:
A rule applied to triangles stating that the ratio of the sine of an angle to the side opposite that angle, is equal to the ratio of the sine of another angle in the triangle to the side opposite that angle.

Limit:
The value that the output of a function approaches as the input of the function approaches a given value.
Limit Notation:
A way of expressing the fact that a function gets arbitrarily close to a value.
Linear Combination:
A set of terms that are added or subtracted from each other with a multiplicative constant in front of each term.
Linear Correlation:
A measure of the strength of the linear relationship between two random variables.
Linear Regression:
In statistics, a process that attempts to model the relationship between two variables by fitting a linear equation to the data.
Linear Velocity:
Of an object, the change in position of an object divided by the change in time.
Local Extrema:
Of a function, the points of the function with y-values that are the highest or lowest of a local neighborhood of the function.
Local Maximum:
The highest point relative to the points around it. A function can have more than one local maximum.
Local Minimum:
The lowest point relative to the points around it. A function can have more than one local minimum.
Logarithmic Function:
Inverse of an exponential function. Recall that $\log_b n = a$ is equivalent to $b^a = n$.
Logistic Function:
A function that grows or decays rapidly for a period of time and then levels out. It takes the form $f(x) = \frac{c}{1+a \cdot b^x}$.

M

Magnitude:
Of a line segment or vector, the length of the line segment or vector.
Major Axis:
Of an ellipse, the longest diameter of the ellipse.
Matrix:
A rectangular arrangement of data elements presented in rows and columns.
Matrix Operations:
The primary matrix operations are addition, subtraction, and multiplication.
Mean:
Often called the average, the mean of a numerical set of data is simply the sum of the data values divided by the number of values.
Mean Absolute Deviation:
An alternate measure of how spread out data are. It involves finding the mean of the distance between each data value and the mean. While this method might seem more intuitive, in statistics it has been found to be too limited and is not commonly used.
Median:
Of a dataset, the middle value of the organized dataset.
Minor Axis:
Of an ellipse, the shortest diameter of the ellipse.
Mode:
Of a dataset, the value or values with greatest frequency in the dataset.

Monotonic:
A function is monotonic if it does not switch between increasing and decreasing at any point.

Multimodal:
When a set of data has more than two values that occur with the same greatest frequency, the set is called multimodal.

Multiplicative Inverse:
Of a number, the reciprocal of the number. The product of a real number and its multiplicative inverse will always be equal to 1 (which is the multiplicative identity for real numbers).

Multiplicity:
Of a term, describes the number of times the given term acts as a zero of the given function.

N

Natural Log:
A logarithm with base e. The natural logarithm is written as "ln."

Normal Curve:
The curve that defines the probability density graph for a normally distributed variable.

Normal Vector:
A vector that is perpendicular to a given surface or plane. A unit normal vector is a normal vector with a magnitude of one.

Normalcdf:
The normal cumulative distribution function. It calculates the area between any two values for data that are normally distributed, as long as you know the mean and standard deviation for the data. Your calculator has this function built in, and it produces an exact answer as opposed to the empirical rule.

N-Roots Theorem:
If $f(x)$ is a polynomial of degree n, where $n \neq 0$, then $f(x)$ has at most n zeros.

O

Oblique Asymptote:
A diagonal line marking a specific range of values toward which the graph of a function may approach, but generally never reach. An oblique asymptote exists when the numerator of the function is exactly one degree greater than the denominator. An oblique asymptote may be found through long division.

Odd Function:
A function with the property that $f(-x) = -f(x)$. Odd functions have rotational symmetry about the origin.

One-Sided Limit:
The value that a function approaches from either the left side or the right side.

One-to-One Function:
A function is one-to-one if its inverse is also a function. A one-to-one function passes both the horizontal and vertical line tests.

Open Interval:
Does not include the endpoints of the interval.

Order:
Of a matrix, describes the number of rows and the number of columns in the matrix.

Orthogonality:
To be orthogonal is to be perpendicular.

P

Parabola:
The characteristic shape of a quadratic function graph, resembling a U. Specifically, the set of points that are equidistant from a fixed point on the interior of the curve, called the "focus," and a line on the exterior, called the "directrix." The directrix is vertical or horizontal, depending on the orientation of the parabola.

Parametric Form:
Means to define x and y as a function of a 3rd variable, often t.

Partial Fraction Decomposition:
A procedure that undoes the operation of adding fractions with unlike denominators. It separates a rational expression into the sum of rational expressions with unlike denominators.

Pascal's Triangle:
A triangular array of numbers constructed with the coefficients of binomials as they are expanded. The ends of each row of Pascal's Triangle are ones, and every other number is the sum of the two nearest numbers in the row above.

Payoff:
Of a game, the expected value of the game minus the cost.

Period:
Of a wave, the horizontal distance traveled before the y-values begin to repeat.

Periodic Function:
A function with a predictable repeating pattern. Sine waves and cosine waves are periodic functions.

Permutation:
An arrangement of objects where order is important.

Phase Shift:
A horizontal translation or shift of a period function.

Pi (π):
π (pi) is the ratio of the circumference of a circle to its diameter. It is an irrational number that is approximately equal to 3.14.

Pie Chart:
A graphic display of categorical data where the relative size of each pie slice corresponds to the frequency of each category.

Piecewise Function:
A function that pieces together two or more parts of other functions to create a new function.

Polar Coordinates:
Describe locations on a grid using the polar coordinate system. The location of each point is determined by its distance from the pole and its angle with respect to the polar axis.

Polynomial Function:
A function defined by an expression with at least one algebraic term.

Polynomial Inequality:
Generally used to describe an inequality with an x-term coefficient of three or greater.

Population:
In statistics, the entire group of interest from which the sample is drawn.

Power Function:
A polynomial of the form $f(x) = ax^n$, where a is a real number and n is an integer with $n \geq 1$.

Power Reducing Identity:
Relates the power of a trigonometric function containing a given argument to a set of trigonometric functions, each containing the original argument.

Probability:
The chance that something will happen. It can be written as a fraction, decimal, or percent.

Product Property of Logarithms:

States that as long as $b \neq 1$, then $\log_b xy = \log_b x + \log_b y$.

Proof:

A series of true statements leading to the acceptance of truth of a more complex statement.

Pythagorean Identity:

A relationship showing that the sine of an angle squared plus the cosine of an angle squared is equal to one.

Pythagorean Theorem:

A mathematical relationship between the sides of a right triangle, given by $a^2 + b^2 = c^2$,where a and b are legs of the triangle and c is the hypotenuse of the triangle.

Pythagorean Triple:

A set of three whole numbers a, b, and c that satisfy the Pythagorean Theorem, $a^2 + b^2 = c^2$.

Q

Quadrantal Angle:

An angle that has its terminal side on one of the four lines of axis: positive x, negative x, positive y, or negative y.

Quadratic Function:

A function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real constants and $a \neq 0$.

Quantitative Variable:

A variable that takes on numerical values that represent a measurable quantity. Examples of quantitative variables are the height of students or the population of a city.

Quartile:

Each of four equal groups that a dataset can be divided into.

Quotient Property of Logarithms:

States that as long as $b \neq 1$, then $\log_b \frac{x}{y} = \log_b x - \log_b y$.

R

Radian:

A unit of angle that is equal to the angle created at the center of a circle whose arc is equal in length to the radius.

Radius:

Of a circle, the distance from the center of the circle to the edge of the circle.

Range:

Of a function, the set of y-values for which the function is defined.

Rank:

Of an observation, the number of observations that are less than or equal to the value of that observation.

Rational Function:

Any function that can be written as the ratio of two polynomial functions.

Rational Inequality:

Ratio of two polynomials, specified to be greater or less than a given value.

Rational Zero Theorem:

States that for a polynomial, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, a_{n-1}, \cdots a_0$ are integers, the rational roots can be determined from the factors of a_n and a_0. More specifically, if p is a factor of a_0 and q is a factor of a_n, then all the rational factors will have the form $\pm \frac{p}{q}$.

Recursive:
The recursive formula for a sequence allows you to find the value of the nth term in the sequence if you know the value of the (n - 1)th term in the sequence.

Reduced Row Echelon Form:
A matrix is in reduced row echelon form if it has a leading one at the start of every non-zero row, zeros below every leading one, all rows containing only zeros at the bottom of the matrix, and only zeros to the right of leading ones in rows with leading ones.

Reference Angle:
The angle formed between the terminal side of the angle and the closest of either the positive or negative x-axis.

Relative Extrema:
Of a function, the points of the function with y-values that are the highest or lowest of a local neighborhood of the function.

Remainder Theorem:
States that if $f(k) = r$, then r is the remainder when dividing $f(x)$ by $(x-k)$.

Restricted Domain:
Refers to the fact that when creating an inverse, you sometimes must cut off the domain of most of the function, saving the largest possible portion so that when the inverse is created, it is also a function.

Resultant:
A vector representing the sum of two or more vectors.

Riemann Sum:
An approximation of the area under a curve, calculated by dividing the region up into shapes that approximate the space.

Right-Hand Rule:
Used to indicate the direction of a cross product. Position the thumb and index finger of our right hand with the 1st vector along your thumb and the 2nd vector along your index finger. Your middle finger, when extended perpendicular to your palm, will indicate the direction of the cross product of the two vectors.

Roots:
Of a function, the values of x that make y equal to zero.

Row Echelon Form:
A matrix is in row echelon form if it has a leading one at the start of every non-zero row, zeros below every leading one, and all rows containing only zeros at the bottom of the matrix.

Row Operations:
Include swapping rows, adding a multiple of one row to another, or scaling a row by multiplying through by a scalar.

S

Sample:
A specified part of a population, intended to represent the population as a whole.

Scalar Projection:
Of a vector onto another vector, equal to the length of the projection of the 1st vector onto the 2nd vector.

Scatter Plot:
A plot of the dependent variable versus the independent variable, used to investigate whether or not there is a relationship or connection between two sets of data.

Secant:
Of an angle in a right triangle, the value found by dividing the length of the hypotenuse by the length of the side adjacent to the given angle. The secant ratio is the reciprocal of the cosine ratio.

Secant Line:
A line that joins two points on a curve.

Second Quartile:
Also known as Q_2, the median of the data.

Sector:
A portion of a circle contained between two radii of the circle. Sectors can be measured in degrees.

Sector Area Formula:
Used to calculate how many degrees of the circle should be allocated to a given value, calculated by dividing the frequency of the data in the sector by the total frequency of the data all multiplied by 360.

Sequence:
An ordered list of numbers or objects.

Series:
The sum of the terms of a sequence.

Sigma Notation:
Also known as summation notation, a way to represent a sum of numbers. It is especially useful when the numbers have a specific pattern or would take too long to write out without abbreviation.

Sine:
Of an angle in a right triangle, a value found by dividing the length of the side opposite the given angle by the length of the hypotenuse.

Square Matrix:
A matrix in which the number of rows equals the number of columns.

Standard Deviation:
The square root of the variance. Standard deviation is one way to measure the spread of a set of data.

Standard Form:
Of a line, $Ax+By=C$, where A, B, and C are real numbers. The standard form of a quadratic function is $f(x)=ax^2+bx+c$.

Standard Normal Distribution:
$\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, a normal distribution with mean of 0 and a standard deviation of 1.

Standard Position:
Of an angle, measures an angle starting from the positive x-axis and going counter-clockwise. It is the typical method for drawing and measuring an angle.

Subtended Arc:
The part of the circle between the two rays that make the central angle.

Symmetrical about the Origin:
A point (x, y) is symmetric about the origin when (-x, -y) is also on the graph.

Symmetrical about the y-axis:
A point (x, y) is symmetric about the y-axis when (-x, y) is also on the graph.

Symmetric Matrix:
A square matrix with reflection symmetry across the main diagonal.

Synthetic Division:
A shorthand version of polynomial long division, where only the coefficients of the polynomial are used.

System of Equations:
A set of two or more equations.

T

Take the Log of Both Sides:
Take the log of both the entire right-hand side of the equation and the entire left-hand side of the equation. As

long as neither side is negative or equal to zero, it maintains the equality of the two sides of the equation.

Tangent:
Of an angle in a right triangle, a value found by dividing the length of the side opposite the given angle by the length of the side adjacent to the given angle.

Tangent Line:
A line that "just touches" a curve at a single point and no others.

Theta (θ):
A Greek letter used in math to stand for an unknown angle.

Third Quartile:
Also known as Q_3, the median of the upper half of the data.

Transcendental Number:
A number that is not the root of any rational polynomial function. Examples include e and π.

Transformation:
Moves a figure in some way on the coordinate plane.

U

Unit Circle:
A circle of radius one, centered at the origin.

Unit Normal Vector:
A vector with a magnitude of one that is perpendicular to the unit tangent vector and the curve.

Unit Vector:
A vector with a magnitude of one.

V

Variance:
A measure of the spread of the dataset equal to the mean of the squared variations of each data value from the mean of the dataset.

Vector:
A mathematical quantity that has both a magnitude and a direction.

Vector Projection:
Of a vector onto a given direction, a vector with a magnitude equal to the scalar projection. The direction of the vector projection is the same as the unit vector of that given direction.

Vertex:
The highest or lowest point on the graph of a parabola. The vertex is the maximum point of a parabola that opens downward and the minimum point of a parabola that opens upward.

Vertex Form:
Of a parabola, $(x-h)^2 = 4p(y-k)$ or $(y-k)^2 = 4p(x-h)$, where (h,k) is the vertex.

Vertical Line Test:
Says that if a vertical line drawn anywhere through the graph of a relation intersects the relation in more than one location, then the relation is *not* a function.

Vertical Shift:
The result of adding a constant term to the value of a function. A positive term results in an upward shift and a negative term in a downward shift.

W

Weighted Average:

An average that multiplies each component by a factor representing its frequency or probability.

Z

Zeros:

Of a function $f(x)$, the values of x that cause $f(x)$ to be equal to zero.